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On fuzzy filters of BE-algebras

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Abstract. Some properties of fuzzy filters and normal fuzzy filters are studied in BE-algebras. A set of equivalent conditions is derived for every fuzzy filter of a BE-algebra to become a normal fuzzy filter. The concept of maximal fuzzy filters is introduced in BE-algebras and some of the properties of this class of normal fuzzy filters are studied.

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1. Introduction

The notion of BE-algebras was introduced and extensively studied by H. S. Kim and Y.H. Kim in [8]. Some properties of filters of BE-algebras were studied by S. S. Ahn and K. S. So in [1] and then by H.S. Kim and Y.H. Kim in [8]. The concepts of a fuzzy set and a fuzzy relation on a set was initially defined by L.A. Zadeh [12]. Fuzzy relations on a group have been studied by Bhattacharya and Mukherjee [3]. In 1996, Y.B. Jun and S.M. Hong [5] discussed the fuzzy deductive systems of Hilbert algebras. Y. B. Jun [6] also studied the properties of fuzzy positive implicative filters in lattice implication algebras. Some properties of sub algebras of BE-algebras were studied by A. Rezaei and A. B. Saeid in [9]. Later, W. A. Dudek and Y. B. Jun [4] considered the fuzzification of ideals in Hilbert algebras and discussed the relation between fuzzy ideals and fuzzy deductive systems. Properties of anti fuzzy ideals and fuzzy quasi-ideals are studied in [2] and [7]. In [10], the author introduced the notion of fuzzy filters in BE-algebras and discussed some related properties. Recently, the concept of fuzzy implicative filters [11] is introduced in BE-algebras and studied their properties.

In this paper, some properties of fuzzy filters and normal fuzzy filters are studied in BE-algebras. For any given filter A of a BE-algebra X, the characteristic function X_A defined on A is proved to be a normal fuzzy filter of a BE-algebra. Fuzzy normal filters of BE-algebras are characterized. An extension property for normal fuzzy filters of BE-algebras is established. The notion of maximal fuzzy filters is introduced and obtained a relation between a maximal fuzzy filter and a normal fuzzy filter of a BE-algebra. Finally, homomorphic properties, cartesian products, and the strongest fuzzy relations of normal fuzzy filters are studied.

2. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [8], [10], [11] and [13] for the ready reference of the reader.

Definition 2.1 ([8]). An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:

1) $x * x = 1$
2) $x * 1 = 1$
3) $1 * x = x$
4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$

Theorem 2.2 ([8]). *Let $(X, *, 1)$ be a BE-algebra. Then we have the following:*

1) $x * (y * x) = 1$
2) $x * ((x * y) * y)) = 1$

We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ implies $x * y = 1$. A BE-algebra X is called self-distributive if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A non-empty subset S of a BE-algebra X is called a subalgebra of X if $x, y \in S$, then $x * y \in S$.

Definition 2.3 ([8]). A BE-algebra X is called commutative if $(x*y)*y = (y*x)*x$ for all $x, y \in X$.

Definition 2.4 ([8]). Let $(X, *, 1)$ be a BE-algebra. A non-empty subset F of X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:

(a) $1 \in F$
(b) $x \in F$ and $x * y \in F$ imply that $y \in F$

Definition 2.5 ([8]). Let $(X_1, *, 1)$ and $(X_2, \circ, 1')$ be two BE-algebras. Then a mapping $f : X_1 \longrightarrow X_2$ is called a homomorphism if $f(x * y) = f(x) \circ f(y)$ for all $x, y \in X_1$.

It it clear that if $f : X_1 \longrightarrow X_2$ is a homomorphism, then $f(1) = 1'$.

Definition 2.6 ([13]). Let X be a set. Then a fuzzy set in X is a function $\mu : X \longrightarrow [0, 1]$.

Definition 2.7 ([10]). Let X be a BE-algebra. A fuzzy set μ of X is called a fuzzy filter if it satisfies the following properties, for all $x, y \in X$:

(F_1) $\mu(1) \geq \mu(x)$
(F_2) $\mu(y) \geq min\{\mu(x), \mu(x * y)\}$

Theorem 2.8 ([10]). *Any filter of a BE-algebra X can be realized as a level filter of some fuzzy filter of X.*

Proposition 2.9 ([10]). *Let μ be a fuzzy filter of a BE-algebra X. Then the following conditions hold:*

(1) $\mu(x * y) = \mu(1)$ *implies* $\mu(x) \leq \mu(y)$
(2) $x \leq y$ *implies* $\mu(x) \leq \mu(y)$
(3) $x * (y * z) = 1$ *implies* $\mu(z) \geq min\{\mu(x), \mu(y)\}$

Theorem 2.10 ([10]). *Let μ be a fuzzy filter of a BE-algebra X. Then two level filters μ_{α_1} and μ_{α_2}(with $\alpha_1 < \alpha_2$) of μ are equal if and only if there is no $x \in X$ such that $\alpha_1 \leq \mu(x) < \alpha_2$.*

Definition 2.11 ([11]). A fuzzy relation on a set S is a fuzzy set $\mu : S \times S \longrightarrow [0, 1]$.

Definition 2.12 ([11]). Let μ be a fuzzy relation on a set S and ν a fuzzy set in S. Then μ is a fuzzy relation on ν if for all $x, y \in S$, it satisfies

$$\mu(x, y) \leq min\{\nu(x), \nu(y)\}$$

Definition 2.13 ([11]). Let μ and ν be fuzzy set s in a BE-algebra X. Then the cartesian product of μ and ν is defined by

$$(\mu \times \nu)(x, y) = min\{\mu(x), \nu(y)\}$$

for all $x, y \in X$.

3. Fuzzy filters of BE-algebras

In this section, some properties of fuzzy filters are studied in BE-algebras. For any non-empty subset F of a BE-algebra, it is proved that the level filter μ_F is a fuzzy filter if and only if F is a filter in X.

Proposition 3.1. *Let μ be a fuzzy filter of a BE-algebra X. If μ is decreasing then it is constant.*

Proof. Since μ is a fuzzy filter of X, we get $\mu(1) \geq \mu(x)$ for all $x \in X$. Suppose μ is decreasing. Since $x \leq 1$, we get that $\mu(x) \leq \mu(1)$. Hence $\mu(x) = \mu(1)$ for all $x \in X$. Therefore μ is constant. □

Theorem 3.2. *Let F be a non-empty subset of a BE-algebra X. Define a fuzzy set $\mu_F : X \longrightarrow [0, 1]$ as follows:*

$$\mu_F(x) = \begin{cases} \alpha & \text{if } x \in F \\ 0 & \text{if } x \notin F \end{cases}$$

where $0 < \alpha < 1$ is fixed. Then μ_F is a fuzzy filter in X if and only if F is a filter in X. Moreover $X_{\mu_F} = F$.

Proof. Assume that μ_F is a fuzzy filter in X. Since $\mu_F(1) \geq \mu_F(x)$ for all $x \in X$, we get $\mu_F(1) = \alpha$ and hence $1 \in F$. Let $x, y \in X$ be such that $x, x * y \in F$. Then $\mu_F(x) = \mu_F(x * y) = \alpha$. Since μ_F is a fuzzy filter, we get

$$\mu_F(y) \geq min\{\mu_F(x), \mu_F(x * y)\} = min\{\alpha, \alpha\} = \alpha$$

Hence $y \in F$. Therefore F is a filter in X. The converse follows from Theorem 2.8. Moreover, it is clear that $X_{\mu_F} = F$. □

Theorem 3.3. *Let μ be a fuzzy filter of a BE-algebra X with $Im(\mu) = \{\alpha_i \mid i \in \Delta\}$ and $\mathcal{F} = \{\mu_{\alpha_i} \mid i \in \Delta\}$ where Δ is an arbitrary indexed set. Then*

(1) there exists a unique $i_0 \in \Delta$ such that $\alpha_{i_0} \geq \alpha_i$ for all $i \in \Delta$

(2) $X_\mu = \bigcap\limits_{i \in \Delta} \mu_{\alpha_i} = \mu_{\alpha_{i_0}}$

(3) $X = \bigcup\limits_{i \in \Delta} \mu_{\alpha_i}$

Proof. (1). Since $\mu(1) \in Im(\mu)$, there exists $i_0 \in \Delta$ such that $\alpha_{i_0} = \mu(1) \geq \mu(x)$ for all $x \in X$. Hence $\alpha_{i_0} \geq \alpha_i$ for all $i \in \Delta$. Suppose $i_1 \in \Delta$ such that $\alpha_{i_1} \geq \alpha_i$ for all $i \in \Delta$. Then there exists $x_1 \in X$ such that $\mu(x_1) = \alpha_{i_1}$. Since $\alpha_{i_0}, \alpha_{i_1} \in \Delta$, it concludes that $\alpha_{i_0} \geq \alpha_{i_1}$ and $\alpha_{i_1} \geq \alpha_{i_0}$. Hence $\alpha_{i_0} = \alpha_{i_1}$. Therefore there exits a unique $\alpha_{i_0} \in \Delta$ such that $\alpha_{i_0} \geq \alpha_i$ for all $i \in \Delta$.
(2). Clearly $\bigcap\limits_{i \in \Delta} \mu_{\alpha_i} \subseteq \mu_{\alpha_{i_0}}$. Since $\alpha_{i_0} \geq \alpha_i$ for all $i \in \Delta$, we get $\mu_{\alpha_{i_0}} \subseteq \mu_{\alpha_i}$ for all $i \in \Delta$. Hence $\mu_{\alpha_{i_0}} \subseteq \bigcap\limits_{i \in \Delta} \mu_{\alpha_i}$. Therefore $\bigcap\limits_{i \in \Delta} \mu_{\alpha_i} = \mu_{\alpha_{i_0}}$. From (1), we observe that α_{i_0} is the unique element in $Im(\mu)$ such that $\mu(1) = \alpha_{i_0}$. Hence we get the following:

$$\begin{aligned} \mu_{\alpha_{i_0}} &= \{x \in X \mid \mu(x) \geq \alpha_{i_0}\} \\ &= \{x \in X \mid \mu(x) \geq \mu(1)\} \\ &= \{x \in X \mid \mu(x) = \mu(1)\} \\ &= X_\mu \end{aligned}$$

(3). Let $x \in X$. Then $\mu(x) \in Im(\mu)$. Hence there exists $i_x \in \Delta$ such that $\mu(x) = \alpha_{i_x}$. Hence $x \in \mu_{\alpha_{i_x}} \subseteq \bigcup\limits_{i \in \Delta} \mu_{\alpha_i}$. Therefore $X \subseteq \bigcup\limits_{i \in \Delta} \mu_{\alpha_i}$. □

Theorem 3.4. *Let μ be a fuzzy filter of a BE-algebra X with $Im(\mu) = \{\alpha_i \mid i \in \Delta\}$ and $\mathcal{F} = \{\mu_{\alpha_i} \mid i \in \Delta\}$ where Δ is an arbitrary indexed set. If μ attains its infimum on all filters of X, then $\mathcal{F}$ contains all level filters of μ.*

Proof. Suppose μ attains its infimum on all filters of X. Let μ_α be a level filter of μ. If $\alpha = \alpha_i$ for some $i \in \Delta$, then clearly $\mu_\alpha \in \mathcal{F}$. Assume that $\alpha \neq \alpha_i$ for all $i \in \Delta$. Then there exists no $x \in X$ such that $\mu(x) = \alpha$. Let $F = \{x \in X \mid \mu(x) > \alpha\}$. Clearly $1 \in F$. Let $x, y \in X$ be such that $x \in F$ and $x * y \in F$. Then $\mu(x) > \alpha$ and $\mu(x * y) > \alpha$. Since μ is a fuzzy filter in X, we get

$$\mu(y) \geq min\{\mu(x), \mu(x * y)\} > \alpha$$

Hence $\mu(y) > \alpha$, which implies that $y \in F$. Therefore F is a filter of X. By the hypothesis, there exists $y \in F$ such that

$$\mu(y) = inf\{\mu(x) \mid x \in X\}$$

Hence $\mu(y) \in Im(\mu)$, which yields that $\mu(y) = \alpha_i$ for some $i \in \Delta$. It is clear that $\alpha_i \geq \alpha$. Hence, by assumption, we get $\alpha_i > \alpha$. Thus there exists no $x \in X$ such that $\alpha \leq \mu(x) < \alpha_i$. Hence by Theorem 2.9, we get $\mu_\alpha = \mu_{\alpha_i}$. Therefore $\mu_\alpha \in \mathcal{F}$. □

4. Normal fuzzy filters

In this section, the notion of normal fuzzy filters is introduces in BE-algebras. These classes of normal fuzzy filters are then characterized. Some properties of normal fuzzy filters are studied with respect to fuzzy relations and cartesian products.

Definition 4.1. A fuzzy filter μ of a BE-algebra X is called a normal fuzzy filter if there exists $x \in X$ such that $\mu(x) = 1$.

For any normal fuzzy filter μ, we obviously have $\mu(1) = 1$.

Proposition 4.2. *For any fuzzy filter μ of a BE-algebra X, define a fuzzy set μ^+ in X as $\mu^+(x) = \mu(x) + 1 - \mu(1)$ for all $x \in X$. Then μ^+ is a normal fuzzy filter of X such that $\mu \subseteq \mu_+$.*

Proof. Let $x \in X$. Then we have $\mu^+(1) = \mu(1) + 1 - \mu(1) = 1 \geq \mu^+(x)$, which proves (F_1). To prove (F_2), let $x, y \in X$. Then

$$\begin{array}{rcl} \mu^+(y) & = & \mu(y) + 1 - \mu(1) \\ & \geq & min\{\mu(x), \mu(x * y)\} + 1 - \mu(1) \\ & = & min\{\mu(x) + 1 - \mu(1), \mu(x * y) + 1 - \mu(1)\} \\ & = & min\{\mu^+(x), \mu^+(x * y)\} \end{array}$$

Therefore μ^+ is a fuzzy filter in X. Now, for any $x \in X$, it is clear that $\mu(x) \leq \mu^+(x)$. Therefore it concludes that $\mu \subseteq \mu^+$. □

Corollary 4.3. *If $\mu^+(x_0) = 0$ for some $x_0 \in X$, then so $\mu(x_0) = 0$.*

Proposition 4.4. *Let A be a filter of a BE-algebra X and X_A a fuzzy set of X defined by*

$$X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Then X_A is a normal fuzzy filter of X.

Proof. Since A is a filter, we get $1 \in A$. Hence $X_A(1) = 1 \geq X_A(x)$ for all $x \in X$. Let $x, y \in X$. Suppose $x \in A$ and $x * y \in A$. Since A is a filter, we get $y \in A$. Then $X_A(x) = X_A(y) = X_A(x * y) = 1$. Hence

$$X_A(y) \geq 1 = min\{X_A(x), X_A(x * y)\}$$

Suppose $x \notin A$ and $x * y \notin A$. Then $X_A(x) = X_A(x * y) = 0$. Hence

$$X_A(y) \geq 0 = min\{X_A(x), X_A(x * y)\}$$

If exactly one of x and $x * y$ belongs to A, then exactly one of $X_A(x)$ and $X_A(x * y)$ is equal to 0. Hence

$$X_A(y) \geq min\{X_A(x), X_A(x * y)\}$$

Hence we can conclude that $X_A(y) \geq min\{X_A(x), X_A(x * y)\}$ for all $x, y \in X$. Therefore X_A is a fuzzy filter of X. Since $1 \in A$, we get $X_A(1) = 1$. Therefore X_A is a normal fuzzy filter of X. □

Theorem 4.5. *Let μ be a fuzzy filter of a BE-algebra X. Then the following conditions are equivalent.*

(1) *μ is normal*
(2) *$\mu(1) = 1$*
(3) *$\mu = \mu^+$*

Proof. (1) $\Rightarrow$ (2): Assume that μ is a normal fuzzy filter of X. Then there exists some $x \in X$ such that $\mu(x) = 1$. Since μ is a filter, we can obtain that $\mu(1) \geq \mu(x) = 1$. Therefore $\mu(1) = 1$.
(2) $\Rightarrow$ (3): Assume that $\mu(1) = 1$. Then for any $x \in X$, we get $\mu^+(x) = \mu(x) + 1 - \mu(1) = \mu(x)$. Hence it concludes that $\mu = \mu^+$.
(3) $\Rightarrow$ (1): Assume that $\mu = \mu^+$. Then for any $x \in X$, we get $\mu^+(x) = \mu(x)+1-\mu(1)$. Since $\mu^+(x) = \mu(x)$, we get $\mu(1) = 1$. Therefore μ is normal. □

Corollary 4.6. *If μ is a fuzzy filter of X, then $(\mu^+)^+ = \mu^+$. Moreover, if μ is normal, then $(\mu^+)^+ = \mu$.*

Proof. By Proposition 4.2, μ^+ is a normal fuzzy filter. Hence by main theorem, we get $(\mu^+)^+ = \mu^+$. If μ is normal, then $\mu = \mu^+ = (\mu^+)^+$. □

Definition 4.7. Let μ be a fuzzy set in a BE-algebra X. Then define the sets X_μ and Δ_μ as follows:

(1) $X_\mu = \{x \in X \mid \mu(x) = \mu(1)\}$
(2) $\Delta_\mu = \{x \in X \mid \mu(x) = 1\}$

If μ is normal, then it can be easily observed that $X_\mu = \Delta_\mu$.

Proposition 4.8. *Let μ be a fuzzy filter of a BE-algebra X. Then we have*

(1) *If μ is normal, then X_μ is a filter in X*
(2) *μ is normal if and only if Δ_μ is a filter in X*

Proof. (1) Clearly $1 \in X_\mu$. Let $x, y \in X$ be such that $x, x * y \in X_\mu$. Then $\mu(x) = \mu(x * y) = \mu(1)$. Since μ is a fuzzy filter, we get

$$\mu(y) \geq min\{\mu(x), \mu(x * y)\} = \mu(1)$$

Since μ is normal, by Theorem 4.5, we get $\mu(y) \geq \mu(1) = 1$. Hence $\mu(y) = 1 = \mu(1)$. Thus $y \in X_\mu$, which yields that X_μ is a filter in X.

(2) Assume that μ is normal. Then $\mu(1) = 1$. Hence $1 \in \Delta_\mu$. Let $x, x * y \in \Delta_\mu$. Then $\mu(x) = \mu(x * y) = 1$. Since μ is a fuzzy filter, we get

$$\mu(y) \geq min\{\mu(x), \mu(x * (y)\} = 1$$

Hence $\mu(y) = 1$, which yields that $y \in \Delta_\mu$. Hence Δ_μ is a filter of X. Conversely, assume that Δ_μ is a fuzzy filter. Hence $1 \in \Delta_\mu$. Thus $\mu(1) = 1$. Therefore, by Theorem 4.5, μ is a normal fuzzy filter in X. □

Proposition 4.9. *Let μ and ν be two fuzzy filters of X such that $\mu \subseteq \nu$. Then $\Delta_\mu \subseteq \Delta_\nu$. Moreover, if μ and ν are normal and $\mu \subseteq \nu$, then $X_\mu \subseteq X_\nu$.*

Proof. Let $x \in \Delta_\mu$. Then $\nu(x) \geq \mu(x) = 1$. Hence $\nu(x) = 1$, which implies that $x \in \Delta_\nu$. Therefore $\Delta_\mu \subseteq \Delta_\nu$. Again, let $x \in X_\mu$. Then $\nu(x) \geq \mu(x) = \mu(1) = 1$. Hence $\nu(x) = 1 = \nu(1)$, which concludes that $x \in X_\nu$. Therefore $X_\mu \subseteq X_\nu$. □

Proposition 4.10. *Let μ and ν be two fuzzy filter of X such that $\mu \subseteq \nu$. If μ is normal, then ν is also normal.*

Proof. Let μ and ν be two fuzzy filter of X such that $\mu \subseteq \nu$. Suppose μ is normal. Then by Theorem 4.5, $\mu(1) = 1$. Hence $\nu(1) \geq \mu(1) = 1$. Hence $\nu(1) = 1$, which concludes that ν is normal. □

Let $\mathcal{ND}(X)$ be the class of all normal fuzzy filters of a BE-algebra X. Then it can be easily observed that $\mathcal{ND}(X)$ is a partially ordered set under the set inclusion.

Definition 4.11. A non-constant fuzzy filter μ of X is called maximal if there exists no non-constant fuzzy filter ν such that $\mu \subseteq \nu$.

Proposition 4.12. *Every maximal fuzzy filter is normal.*

Proof. Let μ be a maximal fuzzy filter of a BE-algebra X. Then μ is non-constant and hence μ^+ is non-constant. Otherwise, suppose $\mu^+(x) = c$ for all $x \in X$, where c is a constant. Then for all $x \in X, c = \mu^+(x) = \mu(x) + 1 - \mu(1)$, which shows that μ is constant. Since $\mu \subseteq \mu^+$ and μ is maximal, we get that $\mu = \mu^+$. Therefore, by Theorem 4.5, we get that μ is normal. □

Theorem 4.13. *Let μ be a maximal fuzzy filter of a BE-algebra X. Then μ takes only the values* 0 *and* 1.

Proof. Since μ is maximal, by above proposition, μ is normal and hence $\mu(1) = 1$. Let $x \in X$ be such that $\mu(x) \neq 0$. Suppose $\mu(x) \neq 1$. Then there exists some $x_0 \in X$ such that $0 < x_0 < 1$. Then define a fuzzy set ν in X as follows:

$$\nu(x) = \tfrac{1}{2}(\mu(x) + \mu(x_0)) \quad \text{for all } x \in X$$

Clearly ν is well-defined. Let $x \in X$. Then $\nu(1) = \frac{1}{2}(\mu(1)+\mu(x_0)) = \frac{1}{2}(1+\mu(x_0)) \geq \frac{1}{2}(\mu(x) + \mu(x_0)) = \nu(x)$. Let $x, y \in X$. Then we have

$$\begin{aligned}
\nu(y) &= \frac{1}{2}\{\mu(y) + \mu(x_0)\} \\
&\geq \frac{1}{2}\{min\{\mu(x), \mu(x * y)\} + \mu(x_0)\} \\
&= \frac{1}{2}\{min\{\mu(x) + \mu(x_0), \mu(x * y) + \mu(x_0)\} \\
&= min\ \{\frac{1}{2}(\mu(x) + \mu(x_0)), \frac{1}{2}(\mu(x * y) + \mu(x_0))\} \\
&= min\{\nu(x), \nu(x * y)\}
\end{aligned}$$

Therefore ν is a fuzzy filter of X. Hence by Proposition 4.2, we get that ν^+ is a normal fuzzy filter of X. Clearly $\nu^+(x) \geq \mu(x)$ for all $x \in X$. Now

$$\begin{aligned} \nu^+(x_0) &= \nu(x_0) + 1 - \nu(1) \\ &= \frac{1}{2}\{\mu(x_0) + \mu(x_0)\} + 1 - \frac{1}{2}\{\mu(1) + \mu(x_0)\} \\ &= \frac{1}{2}\{\mu(x_0) + 1\} \\ &> \mu(x_0) \end{aligned}$$

and also $\nu^+(x_0) < 1 = \nu^+(1)$. Hence ν^+ is non-constant such that $\mu \subseteq \nu^+$. Therefore μ is not maximal, which is a contradiction. Hence $\mu(x) = 1$. □

Theorem 4.14. *Let μ be a non-constant fuzzy filter of a BE-algebra X. Then we have the following conditions.*

(1) If μ is maximal then X_μ is a maximal filter in X

(2) μ is maximal if and only if Δ_μ is a maximal filter

Proof. (1). Assume that μ is a maximal fuzzy filter. Then by Proposition 4.8, X_μ is a filter. Suppose $X_\mu = X$. Then $\mu(x) = \mu(1)$ for all $x \in X$. Thus μ is constant, which is a contradiction. Therefore X_μ is proper. Let F be a filter of X such that $X_\mu \subseteq F$. Then by Theorem 3.2, we get that $\mu = \mu_{X_\mu} \subseteq \mu_F$. Since μ is maximal, we get either $\mu = \mu_F$ or μ_F is constant. Suppose μ_F is constant. Then $F = X$, which is a contradiction. Suppose $\mu = \mu_F$. Then we get $X_\mu = X_{\mu_F} = F$. Therefore X_μ is a maximal filter of X.

(2). Assume that μ is maximal. Then μ is normal and hence by (1), we get $X_\mu = \Delta_\mu$ is a maximal filter on X. Conversely, assume that Δ_μ is a maximal filter of X. Let ν be a non-constant fuzzy filter of X such that $\mu \subset \nu$. Then we get $\Delta_\mu \subseteq \Delta_{nu}$. Since Δ_μ is maximal, we get either $\Delta_\nu = X$. Hence for all $x \in X = \Delta_\nu$, we get $\nu(x) = 1$. Thus ν is constant, which is a contradiction. Hence μ is a maximal fuzzy filter. □

Definition 4.15. Let $(X, *, 1)$ and $(Y, *, 1')$ be two BE-algebras and $f : X \longrightarrow Y$ an onto homomorphism. For any fuzzy set μ in Y, define a mapping $\mu^f : X \longrightarrow [0, 1]$ such that $\mu^f(x) = \mu(f(x))$ for all $x \in X$.

Clearly the above map μ^f is well-defined and fuzzy set in X.

Theorem 4.16. *Let $f : X \longrightarrow Y$ be onto homomorphism. For a fuzzy set μ in Y, μ is a normal fuzzy filter in Y if and only if μ^f is a normal fuzzy filter in X.*

Proof. Assume that μ is a normal fuzzy filter of Y. For any $x \in X$, we have $\mu^f(1) = \mu(f(1)) = \mu(1) \geq \mu(f(x)) = \mu^f(x)$. Let $x, y \in X$. Then

$$\begin{aligned} \mu^f(y) &= \mu(f(y)) \\ &\geq min\{\mu(f(x)), \mu(f(x) * f(y))\} \\ &= min\{\mu(f(x)), \mu(f(x * y))\} \\ &= min\{\mu^f(x), \mu^f(x * y)\} \end{aligned}$$

Hence μ^f is a fuzzy filter of X. We now show that μ^f is normal. Since μ is normal in Y, we get $\mu^f(1) = \mu(f(1)) = \mu(1') = 1$. Hence μ^f is a normal fuzzy filter in X.

Conversely, assume that μ^f is a normal fuzzy filter of X. Let $x \in Y$. Since f is onto, there exists $y \in X$ such that $f(y) = x$. Then $\mu(1) = \mu(f(1)) = \mu^f(1) \geq \mu^f(y) = \mu(f(y)) = \mu(x)$. Let $x, y \in Y$. Then there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Hence we get

$$\begin{aligned}
\mu(y) &= \mu(f(b)) \\
&= \mu^f(b) \\
&\geq min\{\mu^f(a), \mu^f(a * b)\} \\
&= min\{\mu(f(a)), \mu(f(a * b))\} \\
&= min\{\mu(f(a)), \mu(f(a) * f(b))\} \\
&= min\{\mu(x), \mu(x * y)\}
\end{aligned}$$

Therefore μ is a fuzzy filter in X. Since μ^f is normal, we get $\mu(1') = \mu(f(1)) = \mu^f(1) = 1$. Therefore μ is a normal fuzzy filter in Y. □

The proof of the following lemma is routine.

Lemma 4.17. *Let μ and ν be two fuzzy sets in a BE-algebra X. Then the following hold.*

(1) $\mu \times \nu$ *is a fuzzy relation on* X
(2) $(\mu)_\alpha = \mu_\alpha \times \nu_\alpha$ *for all* $\alpha \in [0, 1]$

Proposition 4.18. *Let μ and ν be two normal fuzzy filters of a BE-algebra X. Then $\mu \times \nu$ is a normal fuzzy filters in $X \times X$.*

Proof. Let $(x, y) \in X \times X$. Then $(\mu \times \nu)(1, 1) = min\{\mu(1), \nu(1)\} \geq min\{\mu(x), \nu(y)\} = (\mu \times \nu)(x, y)$. Now let $(x, y), (z, w) \in X \times X$. Then

$$\begin{aligned}
(\mu \times \nu)(z, w) &= min\{\mu(z), \nu(w)\} \\
&\geq min\{min\{\mu(x), \mu(x * z)\}, min\{\nu(y), \nu(y * w)\}\} \\
&= min\{min\{\mu(x), \nu(y)\}, min\{\mu(x * z), \nu(y * w)\}\} \\
&= min\{(\mu \times \nu)(x, y), (\mu \times \nu)(x * z, y * w)\} \\
&= min\{(\mu \times \nu)(x, y), (\mu \times \nu)((x, y) * (z, w))\}
\end{aligned}$$

Therefore $\mu \times \nu$ is a fuzzy filter in $X \times X$. Now

$$(\mu \times \nu)(1, 1) = min\{\mu(1), \nu(1)\} = min\{1, 1\} = 1$$

Therefore $\mu \times \nu$ is a normal fuzzy filter in $X \times X$. □

Definition 4.19. Let ν be a fuzzy set in a BE-algebra X. Then the strongest fuzzy relation μ_ν is a fuzzy relation on X defined by

$$\mu_\nu(x, y) = min\{\nu(x), \nu(y)\}$$

for all $x, y \in X$.

Theorem 4.20. *Let ν be a fuzzy set in a BE-algebra X and μ_ν the strongest fuzzy relation on X. If ν is a normal fuzzy filter in X, then μ_ν is a normal fuzzy filter of $X \times X$.*

Proof. Assume that ν is a normal fuzzy filter of X. For any $(x, y) \in X \times X$, we get $\mu_\nu(1,1) = min\{\nu(1), \nu(1)\} \geq min\{\nu(x), \nu(y)\} = \mu_\nu(x, y)$. Let $(x, y), (z, w) \in X \times X$. Then we have the following:

$$\begin{aligned}
\mu_\nu(z, w) &= min\{\nu(z), \nu(w)\} \\
&\geq min\{min\{\nu(x), \nu(x * z)\}, min\{\nu(y), \nu(y * w)\}\} \\
&= min\{min\{\nu(x), \nu(y)\}, min\{\nu(x * z), \nu(y * w)\}\} \\
&= min\{\mu_\nu(x, y), \mu_\nu(x * z, y * w)\} \\
&= min\{\mu_\nu(x, y), \mu_\nu((x, y) * (z, w))\}
\end{aligned}$$

Therefore μ_ν is a fuzzy filter in $X \times X$. Again

$$\mu_\nu(1, 1) = min\{\nu(1), \nu(1)\} = min\{1, 1\} = 1$$

Therefore μ_ν is a normal fuzzy filter in $X \times X$. □

References

[1] S. S. Ahn and K. S. So, On ideals and upper sets in BE-algebras, Sci. Math. Jpn. 68(2) (2008) 279–285.

[2] F. A. Azam, A. A. Mamun and F. Nasrin, Anti fuzzy ideals of a ring, Ann. Fuzzy Math. Inform. 5(2)(2013) 349–360.

[3] P. Bhattacharya and N. P. Mukherjee, Fuzzy relations and fuzzy groups, Inform. Sci. 36 (1985) 267–282.

[4] W. A. Dudek and Y. B. Jun, On fuzzy ideals in Hilbert algebras, Novi Sad J. Math. 29(2) (1999) 193–207.

[5] Y. B. Jun and S. M. Hong, Fuzzy deductive systems of Hilbert algebras, Indian J. Pure Appl. Math. 27(2) (1996) 141–151.

[6] Y. B. Jun, Fuzzy positive implicative filters and fuzzy associative filters of lattice implication algebras, Fuzzy Sets and Systems 121 (2001) 353–357.

[7] S. Kar and P. Sarkar, Fuzzy quasi-ideals and fuzzy bi-ideals of ternary semigroups, Ann. Fuzzy Math. Inform. 4(2) (2012) 407–423.

[8] H. S. Kim and Y. H. Kim, On BE-algebras, Sci. Math. Jpn. 66(1) (2007) 113–116.

[9] A. Rezaei and A. B. Saeid, On fuzzy subalgebras of BE-algebras, Afr. Mat. 22 (2011) 115–127.

[10] M. Sambasiva Rao, Fuzzy filters of BE-algebras, International Journal of Mathematical Archive 4(6) (2013) 181–187.

[11] M. Sambasiva Rao, Fuzzy implicative filters of BE-algebras, Ann. Fuzzy Math. Inform. 6(3) (2013) 755-765.

[12] Y. Xu and K. Y. Qin, On filters of lattice implication algebras, J. Fuzzy Math. 1(2) (1993) 251–260.

[13] L. A. Zadeh, Fyzzy sets, Information and Control 8 (1965) 338–353.

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