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On vague soft groups

Banu Pazar Varol, Abdülkadir Aygünoğlu, Halis Aygün

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Abstract. Molodtsov proposed the concept of soft sets, which can be seen as a new mathematical tool for dealing with uncertainty. In this paper, we introduce the notion of vague soft groups and study some of their properties. The concept of vague soft homomorphism is defined and the theorems of homomorphic image and pre-image are given. Furthermore, the definition of normal vague soft group is given and basic properties are studied.

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Corresponding Author: Banu Pazar Varol (banupazar@kocaeli.edu.tr)

1. Introduction

In real life situation the problem in economics, engineering, environment, social science, medical science etc. do not always involve crisp data. We can not successfully use the classical methods because of various types of uncertainties presented in these problems. There are several theories to exceed these uncertainties like theory of fuzzy sets [21], intuitionistic fuzzy sets [3], rough sets [16],ie., which we can use as a mathematical tools for describing uncertainties. In 1999, Molodtsov [15] initiated the concept of soft theory as a new approach for modeling uncertainties.

A soft set is parameterized family of subsets of the universal set. In soft set theory the problem of setting the membership function simply does not arise. This makes the theory convenient and easy to apply in practice. The soft set theory has a rich potential for applications in several directions few of which had been demonstrated by Molodtsov [15]. Based on the work of Molodtsov, Maji et al. [12, 13, 14] introduced concept of fuzzy soft set and they also studied the theory of soft sets and used this theory to solve some decision making problems. In 2007, Aktaş et al. [1] introduced to the basic properties of soft sets and then they gave the definition of soft groups. Chen et al. [7] studied the the parametrization reduction of soft sets. Liu et al. [11] discussed the algebraic structure of fuzzy soft sets and gave the definition of fuzzy soft groups. Feng et al. [8] introduced the notion of soft semirings. Irfan Ali et al. [2] pointed out some errors of former studies and gave some new operations on soft sets. Yang et al. [20] combined the interval-valued fuzzy set and soft set models and introduced the concept of interval-valued fuzzy soft set. In 2009, Aygünoğlu and Aygün [4] introduced the concept of fuzzy soft group by using $t - norm$.

The notion of vague sets which is generalization of fuzzy sets was proposed by Gau and Buehrer [9]. A vague set A in the universe of of discourse X is defined by a truth-membership function t_A and a false-membership function f_A, where $t_A(x)$ is a lower bound on the grade of membership of x derived from the "evidence for x" and $f_A(x)$ is a lower bound on the grade of membership of the negation of x derived from the "evidence against x". These lower bounds are both defined on the closed interval $[0, 1]$, where $t_A(x) + f_A(x) \leqslant 1$. In 2010, Xu et al. [19] combined the soft set and vague set models and introduced the notion of vague soft set and presented its basic properties.

The algebraic structure of set theories dealing with uncertainties had been studied by some authors. Rosenfeld [18] proposed the concept of fuzzy group and since then many contributions made on these main direction. For instance, rough groups were defined by Biswas et al. [6] and vague groups were defined by Ranjit Biswas [5].

In this paper, first of all we introduce vague soft group which is a generelization of soft groups introduced by Aktaş and Cağman [1] and then we study their structural characteristics. This paper is organized as follows. In section 2, as preliminaries we give the concept of soft sets, vague sets and vague soft sets. In section 3, we introduce vague soft group and study its basic properties. In section 4, we define soft homomorphism between vague soft groups and according to this definition we prove that image and pre-image of a vague soft group are also vague soft group. In last section, we give the definition of a normal vague soft group and study some of its basic properties.

2. Soft sets, vague sets and vague soft sets

In this section as a preparation, for convenience of subsequent study, we will present the concepts of soft set theory [15], vague set theory [9] and vague soft set theory [19].

2.1. **Soft sets.** Let X be an initial universe set and E be a set of parameters.

Definition 2.1 ([15])**.** A pair (F, E) is called a soft set over X if only if F is a mapping from E into the set of all subsets of the set X, i.e., $F : E \longrightarrow P(X)$, where $P(X)$ is the power set of X.

In other words, the soft set is a parameterized family of subsets of the set X. Every set $F(e)$, for every $e \in E$, from this family may be considered as the set of e-elements of the soft set (F, E), or considered as the set of e-approximate elements of the soft set. According to this manner, we can view a soft set (F, E) as consisting of collection of approximations:

$$(F, E) = \{F(e) : e \in E\}.$$

Example 2.2. (1) A soft set (F, E) describes the attractiveness of the houses which Mr. A is going to buy.

X = The set of houses under consideration.

E = The set of parameters. Each parameter is a word or sentence.

E = {expensive; beautiful; wooden; cheap; in the green surrounding; modern; in good repair; in bad repair}.

In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on.It is worth noting that the sets $F(\varepsilon)$ may be empty for some $\varepsilon \in E$. [15]

2) Suppose A is a fuzzy set of the universe X, we take the parameter set $E = [0, 1]$, and define the mapping $F : E \longrightarrow P(X)$ as follows:

$$F(\alpha) = \{x \in X \ : \ \mu_A(x) \geq \alpha\}, \qquad \alpha \in [0, 1]$$

In other words, $F(\alpha)$ is α-level set of A.

According to this manner and by using the decomposition theorem of fuzzy sets [10], we see that a fuzzy set can be uniquely represented as a soft set. [17]

3) For a topological space (X, τ), if $F(x)$ is the family of all open neighborhoods of a point $x \in X$, i.e., $F(x) = \{V \in \tau \mid x \in V\}$, then the ordered pair (F, X) indeed a soft set over $P(X)$. [17]

Definition 2.3 ([17])**.** For two soft sets (F, A) and $G, B)$ over a common universe X, we say that (F, A) is a soft subset of (G, B) and write $(F, A)\widetilde{\subseteq}(G, B)$ if

(i) $A \subset B$ and

(ii) For each $a \in A$, $F(a) \subseteq G(a)$.

Definition 2.4 ([17])**.** Two soft sets (F, A) and (G, B) over a common universe X are said to be equal if $(F, A)\widetilde{\subseteq}(G, B)$ and $(G, B)\widetilde{\subseteq}(F, A)$.

Definition 2.5 ([17])**.** The union of two soft sets (F, A) and (G, B) over a common universe X is the soft set (H, C), where $C = A \cup B$ and

$$H(c) = \begin{cases} F(c), & \text{if } c \in A - B \\ G(c), & \text{if } c \in B - A \\ F(c) \cup G(c), & \text{if } c \in A \cap B \end{cases}, \qquad \forall c \in C.$$

We write $(F, A)\widetilde{\cup}(G, B) = (H, C)$.

Definition 2.6 ([17])**.** The intersection of two soft sets (F, A) and (G, B) over a common universe X is the soft set (H, C), where $C = A \cap B$ and $H(c) = F(c) \cap G(c)$, $\forall c \in C$.

We write $(F, A)\widetilde{\cap}(G, B) = (H, C)$.

Definition 2.7 ([13])**.** If (F, A) and (G, B) are two soft sets, then (F, A) **AND** (G, B) is denoted $(F, A)\widetilde{\wedge}(G, B)$. $(F, A)\widetilde{\wedge}(G, B)$ is defined as $(H, A \times B)$ where $H(a, b) = F(a) \cap G(b)$, $\forall (a, b) \in A \times B$.

2.2. **Vague sets.** Let X be a spaces of points (objects), with a generic element of X denoted by x.

Definition 2.8 ([9]). A vague set A in the universe of discourse X is characterized by a truth-membership function $t_A : X \to [0,1]$ and a false-membership function $f_A : X \to [0,1]$, where where $t_A(x)$ is a lower bound on the grade of membership of x derived from the "evidence for x" and $f_A(x)$ is a lower bound on the grade of membership of the negation of x derived from the "evidence against x" and $t_A(x) + f_A(x) \leqslant 1$.

Thus the grade of membership of x in the vague set A is bounded by a subinterval $[t_A(x), 1 - f_A(x)]$ of $[0,1]$. The vague value $[t_A(x), 1 - f_A(x)]$ indicates that if the exact grade of membership is $\mu(x)$, then $t_A(x) \leqslant \mu(x) \leqslant 1 - f_A(x)$.

The vague value is denoted by $V_A(x)$,i.e., $V_A(x) = [t_A(x), 1 - f_A(x)]$.

When the universe X is continuous, a vague set A can be written as

$$A = \int_X [t_A(x_i), 1 - f_A(x_i)]/x_i, \quad x_i \epsilon X.$$

When the universe X is discrete, a vague set A can be written as

$$A = \sum_{i=1}^{n} [t_A(x_i), 1 - f_A(x_i)]/x_i, \quad x_i \in X.$$

Definition 2.9 ([9]). Let A be a vague set of the universe X. If $t_A(x) = 1$ and $f_A(x) = 0$ for all $x \in X$, then A is called a unit vague set. If $t_A(x) = 0$ and $f_A(x) = 1$ for all $x \in X$, then A is called a zero vague set.

Definition 2.10 ([9]). For two vague sets A and B over a common universe X, we say that A is a vague subset of B and write $A \subseteq B$ if $t_A(x) \leqslant t_B(x)$ and $1 - f_A(x) \leqslant 1 - f_B(x)$ for all $x \in X$.

Definition 2.11 ([9]). Two vague sets A and B over a common universe X are said to be equal if $A \subseteq B$ and $B \subseteq A$.

Definition 2.12 ([9]). The union of two vague sets A and B over a common universe X is the vague set C, written as $C = A \cup B$, whose truth-membership and false-membership functions are related to those of A and B by

$t_C = max\{t_A, t_B\}$ and

$1 - f_C = max\{1 - f_A, 1 - f_B\} = 1 - min\{f_A, f_B\}$.

Definition 2.13 ([9]). The intersection of two vague sets A and B over a common universe X is the vague set C, written as $C = A \cap B$, whose truth-membership and false-membership functions are related to those of A and B by

$t_C = min\{t_A, t_B\}$ and

$1 - f_C = min\{1 - f_A, 1 - f_B\} = 1 - max\{f_A, f_B\}$.

2.3. **Vague soft sets.** A soft set is a parameterized family of subsets of the set X. However, the concept of soft set cannot be used to represent the vagueness of the associated parameters. In [19], W.Xu et al. introduced the concept of vague soft set based on soft set theory and vague set theory. Let X be a universe, E be a set of parameters and $A \subseteq E$.

Definition 2.14 ([19]). A pair $(\widehat{F}, A)$ is called a vague soft set over X if and only if $\widehat{F}$ is a mapping given by $\widehat{F} : A \to V(X)$, where $V(X)$ is the power set of vague sets on X.

That is, for each $a \in A$, $\widehat{F}(a) = \widehat{F}_a$ is a vague set on X.

In other words, a vague soft set over X is a parameterized family of vague set of the universe X. For $a \in A$, $\mu_{\widehat{F}(a)} : X \to [0,1] \times [0,1]$ is regarded as the set of a approximate elements of the vague set $(\widehat{F}, A)$.

Example 2.15. A vague soft set $(\widehat{F}, E)$ describes the attractiveness of the houses which Mr. A is going to buy.

X = The set of houses under consideration $= \{h_1, h_2, ..., h_6\}$.

E = The set of parameters $= \{e_1, e_2, e_3, e_4, e_5\}$
$= \{expensive, beautiful, wooden, cheap, modern\}$.

Suppose that

$\widehat{F}(e_1) = ([0.3, 0.3]/h_1, [0.2, 1]/h_2, [0.5, 0.7]/h_3, [0.2, 0.9]/h_4, [0.1, 0.2]/h_5, [0.4, 0.6]/h_6)$,
$\widehat{F}(e_2) = ([1, 1]/h_1, [0.2, 0.7]/h_2, [0.4, 0.5]/h_3, [0.3, 0.4]/h_4, [0.1, 0.2]/h_5, [0.1, 0.4]/h_6)$,
$\widehat{F}(e_3) = ([0.1, 0.4]/h_1, [0, 0]/h_2, [0.4, 1]/h_3, [1, 1]/h_4, [0.6, 0.9]/h_5, [0.4, 0.7]/h_6)$,
$\widehat{F}(e_4) = ([0, 0.1]/h_1, [0.2, 0.7]/h_2, [0.2, 0.3]/h_3, [1, 1]/h_4, [0, 0]/h_5, [0.3, 0.6]/h_6)$,
$\widehat{F}(e_5) = ([0.1, 0.9]/h_1, [0, 0]/h_2, [1, 1]/h_3, [0.6, 0.8]/h_4, [0.4, 0.4]/h_5, [0.1, 0.7]/h_6)$.

The vague soft set $(\widehat{F}, E)$ is a parameterized family $\{\widehat{F}(e_i), i = \{1, 2, 3, 4, 5\}\}$ of vague sets on X and

$$\begin{aligned}(\widehat{F}, E) = \{&expensive\ houses = ([0.3, 0.3]/h_1, [0.2, 1]/h_2, [0.5, 0.7]/h_3, [0.2, 0.9]/h_4,\\ &\qquad [0.1, 0.2]/h_5, [0.4, 0.6]/h_6),\\ &beautiful\ houses = ([1, 1]/h_1, [0.2, 0.7]/h_2, [0.4, 0.5]/h_3, [0.3, 0.4]/h_4,\\ &\qquad [0.1, 0.2]/h_5, [0.1, 0.4]/h_6), ...\}\end{aligned}$$

Definition 2.16 ([19]). For two vague soft sets $(\widehat{F}, A)$ and $(\widehat{G}, B)$ over a common universe X, we say that $(\widehat{F}, A)$ is a vague soft subset of $(\widehat{G}, B)$ and write $(\widehat{F}, A)\tilde{\subseteq}(\widehat{G}, B)$ if

(i) $A \subset B$ and

(ii) For each $a \in A$, $\widehat{F}(a) \subseteq \widehat{G}(a)$.

Definition 2.17 ([19]). For two vague soft sets $(\widehat{F}, A)$ and $(\widehat{G}, B)$ over a common universe X are said to be vague soft equal if $(\widehat{F}, A)\tilde{\subseteq}(\widehat{G}, B)$ and $(\widehat{G}, B)\tilde{\subseteq}(\widehat{F}, A)$.

Definition 2.18 ([19]). The union of two vague soft sets $(\widehat{F}, A)$ and $(\widehat{G}, B)$ over a common universe X is the vague soft set $(\widehat{H}, C)$, where $C = A \cup B$ and for all $c \in C$,

$$t_{\widehat{H}(c)}(x) = \begin{cases} t_{\widehat{F}(c)}(x), & \text{if } c \in A - B, x \in X \\ t_{\widehat{G}(c)}(x), & \text{if } c \in B - A, x \in X \\ max\{t_{\widehat{F}(c)}(x), t_{\widehat{G}(c)}(x)\}, & \text{if } c \in A \cap B, x \in X \end{cases}$$

$$1 - f_{\widehat{H}(c)}(x) = \begin{cases} 1 - f_{\widehat{F}(c)}(x), & \text{if } c \in A - B, x \in X \\ 1 - f_{\widehat{G}(c)}(x), & \text{if } c \in B - A, x \in X \\ max\{1 - f_{\widehat{F}(c)}(x), 1 - f_{\widehat{G}(c)}(x)\}, & \text{if } c \in A \cap B, x \in X \end{cases}$$

We write $(\widehat{F}, A)\tilde{\cup}(\widehat{G}, B) = (\widehat{H}, C)$.

Definition 2.19. Let $(\widehat{F}, A)$ and $(\widehat{G}, B)$ be two vague soft set over a common universe X with $A \cap B \neq \varnothing$. Then their intersection is the vague soft set $(\widehat{H}, C)$, where $C = A \cap B$ and for all $c \in C, x \in X$,

$t_{\widehat{H}(c)}(x) = min\{t_{\widehat{F}(c)}(x), t_{\widehat{G}(c)}(x)\}$ and

$1 - f_{\widehat{H}(c)}(x) = min\{1 - f_{\widehat{F}(c)}(x), 1 - f_{\widehat{G}(c)}(x)\}$.

We write $(\widehat{F}, A)\widetilde{\cap}(\widehat{G}, B) = (\widehat{H}, C)$.

Definition 2.20 ([19])**.** If $(\widehat{F}, A)$ and $(\widehat{G}, B)$ are two vague soft sets, then $(\widehat{F}, A)$ **AND** $(\widehat{G}, B)$ is denoted by $(\widehat{F}, A)\widetilde{\wedge}(\widehat{G}, B)$. $(\widehat{F}, A)\widetilde{\wedge}(\widehat{G}, B))$ is defined as $(\widehat{H}, A \times B)$ where

$t_{\widehat{H}(a,b)}(x) = min\{t_{\widehat{F}(a)}(x), t_{\widehat{G}(b)}(x)\}$ and

$1 - f_{\widehat{H}(a,b)}(x) = min\{1 - f_{\widehat{F}(a)}(x), 1 - f_{\widehat{G}(b)}(x)\}, \ \forall (a, b) \in A \times B, \, x \in X$.

3. Vague soft groups

The notion of fuzzy groups defined by Rosenfeld [18] is the first application of fuzzy set theory in algebra. Biswas [5] defined the notion of vague groups analogous the idea of Rosenfeld. Aktaş and Cağman [1] introduced the notion of soft groups, which extends the notion of group to include the algebraic structure of soft sets. A soft group is a parameterized family of subgroups. In this section, we introduce the definition of vague soft group and give some fundamental properties of vague soft groups.

Definition 3.1 ([1])**.** Let X be a group and (F, A) be a soft set over X. Then (F, A) is said to be a soft group over X iff $F(a) < X$, for each $a \in A$.

Definition 3.2. Let X be a group and $(\widehat{F}, A)$ be a vague soft set over X. Then $(\widehat{F}, A)$ is said to be a vague soft group over X iff for each $a \in A$ and $x, y \in X$,

(1) $V_{\widehat{F}_a}(x \cdot y) \geqslant min\{V_{\widehat{F}_a}(x), V_{\widehat{F}_a}(y)\}$

(2) $V_{\widehat{F}_a}(x^{-1}) \geqslant V_{\widehat{F}_a}(x)$. i.e.,

(1) $t_{\widehat{F}_a}(x \cdot y) \geqslant min\{t_{\widehat{F}_a}(x), t_{\widehat{F}_a}(y)\}$ and $1 - f_{\widehat{F}_a}(x \cdot y) \geqslant min\{1 - f_{\widehat{F}_a}(x), 1 - f_{\widehat{F}_a}(y)\}$

(2) $t_{\widehat{F}_a}(x^{-1}) \geqslant t_{\widehat{F}_a}(x)$ and $1 - f_{\widehat{F}_a}(x^{-1}) \geqslant 1 - f_{\widehat{F}_a}(x)$.

That is, for each $a \in A$, $\widehat{F}_a$ is a vague group in Biswas's sense [5] and $t_{\widehat{F}_a}, 1 - f_{\widehat{F}_a}$ are fuzzy groups of X in Rosenfeld's sense [18].

Example 3.3. Let $\mathbb{N}$ be the set of all natural numbers. Let define $t_A : \mathbb{N} \to I^{\mathbb{R}}$ by $t_A(n) : \mathbb{R} \to I$ and $1 - f_A : \mathbb{N} \to I^{\mathbb{R}}$ by $1 - f_A(n) : \mathbb{R} \to I$ for each $n \in \mathbb{N}$ where

$$t_A(n)(x) = \begin{cases} \frac{1}{n}, & \text{if } x = k2^n, \ \exists k \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}, \qquad \forall n \in \mathbb{N}$$

and

$$(1 - f_A)(n)(x) = \begin{cases} 1 - \frac{1}{n}, & \text{if } x = k2^n, \ \exists k \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}, \qquad \forall n \in \mathbb{N}$$

Then the pair $(\widehat{F}, \mathbb{N})$ forms a vague soft set over $\mathbb{R}$ and the vague soft set $(\widehat{F}, \mathbb{N})$ is a vague soft group over $\mathbb{R}$.

From now on, X will be a group.

Proposition 3.4. *Let* $(\widehat{F}, A)$ *is a vague soft set over* X*. Then* $(\widehat{F}, A)$ *is a vague soft group iff for each* $a \in A$ *and* $x, y \in X$*,* $t_{\widehat{F}_a}(x \cdot y^{-1}) \geqslant min\{t_{\widehat{F}_a}(x), t_{\widehat{F}_a}(y)\}$ *and* $1 - f_{\widehat{F}_a}(x \cdot y^{-1}) \geqslant min\{1 - f_{\widehat{F}_a}(x), 1 - f_{\widehat{F}_a}(y)\}$.

Proof. For each $a \in A$ and $x, y \in X$, we have
$t_{\widehat{F}_a}(x \cdot y^{-1}) \geqslant min\{t_{\widehat{F}_a}(x), t_{\widehat{F}_a}(y^{-1})\} \geqslant min\{t_{\widehat{F}_a}(x), t_{\widehat{F}_a}(y)\}$ and similarly,
$1 - f_{\widehat{F}_a}(x \cdot y^{-1}) \geqslant min\{1 - f_{\widehat{F}_a}(x), 1 - f_{\widehat{F}_a}(y)\}$.

Conversely, first of all we have

$$t_{\widehat{F}_a}(e) = t_{\widehat{F}_a}(x \cdot x^{-1}) \geqslant min(t_{\widehat{F}_a}(x), t_{\widehat{F}_a}(x^{-1})) \geqslant min(t_{\widehat{F}_a}(x), t_{\widehat{F}_a}(x)) = t_{\widehat{F}_a}(x),$$

for each $x \in X$, where e is the unit element of X. That is, for each $x \in X$, we have $t_{\widehat{F}_a}(e) \geqslant t_{\widehat{F}_a}(x)$ and $1 - f_{\widehat{F}_a}(e) \geqslant 1 - f_{\widehat{F}_a}(x)$. Furthermore, $t_{\widehat{F}_a}(x^{-1}) = t_{\widehat{F}_a}(e \cdot x^{-1}) \geqslant min(t_{\widehat{F}_a}(e), t_{\widehat{F}_a}(x)) \geqslant min(t_{\widehat{F}_a}(x), t_{\widehat{F}_a}(x)) = t_{\widehat{F}_a}(x)$ and

$$\begin{aligned} 1 - f_{\widehat{F}_a}(x^{-1}) &= 1 - f_{\widehat{F}_a}(e \cdot x^{-1}) \geqslant min(1 - f_{\widehat{F}_a}(e), 1 - f_{\widehat{F}_a}(x)) \\ &\geqslant min(1 - f_{\widehat{F}_a}(x), 1 - f_{\widehat{F}_a}(x)) = 1 - f_{\widehat{F}_a}(x). \end{aligned}$$

So we have (2).

On the other hand, for each $a \in A$ and $x, y \in X$,
$t_{\widehat{F}_a}(x \cdot y) = t_{\widehat{F}_a}(x \cdot (y^{-1})^{-1}) \geqslant min(t_{\widehat{F}_a}(x), t_{\widehat{F}_a}(y^{-1})) \geqslant min(t_{\widehat{F}_a}(x), t_{\widehat{F}_a}(y))$ and similarly, it can be proved that $1 - f_{\widehat{F}_a}(x \cdot y) \geqslant min(1 - f_{\widehat{F}_a}(x), 1 - f_{\widehat{F}_a}(y))$. Then we have (1). This completes the proof. □ □

Theorem 3.5. *Let* $(\widehat{F}, A)$ *is a vague soft group and* e *is the unit element of* X*. Then for each* $x \in X$*,*

(1) $t_{\widehat{F}_a}(x^{-1}) = t_{\widehat{F}_a}(x)$ *and* $1 - f_{\widehat{F}_a}(x^{-1}) = 1 - f_{\widehat{F}_a}(x)$

(2) $t_{\widehat{F}_a}(e) \geqslant t_{\widehat{F}_a}(x)$ *and* $1 - f_{\widehat{F}_a}(e) \geqslant 1 - f_{\widehat{F}_a}(x)$.

Proof. Straightforward. □ □

Theorem 3.6. *Let* $(\widehat{F}, A)$ *and* $(\widehat{G}, B)$ *be two vague soft groups over* X*. Then their intersection* $(\widehat{F}, A)\widetilde{\cap}(\widehat{G}, B)$ *is a vague soft group over* X*.*

Proof. Let $(\widehat{F}, A)\widetilde{\cap}(\widehat{G}, B) = (\widehat{H}, C)$, where $C = A \cap B$ and for all $c \in C, x \in X$,

$$t_{\widehat{H}(c)}(x) = min\{t_{\widehat{F}(c)}(x), t_{\widehat{G}(c)}(x)\}$$

and

$$1 - f_{\widehat{H}(c)}(x) = min\{1 - f_{\widehat{F}(c)}(x), 1 - f_{\widehat{G}(c)}(x)\}.$$

Since $(\widehat{F}, A)$ and $(\widehat{G}, B)$ are two vague soft groups over X for arbitrary $c \in C$, we have

$$\begin{aligned} t_{\widehat{F}(c)}(x \cdot y^{-1}) &\geqslant min\{t_{\widehat{F}(c)}(x), t_{\widehat{F}(c)}(y)\}, 1 - f_{\widehat{F}(c)}(x \cdot y^{-1}) \\ &\geqslant min\{1 - f_{\widehat{F}(c)}(x), 1 - f_{\widehat{F}(c)}(y)\} \end{aligned}$$

and

$$\begin{aligned} t_{\widehat{G}(c)}(x \cdot y^{-1}) &\geqslant min\{t_{\widehat{G}(c)}(x), t_{\widehat{G}(c)}(y)\}, 1 - f_{\widehat{G}(c)}(x \cdot y^{-1}) \\ &\geqslant min\{1 - f_{\widehat{G}(c)}(x), 1 - f_{\widehat{G}(c)}(y)\} \end{aligned}$$

for all $x, y \in X$. Then we obtain,

$$\begin{aligned} t_{\widehat{H}(c)}(x \cdot y^{-1}) &= min\{t_{\widehat{F}(c)}(x \cdot y^{-1}), t_{\widehat{G}(c)}(x \cdot y^{-1})\} \\ &\geqslant min\{min\{t_{\widehat{F}(c)}(x), t_{\widehat{F}(c)}(y)\}, min\{t_{\widehat{G}(c)}(x), t_{\widehat{G}(c)}(y)\}\} \\ &= min\{min\{t_{\widehat{F}(c)}(x), t_{\widehat{G}(c)}(x)\}, min\{t_{\widehat{F}(c)}(y), t_{\widehat{G}(c)}(y)\}\} \\ &= min\{t_{\widehat{H}(c)}(x), t_{\widehat{H}(c)}(y)\} \end{aligned}$$

Similarly, $1 - f_{\widehat{H}(c)}(x \cdot y^{-1}) \geqslant min\{1 - f_{\widehat{H}(c)}(x), 1 - f_{\widehat{H}(c)}(y)\}$. □

Theorem 3.7. *Let $(\widehat{F}, A)$ and $(\widehat{G}, B)$ be two vague soft groups over X. If $A \cap B = \varnothing$, then $(\widehat{F}, A)\widetilde{\cup}(\widehat{G}, B)$ is a vague soft group over X.*

Proof. Let $(\widehat{F}, A)\widetilde{\cup}(\widehat{G}, B) = (\widehat{H}, C)$. Since $A \cap B = \varnothing$, it follows that either $c \in A - B$ or $c \in B - A$ for all $c \in C$. If $c \in A - B$, then $t_{\widehat{H}(c)} = t_{\widehat{F}(c)}$ and $1 - f_{\widehat{H}(c)} = 1 - f_{\widehat{F}(c)}$ are fuzzy groups of X. If $c \in B - A$, then $t_{\widehat{H}(c)} = t_{\widehat{G}(c)}$ and $1 - f_{\widehat{H}(c)} = 1 - f_{\widehat{G}(c)}$ are fuzzy groups of X. Thus, $(\widehat{F}, A)\widetilde{\cup}(\widehat{G}, B)$is a vague soft group over X. □

Theorem 3.8. *Let $(\widehat{F}, A)$ and $(\widehat{G}, B)$ be two vague soft groups over X. Then $(\widehat{F}, A)\widetilde{\wedge}(\widehat{G}, B)$ is a vague soft group over X.*

Proof. Let $(\widehat{F}, A)\widetilde{\wedge}(\widehat{G}, B) = (\widehat{H}, A \times B)$. We know that $\widehat{F}_a$, $\forall a \in A$, and $\widehat{G}_b$, $\forall b \in B$, are vague groups of X and so is $\widehat{H}_{(a,b)}$, i.e., $t_{\widehat{H}(a,b)}(x) = min\{t_{\widehat{F}(a)}(x), t_{\widehat{G}(b)}(x)\}$, and $1 - f_{\widehat{H}(a,b)}(x) = min\{1 - f_{\widehat{F}(a)}(x), 1 - f_{\widehat{G}(b)}(x)\}$, $\forall (a, b) \in A \times B$, because intersection of two vague groups is also a vague group [5]. Thus, $(\widehat{F}, A)\widetilde{\wedge}(\widehat{G}, B) = (\widehat{H},\ A \times B)$ is vague soft group over X. □

□

Definition 3.9. Let $(\widehat{F}, A)$ be a vague soft set over X. The soft set $(\widehat{F}, A)_{(\alpha,\beta)} = \{(\widehat{F}_a)_{(\alpha,\beta)} : a \in A\}$, for each $\alpha, \beta \in [0, 1]$ with $\alpha \leqslant \beta$, is called an (α, β)-level soft set of the vague soft set $(\widehat{F}, A)$, where $(\widehat{F}_a)_{(\alpha,\beta)}$ is an (α, β)-level set of the vague set $\widehat{F}_a$.

That is, for each $\alpha, \beta \in [0, 1]$, $(\widehat{F}, A)_{(\alpha,\beta)}$ is a soft set in classical case.

Theorem 3.10. *Let $(\widehat{F}, A)$ be a vague soft set over X. Then $(\widehat{F}, A)$ is a vague soft group over X iff for all $a \in A$ and for arbitrary $\alpha, \beta \in [0, 1]$ with $\alpha \leqslant \beta$ and $(\widehat{F}_a)_{(\alpha,\beta)} \neq \varnothing$, the (α, β)-level soft set $(\widehat{F}, A)_{(\alpha,\beta)}$ is a soft group over X in Aktaş and Cağman's sense* [1].

Proof. Let $(\widehat{F}, A)$ be a vague soft group over X. Then for each $a \in A$, $\widehat{F}_a$ is a vague subgroup of X. For arbitrary $\alpha, \beta \in [0, 1]$ with $(\widehat{F}_a)_{(\alpha,\beta)} \neq \varnothing$, let $x, y \in (\widehat{F}_a)_{(\alpha,\beta)}$. Then, we have

$$t_{\widehat{F}(a)}(x) \geqslant \alpha,\ \ 1 - f_{\widehat{F}(a)}(x) \geqslant \beta \text{ and } t_{\widehat{F}(a)}(y) \geqslant \alpha,\ \ 1 - f_{\widehat{F}(a)}(y) \geqslant \beta.$$

Therefore, $t_{\widehat{F}(a)}(x \cdot y^{-1}) \geqslant min\{t_{\widehat{F}(a)}(x), t_{\widehat{F}(a)}(y)\} \geqslant min\{\alpha, \alpha\} = \alpha$. Similarly, we see that $1 - f_{\widehat{F}(a)}(x \cdot y^{-1}) \geqslant \beta$. Hence $x \cdot y^{-1} \in (\widehat{F}_a)_{(\alpha,\beta)}$. We obtain that, $(\widehat{F}_a)_{(\alpha,\beta)}$

is a subgroup of X, $\forall a \in A$. Hence $(\widehat{F}, A)_{(\alpha,\beta)}$ is a soft group over X in classical case.

Conversely, let assume that $(\widehat{F}, A)$ is not a vague soft group over X. Then there exist $a \in A$ such that $\widehat{F}_a$ is not a vague subgroup of X. Then there exist $x_0, y_0 \in X$ such that $V_{\widehat{F}(a)}(x_0 \cdot y_0^{-1}) < min\{V_{\widehat{F}(a)}(x_0), V_{\widehat{F}(a)}(y_0)\}$ i.e.,

$$t_{\widehat{F}(a)}(x_0 \cdot y_0^{-1}) < min\{t_{\widehat{F}(a)}(x_0), t_{\widehat{F}(a)}(y_0)\}$$

or

$$1 - f_{\widehat{F}(a)}(x_0 \cdot y_0^{-1}) < min\{1 - f_{\widehat{F}(a)}(x_0), 1 - f_{\widehat{F}(a)}(y_0)\}.$$

Let $t_{\widehat{F}(a)}(x_0 \cdot y_0^{-1}) < min\{t_{\widehat{F}(a)}(x_0), t_{\widehat{F}(a)}(y_0)\}$. Let $t_{\widehat{F}(a)}(x_0 \cdot y_0^{-1}) = \lambda$, $t_{\widehat{F}(a)}(x_0) = \theta$, $t_{\widehat{F}(a)}(y_0) = \delta$. We have $\lambda < min\{\theta, \delta\}$. Let $\alpha = \dfrac{\lambda + min\{\theta, \delta\}}{2}$, then $\lambda < \alpha < min\{\theta, \delta\}$. Now, we have $t_{\widehat{F}(a)}(x_0 \cdot y_0^{-1}) = \lambda < \alpha$. Hence $x_0 \cdot y_0^{-1} \notin (\widehat{F}_a)_{(\alpha,\beta)}$. But, since $\theta > min\{\theta, \delta\} > \alpha$ and $\delta > min\{\theta, \delta\} > \alpha$, then we obtain $t_{\widehat{F}(a)}(x_0) > \alpha$ and $t_{\widehat{F}(a)}(y_0) > \alpha$. For $1 - f_{\widehat{F}(a)}(x_0) > \beta$ and $1 - f_{\widehat{F}(a)}(y_0) > \beta$, we obtain $x_0, y_0 \in (\widehat{F}_a)_{(\alpha,\beta)}$. This contradicts with the fact that $(\widehat{F}, A)_{(\alpha,\beta)}$ is soft group over X.

For the case $1 - f_{\widehat{F}(a)}(x_0 \cdot y_0^{-1}) < min\{1 - f_{\widehat{F}(a)}(x_0), 1 - f_{\widehat{F}(a)}(y_0)\}$, the proof can be made in similar way. □

4. Homomorphism of vague soft groups

In this section, we define vague soft function, and then define the image and preimage of a vague soft set under vague soft function. Furthermore we define vague soft homomorphism and show that the homomorphic image and preimage of a vague soft group are also vague soft group.

Definition 4.1. Let $\varphi : X \longrightarrow Y$ and $\psi : A \longrightarrow B$ be two functions, where A and B are parameter sets for the crisp sets X and Y, respectively. Then the pair (φ, ψ) is called a vague soft function from X to Y.

Definition 4.2. Let $(\widehat{F}, A)$ and $(\widehat{G}, B)$ be two vague soft sets over X and Y, respectively and let (φ, ψ) be a vague soft function from X to Y.

(1) The image of $(\widehat{F}, A)$ under the vague soft function (φ, ψ), denoted by

$$(\varphi, \psi)(\widehat{F}, A),$$

is the vague soft set over Y defined by $(\varphi, \psi)(\widehat{F}, A) = (\varphi(\widehat{F}), \psi(A))$, where

$\varphi(\widehat{F}) = (\varphi(t_{\widehat{F}}), \varphi(1 - f_{\widehat{F}}))$ and

$$\varphi(t_{\widehat{F}})_k(y) = \begin{cases} max_{\varphi(x)=y}\{max_{\psi(a)=k}\ t_{\widehat{F_a}}(x)\}, & \text{if } x \in \varphi^{-1}(y); \\ 0, & \text{otherwise.} \end{cases},$$

for all $k \in \psi(A)$ and $y \in Y$.

$$\varphi(1 - f_{\widehat{F}})_k(y) = \begin{cases} max_{\varphi(x)=y}\{max_{\psi(a)=k}\ 1 - f_{\widehat{F_a}}(x)\}, & \text{if } x \in \varphi^{-1}(y); \\ 0, & \text{otherwise.} \end{cases},$$

for all $k \in \psi(A)$ and $y \in Y$.

(2) The preimage of $(\widehat{G}, B)$ under the vague soft function (φ, ψ), denoted by $(\varphi, \psi)^{-1}(\widehat{G}, B)$, is the vague soft set over X defined by

$$(\varphi, \psi)^{-1}(\widehat{G}, B) = (\varphi^{-1}(\widehat{G}), \psi^{-1}(B)),$$

where $\varphi^{-1}(\widehat{G}) = (\varphi^{-1}(t_{\widehat{G}}), \varphi^{-1}(1 - f_{\widehat{G}}))$ and $\varphi^{-1}(t_{\widehat{G}})_a(x) = t_{\widehat{G}_{\psi(a)}}(\varphi(x))$, for all $a \in \psi^{-1}(B)$ and $x \in X$.

$\varphi^{-1}(1 - f_{\widehat{G}})_a(x) = 1 - f_{\widehat{G}_{\psi(a)}}(\varphi(x))$, for all $a \in \psi^{-1}(B)$ and $x \in X$.

If φ and ψ is injective (surjective), then (φ, ψ) is said to be injective (surjective).

Definition 4.3. Let (φ, ψ) be a vague soft function from X to Y. If φ is a homomorphism from X to Y then (φ, ψ) is said to be vague soft homomorphism. If φ is a isomorphism from X to Y and ψ is one-to-one mapping from A onto B then (φ, ψ) is said to be vague soft isomorphism.

Theorem 4.4. *Let $(\widehat{F}, A)$ be a vague soft group over X, (φ, ψ) be a vague soft homomorphism from X to Y. Then $(\varphi, \psi)(\widehat{F}, A)$ is a vague soft group over Y.*

Proof. Let $k \in \psi(A)$ and $y_1, y_2 \in Y$. If $\varphi^{-1}(y_1) = \varnothing$ or $\varphi^{-1}(y_2) = \varnothing$ the proof is straightforward. Let assume that there exist $x_1, x_2 \in X$ such that $\varphi(x_1) = y_1$, $\varphi(x_2) = y_2$.

$$\begin{aligned}
\varphi(t_{\widehat{F}})_k(y_1 \cdot y_2^{-1}) &= max_{\varphi(t)=y_1 \cdot y_2^{-1}}\{max_{\psi(a)=k}\ t_{\widehat{F}_a}(t)\} \\
&\geqslant max_{\psi(a)=k}\ t_{\widehat{F}_a}(x_1 \cdot x_2^{-1}) \\
&\geqslant max_{\psi(a)=k}\{min(t_{\widehat{F}_a}(x_1), t_{\widehat{F}_a}(x_2))\} \\
&= min\left\{max_{\psi(a)=k} t_{\widehat{F}_a}(x_1), max_{\psi(a)=k} t_{\widehat{F}_a}(x_2)\right\}
\end{aligned}$$

This inequality is satisfied for each $x_1, x_2 \in X$, which satisfy $\varphi(x_1) = y_1$, $\varphi(x_2) = y_2$. Then we have

$$\begin{aligned}
\varphi(t_{\widehat{F}})_k(y_1 \cdot y_2^{-1}) &\geqslant min\Big\{max_{\varphi(t_1)=y_1}\{max_{\psi(a)=k} t_{\widehat{F_a}}(t_1)\}, \\
&\qquad max_{\varphi(t_2)=y_2}\{max_{\psi(a)=k} t_{\widehat{F_a}}(t_2)\}\Big\} \\
&= min\{\varphi(t_{\widehat{F}})_k(y_1), \varphi(t_{\widehat{F}})_k(y_2)\}
\end{aligned}$$

Similarly, we obtain that

$$\varphi(1 - f_{\widehat{F}})_k(y_1 \cdot y_2^{-1}) \geqslant min\{\varphi(1 - f_{\widehat{F}})_k(y_1), \varphi(1 - f_{\widehat{F}})_k(y_2)\}. \qquad \square$$

Theorem 4.5. *Let $(\widehat{G}, B)$ be a vague soft group over Y and (φ, ψ) be a vague soft homomorphism from X to Y. Then $(\varphi, \psi)^{-1}(\widehat{G}, B)$ is a vague soft group over X.*

Proof. Let $a \in \psi^{-1}(B)$ and $x_1, x_2 \in X$.

$$\begin{aligned}
\varphi^{-1}(t_{\widehat{G}})_a(x_1 \cdot x_2^{-1}) &= t_{\widehat{G}_{\psi(a)}}(\varphi(x_1 \cdot x_2^{-1})) \\
&= t_{\widehat{G}_{\psi(a)}}(\varphi(x_1) \cdot \varphi(x_2)^{-1}) \\
&\geqslant min\{t_{\widehat{G}_{\psi(a)}}(\varphi(x_1)), t_{\widehat{G}_{\psi(a)}}(\varphi(x_2))\} \\
&= min\{\varphi^{-1}(t_{\widehat{G}})_a(x_1), \varphi^{-1}(t_{\widehat{G}})_a(x_2)\}.
\end{aligned}$$

Similarly, we obtain that

$$\varphi^{-1}(1 - f_{\widehat{G}})_a(x_1 \cdot x_2^{-1}) \geqslant min\{\varphi^{-1}(1 - f_{\widehat{G}})_a(x_1), \varphi^{-1}(1 - f_{\widehat{G}})_a(x_2)\}.$$

Hence, $(\varphi, \psi)^{-1}(\widehat{G}, B)$ is a vague soft group over X. $\square$

5. Normal vague soft groups

In this section, we define normal vague soft group and study their properties.

Definition 5.1. Let X be a classical group and $(\widehat{F}, A)$ be a vague soft group over X. Then $(\widehat{F}, A)$ is said to be a normal vague soft group over X if $V_{\widehat{F}_a}(x\cdot y) = V_{\widehat{F}_a}(y\cdot x)$, for all $x, y \in X$ and $\forall a \in A$, i.e.,
$t_{\widehat{F}_a}(x\cdot y) = t_{\widehat{F}_a}(y\cdot x)$ and $1 - f_{\widehat{F}_a}(x\cdot y) = 1 - f_{\widehat{F}_a}(y\cdot x)$.

Theorem 5.2. *Let $(\widehat{F}, A)$ be a vague soft set over X. Then $(\widehat{F}, A)$ is a normal vague soft group over X iff for all $a \in A$ and for arbitrary $\alpha, \beta \in [0, 1]$ with $\alpha \leqslant \beta$ and $(\widehat{F}_a)_{(\alpha,\beta)} \neq \varnothing$, the (α, β)-level soft set $(\widehat{F}, A)_{(\alpha,\beta)}$ is a normal soft group over X in Aktaş and Cağman's sense [1].*

Proof. From the Theorem 3.10, we need only to show normality. For any $x \in (\widehat{F}_a)_{(\alpha,\beta)}$ and $y \in X$, we have $t_{\widehat{F}_a}(y \cdot x \cdot y^{-1}) = t_{\widehat{F}_a}(y^{-1} \cdot (y \cdot x)) = t_{\widehat{F}_a}(x) \geqslant \alpha$ and $1 - f_{\widehat{F}_a}(y \cdot x \cdot y^{-1}) = 1 - f_{\widehat{F}_a}(y^{-1} \cdot (y \cdot x)) = 1 - f_{\widehat{F}_a}(x) \geqslant \beta$.

It follows that $y \cdot x \cdot y^{-1} \in (\widehat{F}_a)_{(\alpha,\beta)}$, i.e., $(\widehat{F}_a)_{(\alpha,\beta)}$ is a normal subgroup of X, for each $a \in A$. Hence, $(\widehat{F}, A)_{(\alpha,\beta)}$ is a normal soft group over X.

Conversely, let assume that $(\widehat{F}, A)$ is not a normal vague soft group over X. Then there exist $a \in A$ such that $\widehat{F}_a$ is not normal vague group of X. Then there exist $x_0, y_0 \in X$ such that
$V_{\widehat{F}_a}(x_0 \cdot y_0) < V_{\widehat{F}_a}(y_0 \cdot x_0)$ or $V_{\widehat{F}_a}(x_0 \cdot y_0) > V_{\widehat{F}_a}(y_0 \cdot x_0)$, i.e.,
$t_{\widehat{F}_a}(x_0 \cdot y_0) < t_{\widehat{F}_a}(y_0 \cdot x_0)$ or $t_{\widehat{F}_a}(x_0 \cdot y_0) > t_{\widehat{F}_a}(y_0 \cdot x_0)$ or
$1 - f_{\widehat{F}_a}(x_0 \cdot y_0) < 1 - f_{\widehat{F}_a}(y_0 \cdot x_0)$ or $1 - f_{\widehat{F}_a}(x_0 \cdot y_0) > 1 - f_{\widehat{F}_a}(y_0 \cdot x_0)$

In case $t_{\widehat{F}_a}(x_0 \cdot y_0) < t_{\widehat{F}_a}(y_0 \cdot x_0)$, there exist $\alpha \in [0, 1]$ such that $t_{\widehat{F}_a}(x_0 \cdot y_0) < \alpha < t_{\widehat{F}_a}(y_0 \cdot x_0)$. It follows that $x_0 \cdot y_0 \notin (\widehat{F}_a)_{(\alpha,\beta)}$, but for $1 - f_{\widehat{F}_a}(y_0 \cdot x_0) < \beta, x_0 \cdot y_0 \notin (\widehat{F}_a)_{(\alpha,\beta)}$. This contradicts with the fact that $(\widehat{F}, A)_{(\alpha,\beta)}$ is a normal soft group over X. Hence, $(\widehat{F}, A)$ is a normal vague soft group over X. In cases $t_{\widehat{F}_a}(x_0 \cdot y_0) > t_{\widehat{F}_a}(y_0 \cdot x_0)$ and $1 - f_{\widehat{F}_a}(x_0 \cdot y_0) < 1 - f_{\widehat{F}_a}(y_0 \cdot x_0)$ and $1 - f_{\widehat{F}_a}(x_0 \cdot y_0) > 1 - f_{\widehat{F}_a}(y_0 \cdot x_0)$, the proof can be obtained in similar way. □

Theorem 5.3. *Let $(\widehat{F}, A)$ and $(\widehat{G}, B)$ be two normal vague soft groups over X and $A \cap B \neq \varnothing$. Then their intersection $(\widehat{F}, A)\widetilde{\cap}(\widehat{G}, B)$ is a normal vague soft subgroup over X.*

Proof. Straightforward. □

Theorem 5.4. *Let $(\widehat{F}, A)$ and $(\widehat{G}, B)$ be two normal vague soft groups over X. Then $(\widehat{F}, A)\widetilde{\wedge}(\widehat{G}, B)$ is a normal vague soft subgroup over X.*

Proof. Straightforward. □

Theorem 5.5. *Let $(\widehat{F}, A)$ be a normal vague soft group over X. Let $(\widehat{F}, A)|_e = \{(\widehat{F}_a)|_e = \{x \in X : t_{\widehat{F}_a}(x) = t_{\widehat{F}_a}(e)\}$ and $1 - f_{\widehat{F}_a}(x) = 1 - f_{\widehat{F}_a}(e)\} : a \in A\}$, where e is the unit element of X. Then the classical soft set $(\widehat{F}, A)|_e$ is a normal soft group over X.*

Proof. For each $a \in A$ and for arbitrary $x, y \in (\widehat{F}, A)|_e = \{x \in X : t_{\widehat{F}_a}(x) = t_{\widehat{F}_a}(e)$ and $1 - f_{\widehat{F}_a}(x) = 1 - f_{\widehat{F}_a}(e)\}$, we have
$t_{\widehat{F}_a}(x \cdot y^{-1}) \geqslant min\{t_{\widehat{F}_a}(x), t_{\widehat{F}_a}(y)\} = min\{t_{\widehat{F}_a}(e), t_{\widehat{F}_a}(e)\} = t_{\widehat{F}_a}(e)$,

$1 - f_{\widehat{F}_a}(x \cdot y^{-1}) \geqslant min\{1 - f_{\widehat{F}_a}(x), 1 - f_{\widehat{F}_a}(y)\} = min\{1 - f_{\widehat{F}_a}(e), 1 - f_{\widehat{F}_a}(e)\} = 1 - f_{\widehat{F}_a}(e)$

and always $t_{\widehat{F}_a}(e) \geqslant t_{\widehat{F}_a}(x \cdot y^{-1}),\ 1 - f_{\widehat{F}_a}(e) \geqslant 1 - f_{\widehat{F}_a}(x \cdot y^{-1})$

Therefore, $t_{\widehat{F}_a}(x \cdot y^{-1}) = t_{\widehat{F}_a}(e),\ 1 - f_{\widehat{F}_a}(x \cdot y^{-1}) = 1 - f_{\widehat{F}_a}(e)$ and $x \cdot y^{-1} \in (\widehat{F}_a)|_e$.

Let $x \in (\widehat{F}_a)|_e$ and $y \in X$. Since $(\widehat{F}, A)$ is normal vague soft set, we have

$t_{\widehat{F}_a}(y \cdot x \cdot y^{-1}) = t_{\widehat{F}_a}(y^{-1} \cdot y \cdot x) = t_{\widehat{F}_a}(x) = t_{\widehat{F}_a}(e)$ and

$1 - f_{\widehat{F}_a}(y \cdot x \cdot y^{-1}) = 1 - f_{\widehat{F}_a}(y^{-1} \cdot y \cdot x) = 1 - f_{\widehat{F}_a}(x) = 1 - f_{\widehat{F}_a}(e)$

Therefore, $y \cdot x \cdot y^{-1} \in (\widehat{F}_a)|_e$. Hence, $(\widehat{F}_a)|_e$ is a normal soft group over X. □

Theorem 5.6. *Let $(\widehat{F}, A)$ be a vague soft group over X. Then the followings are equivalent for each $a \in A$;*

(1) $t_{\widehat{F}_a}(y \cdot x \cdot y^{-1}) \geqslant t_{\widehat{F}_a}(x)$ *and* $1 - f_{\widehat{F}_a}(y \cdot x \cdot y^{-1}) \geqslant 1 - f_{\widehat{F}_a}(x), \quad \forall x, y \in X$

(2) $t_{\widehat{F}_a}(y \cdot x \cdot y^{-1}) = t_{\widehat{F}_a}(x)$ *and* $1 - f_{\widehat{F}_a}(y \cdot x \cdot y^{-1}) = 1 - f_{\widehat{F}_a}(x), \quad \forall x, y \in X$

(3) $t_{\widehat{F}_a}(x \cdot y) = t_{\widehat{F}_a}(y \cdot x)$ *and* $1 - f_{\widehat{F}_a}(x \cdot y) = 1 - f_{\widehat{F}_a}(y \cdot x), \quad \forall x, y \in X.$

Proof. For each $a \in A$

(1)⇒(2): For arbitrary $x, y \in X$, since $t_{\widehat{F}_a}(y \cdot x \cdot y^{-1}) \geqslant t_{\widehat{F}_a}(x)$, we have

$t_{\widehat{F}_a}(y^{-1} \cdot x \cdot y) = t_{\widehat{F}_a}(y^{-1} \cdot x \cdot (y^{-1})^{-1}) \geqslant t_{\widehat{F}_a}(x)$ and similarly, $1 - f_{\widehat{F}_a}(y^{-1} \cdot x \cdot y) \geqslant 1 - f_{\widehat{F}_a}(x)$.

Hence, $t_{\widehat{F}_a}(x) = t_{\widehat{F}_a}(y^{-1} \cdot (y \cdot x \cdot y^{-1}) \cdot y) \geqslant t_{\widehat{F}_a}(y \cdot x \cdot y^{-1})$ and

$1 - f_{\widehat{F}_a}(x) = 1 - f_{\widehat{F}_a}(y^{-1} \cdot (y \cdot x \cdot y^{-1}) \cdot y) \geqslant 1 - f_{\widehat{F}_a}(y \cdot x \cdot y^{-1})$

Then we have (2).

(2)⇒(3): Substituting x for $x \cdot y$ in (2), we can obtain (3).

(3)⇒(1): Since $t_{\widehat{F}_a}(x \cdot y) = t_{\widehat{F}_a}(y \cdot x)$ and $1 - f_{\widehat{F}_a}(x \cdot y) = 1 - f_{\widehat{F}_a}(y \cdot x)$, we obtain $t_{\widehat{F}_a}(y \cdot x \cdot y^{-1}) = t_{\widehat{F}_a}(y^{-1} y \cdot x) = t_{\widehat{F}_a}(x) \geqslant t_{\widehat{F}_a}(x)$ and similarly, $1 - f_{\widehat{F}_a}(y \cdot x \cdot y^{-1}) \geqslant 1 - f_{\widehat{F}_a}(x)$. □

Theorem 5.7. *Let $(\widehat{F}, A)$ be a normal vague soft group over X, (φ, ψ) be a vague soft homomorphism from X onto Y. Then $(\varphi, \psi)(\widehat{F}, A)$ is a normal vague soft group over Y.*

Proof. From the Theorem 4.4, we need only show normality. From the Theorem 5.6, for each $k \in \psi(A)$ and $y_1, y_2 \in Y$, there exist $x_1, x_2 \in X$ such that $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. Then

$$\begin{aligned}
\varphi(t_{\widehat{F}_a})_k(y_1 \cdot y_2 \cdot y_1^{-1}) &= max_{\varphi(t) = y_1 \cdot y_2 \cdot y_1^{-1}}\{max_{\psi(a)=k}\ t_{\widehat{F}_a}(t)\} \\
&\geqslant max_{\psi(a)=k}\ t_{\widehat{F}_a}(x_1 \cdot x_2 \cdot x_1^{-1}) \\
&\geqslant max_{\psi(a)=k}\ t_{\widehat{F}_a}(x_1^{-1} \cdot x_1 \cdot x_2) \\
&= max_{\psi(a)=k}\ t_{\widehat{F}_a}(x_2)
\end{aligned}$$

Similarly, $\varphi(1 - f_{\widehat{F}_a})_k(y_1 \cdot y_2 \cdot y_1^{-1}) \geqslant max_{\psi(a)=k}\ 1 - f_{\widehat{F}_a}(x_2)$. This inequality holds for each $\varphi(x_2) = y_2$, then we have

$\varphi(t_{\widehat{F}_a})_k(y_1 \cdot y_2 \cdot y_1^{-1}) \geqslant max_{\varphi(t)=y_2}\{max_{\psi(a)=k}\ t_{\widehat{F}_a}(t)\} = \varphi(t_{\widehat{F}_a})_k(y_2)$

$\varphi(1 - f_{\widehat{F}_a})_k(y_1 \cdot y_2 \cdot y_1^{-1}) \geqslant max_{\varphi(t)=y_2}\{max_{\psi(a)=k}\ 1 - f_{\widehat{F}_a}(t)\} = \varphi(1 - f_{\widehat{F}_a})_k(y_2)$. □

Theorem 5.8. *Let $(\widehat{G}, B)$ be a normal vague soft group over Y, (φ, ψ) is a vague soft homomorphism from X to Y. Then $(\varphi, \psi)^{-1}(\widehat{G}, B)$ is a normal vague soft group over X.*

Proof. From the Theorem 4.5, we need only to show normality. Let $a \in \psi^{-1}(B)$ and $x_1, x_2 \in X$.

$$\begin{aligned} \varphi^{-1}(t_{\widehat{G}_a})(x_1 \cdot x_2) &= t_{\widehat{G}_{\psi(a)}}(\varphi(x_1 \cdot x_2)) \\ &= t_{\widehat{G}_{\psi(a)}}(\varphi(x_1) \cdot \varphi(x_2)) \\ &= t_{\widehat{G}_{\psi(a)}}(\varphi(x_2) \cdot \varphi(x_1)) \; . \\ &= t_{\widehat{G}_{\psi(a)}}(\varphi(x_2 \cdot x_1)) \\ &= \varphi^{-1}(t_{\widehat{G}_a})(x_2 \cdot x_1) \end{aligned}$$

Similarly, we obtain $\varphi^{-1}(1 - f_{\widehat{G}_a})(x_1 \cdot x_2) = \varphi^{-1}(1 - f_{\widehat{G}_a})(x_2 \cdot x_1)$. □

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Banu Pazar Varol (`banupazar@kocaeli.edu.tr`)

Department of Mathematics, Kocaeli University, 41380, Kocaeli, Turkey

Abdülkadir Aygünoğlu (abdulkadir.aygunoglu@kocaeli.edu.tr)
Department of Mathematics, Kocaeli University, 41380, Kocaeli, Turkey

Halis Aygün (halis@kocaeli.edu.tr)
Department of Mathematics, Kocaeli University, 41380, Kocaeli, Turkey