Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 7, No. 2, (February 2014), pp. 181–196
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

γ-operation and decompositions of some forms of soft continuity in soft topological spaces

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Received 3 February 2013; Revised 6 April 2013; Accepted 26 June 2013

Abstract. In this paper the authors introduce the notions of γ-operation, pre-open soft sets, α-open soft sets, semi-open soft sets and β-open soft sets to soft topological spaces. The authors study the relations between these different types of subsets of soft topological spaces. The authors also introduce the concepts of pre-soft continuous functions, α-soft continuous functions, semi-soft continuous functions and β-soft continuous functions. Finally, the authors show that a mapping between two soft topological spaces is α-soft continuous if and only if it is pre-soft continuous and semi soft continuous.

2010 AMS Classification: 06D72, 54A40

Keywords: Soft set, Soft topological space, Soft interior, Soft closure, Open soft, Closed soft, γ-operation, Pre-open soft set, α-open soft set, Semi-open soft set, β-open soft set, Soft continuous functions, Pre-soft continuous functions, Semi-soft continuous functions, α-soft continuous functions, β-soft continuous functions.

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1. Introduction

The concept of soft sets was first introduced by Molodtsov [14] in 1999 as a general mathematical tool for dealing with uncertain objects. In [14, 13], Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions,game theory, operations research,Riemann integration, Perron integration, probability, theory of measurement, and so on. After presentation of the operations of soft sets [11], the properties and applications of soft set theory have been studied increasingly [3, 8, 13, 17]. In recent years, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [1, 2, 5, 9, 10, 11, 12, 13, 15, 20]. To develop soft set theory, the operations of the soft sets are redefined and a uni-int decision making method was constructed by using these new operations [6].

Recently, in 2011, Shabir and Naz [18] initiated the study of soft topological spaces. They defined soft topology on the collection τ of soft sets over X. Consequently, they defined basic notions of soft topological spaces such as open and closed soft sets, soft subspace, soft closure, soft nbd of a point, soft separation axioms, soft regular spaces and soft normal spaces and established their several properties. Hussain and Ahmad [7] investigated the properties of open (closed) soft, soft nbd and soft closure. They also defined and discussed the properties of soft interior, soft exterior and soft boundary which are fundamental for further research on soft topology and will strengthen the foundations of the theory of soft topological spaces.

The purpose of this paper is to introduce the notions of γ-operation, pre-open soft sets, α-open soft sets, semi-open soft sets and β-open soft sets to soft topological spaces. We study the relations between these different types of subsets of soft topological spaces. We also introduce the concepts of pre-soft continuous functions, α-soft continuous functions, semi-soft continuous functions and β-soft continuous functions. Finally, the decomposition has given.

2. Preliminaries

Definition 2.1 ([14]). Let X be an initial universe and E be a set of parameters. Let $P(X)$ denote the power set of X and A be a non-empty subset of E. A pair (F, A) denoted by F_A is called a soft set over X , where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parametrized family of subsets of the universe X. For a particular $e \in A$, $F(e)$ may be considered the set of e-approximate elements of the soft set (F, A) and if $e \notin A$, then $F(e) = \phi$ i.e $F_A = \{F(e) : e \in A \subseteq E,\ F : A \to P(X)\}$. The family of all these soft sets denoted by $SS(X)_A$.

Definition 2.2 ([11]). Let $F_A,\ G_B \in SS(X)_E$. Then F_A is soft subset of G_B, denoted by $F_A \tilde{\subseteq} G_B$, if

(1): $A \subseteq B$, and
(2): $F(e) \subseteq G(e), \forall e \in A$.

In this case, F_A is said to be a soft subset of G_B and G_B is said to be a soft superset of F_A, $G_B \tilde{\supseteq} F_A$.

Definition 2.3 ([11]). Two soft subset F_A and G_B over a common universe set X are said to be soft equal if F_A is a soft subset of G_B and G_B is a soft subset of F_A.

Definition 2.4 ([3]). The complement of a soft set (F, A), denoted by $(F, A)'$, is defined by $(F, A)' = (F', A)$, $F' : A \to P(X)$ is a mapping given by $F'(e) = X - F(e)$, $\forall\, e \in A$ and F' is called the soft complement function of F.
Clearly $(F')'$ is the same as F and $((F, A)')' = (F, A)$.

Definition 2.5 ([18]). The difference of two soft sets (F, E) and (G, E) over the common universe X, denoted by $(F, E) - (G, E)$ is the soft set (H, E) where for all $e \in E$, $H(e) = F(e) - G(e)$.

Definition 2.6 ([18]). Let (F, E) be a soft set over X and $x \in X$. We say that $x \in (F, E)$ read as x belongs to the soft set (F, E) whenever $x \in F(e)$ for all $e \in E$.

Definition 2.7 ([11])**.** A soft set (F, A) over X is said to be a NULL soft set denoted by $\tilde{\phi}$ or ϕ_A if for all $e \in A,\ F(e) = \phi$ (null set).

Definition 2.8 ([11])**.** A soft set (F, A) over X is said to be an absolute soft set denoted by $\tilde{A}$ or X_A if for all $e \in A,\ F(e) = X$. Clearly we have $X'_A = \phi_A$ and $\phi'_A = X_A$.

Definition 2.9 ([11])**.** The union of two soft sets (F, A) and (G, B) over the common universe X is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e),\ e \in A - B, \\ G(e),\ e \in B - A, \\ F(e) \cup G(e),\ e \in A \cap B \end{cases}.$$

Definition 2.10 ([11])**.** The intersection of two soft sets (F, A) and (G, B) over the common universe X is the soft set (H, C), where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$. Note that, in order to efficiently discuss, we consider only soft sets (F, E) over a universe X in which all the parameter set E are same. We denote the family of these soft sets by $SS(X)_E$.

Definition 2.11 ([21])**.** Let I be an arbitrary indexed set and $L = \{(F_i, E), i \in I\}$ be a subfamily of $SS(X)_E$.

- **(1):** The union of L is the soft set (H, E), where $H(e) = \bigcup_{i \in I} F_i(e)$ for each $e \in E$. We write $\tilde{\bigcup}_{i \in I}(F_i, E) = (H, E)$.
- **(2):** The intersection of L is the soft set (M, E), where $M(e) = \bigcap_{i \in I} F_i(e)$ for each $e \in E$. We write $\tilde{\bigcap}_{i \in I}(F_i, E) = (M, E)$.

Definition 2.12 ([18])**.** Let τ be a collection of soft sets over a universe X with a fixed set of parameters E, then $\tau \subseteq SS(X)_E$ is called a soft topology on X if

- **(1):** $\tilde{X}, \tilde{\phi} \in \tau$, where $\tilde{\phi}(e) = \phi$ and $\tilde{X}(e) = X,\ \forall e \in E$,
- **(2):** the union of any number of soft sets in τ belongs to τ,
- **(3):** the intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X.

Definition 2.13 ([7])**.** Let (X, τ, E) be a soft topological space. A soft set (F, A) over X is said to be closed soft set in X, if its relative complement $(F, A)'$ is an open soft set.

Definition 2.14 ([7])**.** Let(X, τ, E) be a soft topological space. The members of τ are said to be open soft sets in X. We denote the set of all open soft sets over X by $OS(X, \tau, E)$, or when there can be no confusion by $OS(X)$ and the set of all closed soft sets by $CS(X, \tau, E)$, or $CS(X)$.

Definition 2.15 ([18])**.** Let (X, τ, E) be a soft topological space and $(F, E) \in SS(X)_E$. The soft closure of (F, E), denoted by $cl(F, E)$ is the intersection of all closed soft super sets of (F, E). Clearly $cl(F, E)$ is the smallest closed soft set over X which contains (F, E) i.e
$cl(F, E) = \tilde{\cap}\{(H, E) : (H, E)\ is\ closed\ soft\ set\ and\ (F, E)\tilde{\subseteq}(H, E)\})$.

Definition 2.16 ([21])**.** Let (X, τ, E) be a soft topological space and $(F, E) \in SS(X)_E$. The soft interior of (G, E), denoted by $int(G, E)$ is the union of all open

soft subsets of (G,E). Clearly $int(G,E)$ is the largest open soft set over X which contained in (G,E) i.e
$int(G,E) = \tilde{\cup}\{(H,E) : (H,E)\ is\ an\ open\ soft\ set\ and\ (H,E)\tilde{\subseteq}(G,E)\})$.

Definition 2.17 ([21]). The soft set $(F,E) \in SS(X)_E$ is called a soft point in X_E if there exist $x \in X$ and $e \in E$ such that $F(e) = \{x\}$ and $F(e') = \phi$ for each $e' \in E - \{e\}$, and the soft point (F,E) is denoted by x_e.

Proposition 2.18 ([19]). *The union of any collection of soft points can be considered as a soft set and every soft set can be expressed as union of all soft points belonging to it.*

Definition 2.19 ([21]). The soft point x_e is said to be belonging to the soft set (G,A), denoted by $x_e\tilde{\in}(G,A)$, if for the element $e \in A$, $F(e) \subseteq G(e)$.

Definition 2.20 ([21]). A soft set (G,E) in a soft topological space (X,τ,E) is called a soft neighborhood (briefly: nbd) of the soft point $x_e\tilde{\in}X_E$ if there exists an open soft set (H,E) such that $x_e\tilde{\in}(H,E)\tilde{\subseteq}(G,E)$.
A soft set (G,E) in a soft topological space (X,τ,E) is called a soft neighborhood of the soft (F,E) if there exists an open soft set (H,E) such that $(F,E)\tilde{\in}(H,E)\tilde{\subseteq}(G,E)$. The neighborhood system of a soft point x_e, denoted by $N_\tau(x_e)$, is the family of all its neighborhoods.

Definition 2.21 ([16]). Let (X,τ,E) be a soft topological space and $(F,E) \in SS(X)_E$. Define $\tau_{(F,E)} = \{(G,E)\tilde{\cap}(F,E) : (G,E) \in \tau\}$, which is a soft topology on (F,E). This soft topology is called soft relative topology of τ on (F,E), and $[(F,E),\tau_{(F,E)}]$ is called soft subspace of (X,τ,E).

Definition 2.22 ([21]). Let $SS(X)_A$ and $SS(Y)_B$ be families of soft sets, $u : X \to Y$ and $p : A \to B$ be mappings. Then the mapping $f_{pu} : SS(X)_A \to SS(Y)_B$ is defined as:

(1): Let $(F,A) \in SS(X)_A$. The image of (F,A) under f_{pu}, written as
$$f_{pu}(F,A) = (f_{pu}(F), p(A)),$$
is a soft set in $SS(Y)_B$ such that
$$f_{pu}(F)(y) = \begin{cases} \cup_{x\in p^{-1}(y)\cap A}\ u(F(x)), & p^{-1}(y) \cap A \neq \phi, \\ \phi, & otherwise. \end{cases}$$
for all $y \in B$.

(2): Let $(G,B) \in SS(Y)_B$. The inverse image of (G,B) under f_{pu}, written as $f^{-1}_{pu}(G,B) = (f^{-1}_{pu}(G), p^{-1}(B))$, is a soft set in $SS(X)_A$ such that
$$f^{-1}_{pu}(G)(x) = \begin{cases} u^{-1}(G(p(x))), & p(x) \in B, \\ \phi, & otherwise. \end{cases}$$
for all $x \in A$.

Definition 2.23 ([21]). Let (X,τ,A) and (Y,τ^*,B) be soft topological spaces and $f_{pu} : SS(X)_A \to SS(Y)_B$ be a function. Then

(1): The function f_{pu} is called soft continuous (soft-cts) if $f^{-1}_{pu}(G,B) \in \tau$ $\forall\ (G,B) \in \tau^*$.

(2): The function f_{pu} is called open soft if $f_{pu}(G,A) \in \tau^*\ \forall\ (G,A) \in \tau$.

Definition 2.24 ([4]). Let (X, τ, E) be a soft topological space and $x, y \in X$ such that $x \neq y$. Then (X, τ, E) is called soft Hausdorff space or soft T_2 space if there exist open soft sets (F, E) and (G, E) such that $x \in (F, E)$, $y \in (G, E)$ and $(F, E)\tilde{\cap}(G, E) = \tilde{\phi}$

3. Subsets of soft topological spaces

Definition 3.1. Let (X, τ, E) be a soft topological space. A mapping $\gamma : SS(X)_E \rightarrow SS(X)_E$ is said to be an operation on $OS(X)$ if $F_E \tilde{\subseteq} \gamma(F_E)\ \forall\ F_E \in OS(X)$. The collection of all γ-open soft sets is denoted by $OS(\gamma) = \{F_E : F_E \tilde{\subseteq} \gamma(F_E), F_E \in SS(X)_E\}$. Also, the complement of γ-open soft set is called γ-closed soft set, i.e $CS(\gamma) = \{F'_E : F_E \ is\ a\ \gamma - open\ soft\ set, F_E \in SS(X)_E\}$ is the family of all γ-closed soft sets.

Definition 3.2. Let (X, τ, E) be a soft topological space. Different cases of γ-operations on $SS(X)_E$ are as follows:

- **(1):** If $\gamma = int(cl)$, then γ is called pre-open soft operator. We denote the set of all pre-open soft sets by $POS(X, \tau, E)$, or when there can be no confusion by $POS(X)$ and the set of all pre-closed soft sets by $PCS(X, \tau, E)$, or $PCS(X)$.
- **(2):** If $\gamma = int(cl(int))$, then γ is called α-open soft operator. We denote the set of all α-open soft sets by $\alpha OS(X, \tau, E)$, or $\alpha OS(X)$ and the set of all α-closed soft sets by $\alpha CS(X, \tau, E)$, or $\alpha CS(X)$.
- **(3):** If $\gamma = cl(int)$, then γ is called semi-open soft operator. We denote the set of all semi-open soft sets by $SOS(X, \tau, E)$, or $SOS(X)$ and the set of all semi-closed soft sets by $SCS(X, \tau, E)$, or $SCS(X)$.
- **(4):** If $\gamma = cl(int(cl))$, then γ is called β-open soft operator. We denote the set of all β-open soft sets by $\beta OS(X, \tau, E)$, or $\beta OS(X)$ and the set of all β-closed soft sets by $\beta CS(X, \tau, E)$, or $\beta CS(X)$.

Theorem 3.3. *Let (X, τ, E) be a soft topological space and $\gamma : SS(X)_E \rightarrow SS(X)_E$ be an operation on $OS(X)$.*
If $\gamma \in \{int(cl), int(cl(int)), cl(int), cl(int(cl))\}$. Then

- **(1):** *Arbitrary soft union of γ-open soft sets is γ-open soft.*
- **(2):** *Arbitrary soft intersection of γ-closed soft sets is γ-closed soft.*

Proof. (1) We give the proof for the case of pre-open soft operator i.e $\gamma = int(cl)$. Let $\{F_{jE} : j \in J\} \subseteq POS(X)$. Then $\forall\ j \in J$, $F_{jE} \tilde{\subseteq} int(cl(F_{jE}))$. It follows that $\tilde{\bigcup}_j F_{jE} \tilde{\subseteq} \tilde{\bigcup}_j int(cl(F_{jE})) \tilde{\subseteq} int(\tilde{\bigcup}_j cl(F_{jE})) = int(cl(\tilde{\bigcup}_j F_{jE}))$. Hence $\tilde{\bigcup}_j F_{jE} \in POS(X)\ \forall\ j \in J$. The rest of the proof is similar.

(2) Immediate. □

Remark 3.4. The soft intersection of two pre-open (resp. semi-open, β-open) soft sets need not to be pre-open (resp. semi-open, β-open), as shown in the following examples.

Example 3.5. (1) Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\tilde{X}, \tilde{\phi}, (F, E)\}$ where (F, E) is a soft set over X defined as follows:

$F(e_1) = \{h_1, h_3\}, \quad F(e_2) = \{h_2, h_3\}$.
Then τ defines a soft topology on X, hence (X, τ, E) is a soft topological space over X. Then the sets (G, E) and (H, E) which defined as follows
$G(e_1) = \{h_1, h_2\}, \quad G(e_2) = \{h_1\}$.
$H(e_1) = \{h_2, h_3\}, \quad H(e_2) = \{h_1\}$
are pre-open soft sets of (X, τ, E), but their soft intersection $(G, E)\tilde{\cap}(H, E) = (M, E)$ where $M(e_1) = \{h_2\}, \quad M(e_2) = \{h_1\}$ is not a pre-open soft set.

(2) Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\tilde{X}, \tilde{\phi}, (F_1, E), (F_2, E), (F_3, E)\}$ where $(F_1, E), (F_2, E), (F_3, E)$ are soft sets over X defined as follows:
$F_1(e_1) = \{h_1\}, \quad F_1(e_2) = \{h_1\}$.
$F_2(e_1) = \{h_2\}, \quad F_2(e_2) = \{h_2\}$.
$F_3(e_1) = \{h_1, h_2\}, \quad F_3(e_2) = \{h_1, h_2\}$.
Then τ defines a soft topology on X. Then the sets (G, E) and (H, E) which defined as follows
$G(e_1) = \{h_2, h_3\}, \quad G(e_2) = \{h_2, h_3\}$.
$H(e_1) = \{h_1\}, \quad H(e_2) = \{h_1, h_3\}$
are semi-open soft sets of (X, τ, E), but their soft intersection $(G, E)\tilde{\cap}(H, E) = (M, E)$ where $M(e_1) = \{\phi\}, \quad M(e_2) = \{h_3\}$ is not a semi-open soft set.

(3) Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\tilde{X}, \tilde{\phi}, (F, E)\}$ where (F, E) is a soft set over X defined as follows:
$F(e_1) = \{h_1, h_3\}, \quad F(e_2) = \{h_2, h_3\}$.
Then τ defines a soft topology on X, hence (X, τ, E) is a soft topological space over X. Then the sets (G, E) and (H, E) which defined as follows:
$G(e_1) = \{h_1, h_2\}, \quad G(e_2) = \{h_1\}$.
$H(e_1) = \{h_2, h_3\}, \quad H(e_2) = \{h_1\}$
are β-open soft sets of (X, τ, E), but their soft intersection $(G, E)\tilde{\cap}(H, E) = (M, E)$ where
$M(e_1) = \{h_2\}, \quad M(e_2) = \{h_1\}$
is not a β-open soft set.

Definition 3.6. Let (X, τ, E) be a soft topological space, $(F, E) \in SS(X)_E$ and $x_e \in SS(X)_E$. Then

- **(1):** x_e is called a γ- interior soft point of (F, E) if $\exists$ $(G, E) \in OS(\gamma)$ such that $x_e \in (G, E)\tilde{\subseteq}(F, E)$, the set of all γ-interior soft points of (F, E) is called the γ-soft interior of (F, E) and is denoted by $\gamma S(int(F, E))$ consequently, $\gamma S(int(F, E)) = \tilde{\bigcup}\{(G, E) : (G, E)\tilde{\subseteq}(F, E),\ (G, E) \in OS(\gamma)\}$.
- **(2):** x_e is called a γ-cluster soft point of (F, E) if $(F, E)\tilde{\cap}(H, E) \neq \tilde{\phi}\ \forall\ (H, E) \in OS(\gamma)$. The set of all γ-cluster soft points of (F, E) is called γ-soft closure of (F, E) and is denoted by $\gamma S(cl(F, E))$ consequently, $\gamma S(cl(F, E)) = \tilde{\bigcap}\{(H, E) : (H, E) \in CS(\gamma),\ (F, E)\tilde{\subseteq}(H, E)\}$.

Theorem 3.7. *Let (X, τ, E) be a soft topological space, $\gamma : SS(X)_E \to SS(X)_E$ be one of the operations in Example 3.5 and $(F, E), (G, E) \in SS(X)_E$. Then the following properties are satisfied for the γ S-interior operators, denoted by $\gamma S(int)$.*

- **(1):** $\gamma S(int(\tilde{X})) = \tilde{X}$ *and* $\gamma S(int(\tilde{\phi})) = \tilde{\phi}$.
- **(2):** $\gamma S(int(F, E))\tilde{\subseteq}(F, E)$.

(3): *$\gamma S(int(F,E))$ is the largest γ-open soft set contained in (F,E).*
(4): *if $(F,E)\tilde{\subseteq}(G,E)$, then $\gamma S(int(F,E))\tilde{\subseteq}\gamma S(int(G,E))$.*
(5): *$\gamma S(int(\gamma S(int(F,E)))) = \gamma S(int(F,E))$.*
(6): *$\gamma S(int(F,E))\tilde{\cup}\gamma S(int(G,E))\tilde{\subseteq}\gamma S(int(F,E)\tilde{\cup}(G,E))$.*
(7): *$\gamma S(int((F,E))\tilde{\cap}(G,E))\tilde{\subseteq}\gamma S(int(F,E))\tilde{\cap}\gamma S(int(G,E))$.*

Proof. Immediate. □

Theorem 3.8. *Let (X,τ,E) be a soft topological space, $\gamma : SS(X)_E \to SS(X)_E$ be one of the operations in Example 3.5 and $(F,E),(G,E) \in SS(X)_E$. Then the following properties are satisfied for the γ-soft closure operators, denoted by $\gamma S(cl)$.*

(1): *$\gamma S(cl(\tilde{X})) = \tilde{X}$ and $\gamma S(cl(\tilde{\phi})) = \tilde{\phi}$.*
(2): *$(F,E)\tilde{\subseteq}\gamma S(cl(F,E))$.*
(3): *$\gamma S(cl(F,E))$ is the smallest γ-closed soft set contains (F,E).*
(4): *if $(F,E)\tilde{\subseteq}(G,E)$, then $\gamma S(cl(F,E))\tilde{\subseteq}\gamma S(cl(G,E))$.*
(5): *$\gamma S(cl(\gamma S(cl(F,E)))) = \gamma S(cl(F,E))$.*
(6): *$\gamma S(cl(F,E))\tilde{\cup}\gamma S(cl(G,E))\tilde{\subseteq}\gamma S(cl((F,E)\tilde{\cup}(G,E)))$.*
(7): *$\gamma S(cl((F,E))\tilde{\cap}(G,E))\tilde{\subseteq}\gamma S(cl(F,E))\tilde{\cap}\gamma S(cl(G,E))$.*

Proof. Immediate. □

Remark 3.9. Note that the family of all γ-open soft sets on a soft topological space (X,τ,E) forms a supra soft topology, which is a collection of soft sets contains $\tilde{X},\tilde{\phi}$ and closed under arbitrary soft union.

4. Relations between subsets of soft topological spaces

In this section we introduce the relations between some special subsets of a soft topological space (X,τ,E).

Theorem 4.1. *In a soft topological space (X,τ,E), the following statements hold,*

(1): *every open (resp. closed) soft set is pre-open (resp. pre-closed) soft.*
(2): *every open (resp. closed) soft set is semi-open (resp. semi-closed) soft.*
(3): *every open (resp. closed) soft set is α-open (resp. α-closed) soft.*
(4): *every open (resp. closed) soft set is β-open (resp. β-closed) soft.*

Proof. We prove the assertion in the case of open soft set, the other case is similar.

(1) Let $(F,E) \in OS(X)$. Then $int(F,E) = (F,E)$. Since $(F,E)\tilde{\subseteq}cl(F,E)$, then $(F,E)\tilde{\subseteq}int(cl(F,E))$. Therefore $(F,E) \in POS(X)$.

(2) Let $(F,E) \in OS(X)$. Then $int(F,E) = (F,E)$. Since $(F,E)\tilde{\subseteq}cl(F,E)$, then $(F,E)\tilde{\subseteq}cl(int(F,E))$. Thus $(F,E) \in SOS(X)$.

(3) Let $(F,E) \in OS(X)$. Then $int(F,E) = (F,E)$. Since $(F,E)\tilde{\subseteq}cl(F,E)$, then $(F,E)\tilde{\subseteq}int(cl(F,E)) = int(cl(int(F,E)))$. Hence $(F,E) \in \alpha OS(X)$.

(4) Let $(F,E) \in OS(X)$. Then $int(F,E) = (F,E)$. Since $(F,E)\tilde{\subseteq}cl(F,E)$, then $(F,E)\tilde{\subseteq}int(cl(F,E))$. Hence $(F,E)\tilde{\subseteq}cl(F,E)\tilde{\subseteq}cl(int(cl(F,E)))$. Therefore $(F,E) \in \beta OS(X)$. □

Remark 4.2. The converse of Theorem 4.1 is not true in general as shown in the following examples.

Example 4.3. (1) Let $X = \{h_1, h_2, h_3, h_4\}$, $E = \{e\}$ and

$$\tau = \{\tilde{X}, \tilde{\phi}, (F_1, E), (F_2, E), (F_3, E)\}$$

where $(F_1, E), (F_2, E), (F_3, E)$ are soft sets over X defined as follows:
$F_1(e) = \{h_4\}, F_2(e) = \{h_1, h_2\},\ F_3(e) = \{h_1, h_2, h_4\}$
Then τ defines a soft topology on X, hence the set (G, E) which defined by $G(e) = \{h_1\}$ is a pre-open soft set of (X, τ, E), but it is not open soft.

(2) Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\tilde{X}, \tilde{\phi}, (F, E)\}$ where (F, E) is a soft set over X defined as follows:
$F(e_1) = \{h_1\}, \quad F(e_2) = \{h_2\}$.
Then τ defines a soft topology on X, hence the set (G, E) where
$G(e_1) = \{h_1, h_2\}, \quad G(e_2) = \{h_2\}$.
is a semi-open soft set of (X, τ, E), but it is not open soft.

(3) Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\tilde{X}, \tilde{\phi}, (F, E)\}$ where (F, E) is a soft set over X defined as follows:
$F(e_1) = \{h_1\}, \quad F(e_2) = \{h_2\}$.
Then τ defines a soft topology on X, hence the set (G, E) where
$G(e_1) = \{h_1, h_2\}, \quad G(e_2) = \{h_2\}$.
is an α-open soft set of (X, τ, E), but it is not open soft.

(4) Let $X = \{h_1, h_2, h_3, h_4\}$, $E = \{e\}$ and $\tau = \{\tilde{X}, \tilde{\phi}, (F_1, E), (F_2, E), (F_3, E)\}$ where $(F_1, E), (F_2, E), (F_3, E)$ are soft sets over X defined as follows:
$F_1(e) = \{h_4\}, F_2(e) = \{h_1, h_2\},\ F_3(e) = \{h_1, h_2, h_4\}$
Then τ defines a soft topology on X, hence the set (G, E) where $G(e) = \{h_1\}$ is a β-open soft set of (X, τ, E), but it is not open soft.

Theorem 4.4. *Let (X, τ, E) be a soft topological space,then the following statements are hold,*

- **(1):** *Every α-open (resp. α-closed) soft set is semi-open (resp. semi-closed) soft.*
- **(2):** *Every semi-open (resp. semi-closed) soft set is β-open (resp. β-closed) soft.*
- **(3):** *Every pre-open (resp. pre-closed) soft set is β-open (resp. β-closed) soft.*
- **(4):** *Every α-open (resp. α-closed) soft set is pre-open (resp. pre-closed) soft.*

Proof. We prove the assertion in the case of open soft set, the other case is similar.

(1) Let $(F, E) \in \alpha OS(X)$. Then

$$(F, E)\tilde{\subseteq} int(cl(int(F, E)))\tilde{\subseteq} cl(int(F, E)).$$

Hence $(F, E) \in SOS(X)$.

(2) Let $(F, E) \in SOS(X)$. Then $(F, E)\tilde{\subseteq} cl(int(F, E))$. Since $(F, E)\tilde{\subseteq} cl(F, E)$, then $(F, E)\tilde{\subseteq} cl(int(F, E))\tilde{\subseteq} cl(int(cl(F, E)))$. Thus $(F, E) \in \beta OS(X)$.

(3) Let $(F, E) \in POS(X)$. Then

$$(F, E)\tilde{\subseteq} int(cl(F, E))\tilde{\subseteq} cl(int(cl(F, E))).$$

Hence $(F, E) \in \beta OS(X)$.

(4) Let $(F,E) \in \alpha OS(X)$. Since $int(F,E)\tilde{\subseteq}cl(F,E)$. Then $cl(int(F,E))\tilde{\subseteq}cl(F,E)$. Hence $(F,E)\tilde{\subseteq}int(cl(int(F,E)))\tilde{\subseteq}int(cl(F,E))$. Thus $(F,E)\tilde{\subseteq}int(cl(F,E))$. It follows that $(F,E) \in POS(X)$. $\square$

The converse of Theorem 4.4 is not true in general as shown by the following examples.

Example 4.5. (1) Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and

$$\tau = \{\tilde{X}, \tilde{\phi}, (F_1,E), (F_2,E), (F_3,E)\}$$

where $(F_1,E), (F_2,E), (F_3,E)$ are soft sets over X defined as follows:
$F_1(e_1) = \{h_1\}$, $F_1(e_2) = \{h_1\}$.
$F_2(e_1) = \{h_2\}$, $F_2(e_2) = \{h_2\}$.
$F_3(e_1) = \{h_1, h_2\}$, $F_3(e_2) = \{h_1, h_2\}$.
Then τ defines a soft topology on X, hence the set (G,E) which defined as follows:
$G(e_1) = \{h_2, h_3\}$, $G(e_2) = \{h_2, h_3\}$.
is a semi-open soft set of (X, τ, E), but it is not an α-open soft.

(2) Let $X = \{h_1, h_2, h_3, h_4\}$, $E = \{e\}$ and

$$\tau = \{\tilde{X}, \tilde{\phi}, (F_1,E), (F_2,E), (F_3,E)\}$$

where $(F_1,E), (F_2,E), (F_3,E)$ are soft sets over X defined as follows:
$F_1(e) = \{h_4\}, F_2(e) = \{h_1, h_2\},\ F_3(e) = \{h_1, h_2, h_4\}$
Then τ defines a soft topology on X, hence the set (G,E) which defined by $G(e) = \{h_1\}$ is a β-open soft set of (X, τ, E), but it is not a semi-open soft.

(3) Let $X = \{h_1, h_2, h_3, h_4\}$, $E = \{e\}$ and

$$\tau = \{\tilde{X}, \tilde{\phi}, (F_1,E), (F_2,E), (F_3,E), (F_4,E)\}$$

where $(F_1,E), (F_2,E), (F_3,E), (F_4,E)$ are soft sets over X defined as follows:
$F_1(e) = \{h_1\}, F_2(e) = \{h_2\},\ F_3(e) = \{h_1, h_2\}, F_4(e) = \{h_1, h_2, h_4\}$
Then τ defines a soft topology on X, hence the set (G,E) which defined by $G(e) = \{h_1, h_4\}$ is a β-open soft set of (X, τ, E), but it is not a pre-open soft.

(4) Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\tilde{X}, \tilde{\phi}, (F_1,E), (F_2,E), (F_3,E)\}$ where $(F_1,E), (F_2,E), (F_3,E)$ are soft sets over X defined as follows:
$F_1(e_1) = \{h_1\}$, $F_1(e_2) = \{h_1\}$.
$F_2(e_1) = \{h_2\}$, $F_2(e_2) = \{h_2\}$.
$F_3(e_1) = \{h_1, h_2\}$, $F_3(e_2) = \{h_1, h_2\}$.
Then τ defines a soft topology on X, hence the set (G,E) which defined as follows:
$G(e_1) = \{h_1, h_2\}$, $G(e_2) = \{h_1\}$.
is a pre-open soft set of (X, τ, E), but it is not an α-open soft.

Remark 4.6. The following implications hold from Theorem 4.1 and Theorem 4.4 for a soft topological space (X, τ, E). These implications are not reversible.

$$\begin{tikzcd}
OS(X) \arrow[r] \arrow[d] & \alpha OS(X) \arrow[r] \arrow[dl] & SOS(X) \arrow[d] \\
POS(X) \arrow[rr] & & \beta OS(X)
\end{tikzcd}$$

Theorem 4.7. *Let (X, τ, E) be a soft topological space, $\gamma : SS(X)_E \to SS(X)_E$ be one of the operations in Definition 3.5 and $F_E \in SS(X)_E$. Then the following hold:*

(1): $\gamma S(int(F'_E)) = \tilde{X} - \gamma S(cl(F_E))$.
(2): $\gamma S(cl(F'_E)) = \tilde{X} - \gamma S(int(F_E))$.

Proof. We give the proof for the case of pre-open soft operator i.e $\gamma = int(cl)$, the other cases is similar.

(1) Let $x_e \notin Pcl(F_E)$. Then $\exists\ G_E \in PO(\tilde{X}, x_e)$ such that $G_E \tilde{\cap} F_E = \tilde{\phi}$, hence $x_e \in G_E \tilde{\subseteq} F'_E$. Thus $x_e \in Pint(F'_E)$. This means that $\tilde{X} - Pcl(F_E) \tilde{\subseteq} Pint(F'_E)$. Let $x_e \in Pint(F'_E)$. Since $Pint(F'_E) \tilde{\cap} F_E = \tilde{\phi}$, so $x_e \notin Pcl(F_E)$. It follows that $x_e \in \tilde{X} - Pcl(F_E)$. Therefore $Pint(F'_E) \tilde{\subseteq} \tilde{X} - Pcl(F_E)$.

(2) Let $x_e \in Pint(F_E)$. Then $\forall\ G_E \in PO(\tilde{X}, x_e)$, $x_e \in G_E \tilde{\subseteq} F_E$, hence $G_E \tilde{\cap} F'_E = \tilde{\phi}$. Thus $x_e \notin Pint(F'_E)$. This means that $\tilde{X} - Pint(F_E) \tilde{\subseteq} Pcl(F'_E)$. Let $x_e \notin Pcl(F'_E)$. Then $\exists\ G_E \in PO(\tilde{X}, x_e)$ such that $G_E \tilde{\cap} F'_E = \tilde{\phi}$, hence $x_e \in G_E \tilde{\subseteq} F_E$. It follows that $x_e \in Pint(F_E)$. This means that $Pcl(F'_E) \tilde{\subseteq} \tilde{X} - Pint(F_E)$. This completes the proof. □

Theorem 4.8. *Let (X, τ, E) be a soft topological space and $F_E \in SS(X)_E$. Then*

(1): *$F_E \in SOS(X)$ if and only if $cl(F_E) = cl(int(F_E))$.*
(2): *If $G_E \in OS(X)$, then $G_E \tilde{\cap} cl(F_E) \tilde{\subseteq} cl(G_E \tilde{\cap} F_E)$.*

Proof. Immediate. □

Theorem 4.9. *Let (X, τ, E) be a soft topological space, $F_E \in OS(X)$ and $G_E \in POS(X)$. Then $F_E \tilde{\cap} G_E \in POS(X)$.*

Proof. Let $F_E \in OS(X)$ and $G_E \in POS(X)$. Then $F_E \tilde{\cap} G_E \tilde{\subseteq}$ $int(F_E) \tilde{\cap} int(cl(G_E)) = int((F_E) \tilde{\cap} cl(G_E)) \tilde{\subseteq} int(cl((F_E) \tilde{\cap} (G_E)))$ from Theorem 4.8 (2). Hence $F_E \tilde{\cap} G_E \in POS(X)$. □

Theorem 4.10. *Let (X, τ, E) be a soft topological space and $F_E, G_E \in SS(X)_E$. If either $F_E \in SOS(X)$ or $G_E \in SOS(X)$. Then $int(cl(F_E \tilde{\cap} G_E)) = int(cl(F_E)) \tilde{\cap} int(cl(F_E))$.*

Proof. Let $F_E, G_E \in SS(X)_E$. Then we generally have

$$int(cl(F_E \tilde{\cap} G_E)) \tilde{\subseteq} int(cl(F_E)) \tilde{\cap} int(cl(F_E)).$$

Suppose that $F_E \in SOS(X)$. Then $cl(F_E) = cl(int(F_E))$ from Theorem 4.8 (1). Hence

$$\begin{aligned}
int(cl(F_E))\tilde{\cap}int(cl(G_E))&\tilde{\subseteq}int[cl(F_E)\tilde{\cap}int(cl(G_E))]\\
&= int[cl(int(F_E))\tilde{\cap}int(cl(G_E))]\\
&\tilde{\subseteq}int(cl[int(F_E)\tilde{\cap}int(cl(G_E))])\\
&\tilde{\subseteq}int(cl(int[int(F_E)\tilde{\cap}cl(G_E)]))\\
&\tilde{\subseteq}int(cl(int(cl[int(F_E)\tilde{\cap}(G_E)])))\\
&\tilde{\subseteq}int(cl(int(cl[(F_E)\tilde{\cap}(G_E)])))\\
&\tilde{\subseteq}int(cl[(F_E)\tilde{\cap}(G_E)])
\end{aligned}$$

from Theorem 4.8(2). This complete the proof. □

Theorem 4.11. *Let (X, τ, E) be a soft topological space, $F_E \in OS(X)$ and $G_E \in SOS(X)$. Then $F_E\tilde{\cap}G_E \in SOS(X)$.*

Proof. Let $F_E \in OS(X)$ and $G_E \in SOS(X)$. Then

$$F_E\tilde{\cap}G_E\tilde{\subseteq}int(F_E)\tilde{\cap}cl(int(G_E)) = cl(int(F_E)\tilde{\cap}int(G_E)) = cl(int((F_E)\tilde{\cap}(G_E)))$$

from Theorem 4.8(2). Hence $F_E\tilde{\cap}G_E \in SOS(X)$. □

Theorem 4.12. *Let (X, τ, E) be a soft topological space and F_E, $G_E \in SS(X)_E$. Then*

(1): *$F_E \in \alpha OS(X)$ if and only if $\exists\, H_E \in OS(X)$ such that*

$$H_E\tilde{\subseteq}F_E\tilde{\subseteq}int(cl(H_E)).$$

(2): *If $F_E \in \alpha OS(X)$ and $F_E\tilde{\subseteq}G_E\tilde{\subseteq}int(cl(F_E))$, then $G_E \in \alpha OS(X)$.*

Proof. (1) ⇒: Suppose that $int(F_E) = H_E \in OS(X)$. Then $H_E\tilde{\subseteq}F_E\tilde{\subseteq}int(cl(H_E))$.

⇐: Let $H_E\tilde{\subseteq}F_E\tilde{\subseteq}int(cl(H_E))$, $H_E \in OS(X)$. Then $int(H_E) = H_E\tilde{\subseteq}int(F_E)$. It follows that $F_E\tilde{\subseteq}int(cl(int(H_E)))\tilde{\subseteq}int(cl(int(F_E)))$. Thus $F_E \in \alpha OS(X)$.

(2) Let $F_E \in \alpha OS(X)$, then $F_E\tilde{\subseteq}int(cl(int(F_E)))$. Hence

$$F_E\tilde{\subseteq}G_E\tilde{\subseteq}int(cl(int(cl(int(F_E)))))\tilde{\subseteq}int(cl(int(F_E)))\tilde{\subseteq}int(cl(int(G_E))).$$

Thus $G_E \in \alpha OS(X)$. □

Theorem 4.13. *Let (X, τ, E) be a soft topological space and $F_E \in SS(X)_E$. Then*

(1): *$F_E \in \alpha OS(X)$ if and only if $F_E \in POS(X)\tilde{\cap}SOS(X)$.*

(2): *$F_E \in \alpha CS(X)$ if and only if $F_E \in PCS(X)\tilde{\cap}SCS(X)$.*

Proof. (i) ⇒: Let $F_E \in \alpha OS(X)$, Then $F_E\tilde{\subseteq}int(cl(int(F_E)))$. Hence $F_E\tilde{\subseteq}cl(int(F_E))$ and $F_E\tilde{\subseteq}int(cl(F_E))$. Thus $F_E \in POS(X)\tilde{\cap}SOS(X)$.

⇐: Let $F_E \in POS(X)\tilde{\cap}SOS(X)$. Then $F_E\tilde{\subseteq}cl(int(F_E))$ and $F_E\tilde{\subseteq}int(cl(F_E))$. Thus $F_E\tilde{\subseteq}int(cl(cl(int(F_E)))) = int(cl(int(F_E)))$. It follows that $F_E \in \alpha OS(X)$.

(2)By a similar way □

Theorem 4.14. *Let (X, τ, E) be a soft topological space, $F_E \in \alpha OS(X)$ and $G_E \in \beta OS(X)$. Then $F_E\tilde{\cap}G_E \in \beta OS(X)$.*

Proof. Let $F_E \in \alpha OS(X)$ and $G_E \in \beta OS(X)$. Then

$$\begin{aligned}F_E\tilde{\cap}G_E &\tilde{\subseteq} int(cl(int(F_E)))\tilde{\cap}cl(int(cl(G_E)))\\ &\tilde{\subseteq} cl[int(cl(int(F_E)))\tilde{\cap}int(cl(G_E))]\\ &= cl(int[cl(int(F_E))\tilde{\cap}int(cl(G_E))])\\ &\tilde{\subseteq} cl(int(cl[int(F_E)\tilde{\cap}int(cl(G_E))]))\\ &= cl(int(cl(int[int(F_E)\tilde{\cap}cl(G_E)])))\\ &\tilde{\subseteq} cl(int(cl(int(cl[int(F_E)\tilde{\cap}G_E]))))\\ &\tilde{\subseteq} cl(int(cl(int(cl[F_E\tilde{\cap}G_E]))))\\ &\tilde{\subseteq} cl(int(cl[F_E\tilde{\cap}G_E]))\end{aligned}$$

from Theorem 4.8(2). Hence $F_E\tilde{\cap}G_E \in \beta OS(X)$. □

Theorem 4.15. *Let (X, τ, E) be a soft topological space and $F_E \in SS(X)_E$. Then $F_E \in PCS(X)$ if and only if $cl(int(F_E))\tilde{\subseteq}F_E$.*

Proof. Let $F_E \in PCS(X)$, then F'_E is a pre-open soft set, this means that

$$F'_E\tilde{\subseteq}int(cl(\tilde{X} - F_E)) = \tilde{X} - (cl(int(F_E))).$$

Therefore $cl(int(F_E))\tilde{\subseteq}F_E$.

Conversely, let $cl(int(F_E))\tilde{\subseteq}F_E$. Then $\tilde{X} - F_E\tilde{\subseteq}int(cl(\tilde{X} - F_E))$, hence $\tilde{X} - F_E$ is pre-open soft set. Therefore, F_E is pre-closed soft set. □

Theorem 4.16. *Let (X, τ, E) be a soft topological space. If $F_E \in \alpha OS(X)$ and $F'_E \in POS(X)$. Then $F_E \in OS(X)$.*

Proof. Let $F_E \in \alpha OS(X)$ and $F'_E \in POS(X)$. Then $F_E \in PCS(X)$. Hence

$$cl(int(F_E)\tilde{\subseteq}F_E\tilde{\subseteq}int(cl(int(F_E)))\tilde{\subseteq}cl(int(F_E).$$

This means that $cl(int(F_E) = F_E$. Thus $F_E\tilde{\subseteq}int(cl(int(F_E))) = int(F_E)$. Therefore $F_E \in OS(X)$. □

Theorem 4.17. *Let (X, τ, E) be a soft topological space and $F_E \in SS(X)_E$. Then $F_E \in \alpha CS(X)$ if and only if $cl(int(cl(F_E)))\tilde{\subseteq}F_E$.*

Proof. Let $F_E \in \alpha SCS(X)$, then F'_E is α-open soft set, this means that

$$F'_E\tilde{\subseteq}int(cl(int(\tilde{X} - F_E))) = \tilde{X} - (cl(int(cl(F_E)))).$$

Therefore $cl(int(cl(F_E)))\tilde{\subseteq}F_E$.

Conversely, let $cl(int(cl(F_E)))\tilde{\subseteq}F_E$. Then $\tilde{X} - F_E\tilde{\subseteq}int(cl(int(\tilde{X} - F_E)))$, hence $\tilde{X} - F_E$ is α-open soft set. Therefore, F_E is an α-closed soft set. □

Theorem 4.18. *Let (X, τ, E) be a soft topological space and $F_E \in SS(X)_E$. Then $F_E \in SCS(X)$ if and only if $int(cl(F_E))\tilde{\subseteq}F_E$.*

Proof. Let $F_E \in SCS(X)$, then F'_E is semi-open soft set, this means that

$$F'_E\tilde{\subseteq}cl(int(\tilde{X} - F_E)) = \tilde{X} - (int(cl(F_E))).$$

Therefore $int(cl(F_E))\tilde{\subseteq}F_E$.

Conversely, let $int(cl(F_E))\tilde{\subseteq}F_E$. Then $\tilde{X}-F_E\tilde{\subseteq}cl(int(\tilde{X}-F_E))$, hence $\tilde{X}-F_E$ is semi-open soft set. Therefore, F_E is semi-closed soft set. □

Corollary 4.19. *Let (X,τ,E) be a soft topological space and $F_E\in SS(X)_E$. Then $F_E\in SCS(X)$ if and only if $F_E=F_E\tilde{\cup}int(cl(F_E))$.*

Proof. Immediate from Theorem 4.18. □

Theorem 4.20. *Let (X,τ,E) be a soft topological space, $F_E\in\alpha OS(X)$ and $G_E\in SOS(X)$. Then $F_E\tilde{\cap}G_E\in SOS(X)$.*

Proof. Let $F_E\in\alpha OS(X)$ and $G_E\in SOS(X)$. Then

$$\begin{aligned}F_E\tilde{\cap}G_E&\tilde{\subseteq}int(cl(int(F_E)))\tilde{\cap}cl(int(G_E))\\&\tilde{\subseteq}cl[int(cl(int(F_E)))\tilde{\cap}int(G_E)]\\&\tilde{\subseteq}cl(int[cl(int(F_E))\tilde{\cap}int(G_E)])\\&\tilde{\subseteq}cl(int(cl[int(F_E)\tilde{\cap}int(G_E)]))\\&=cl(int(cl(int(F_E\tilde{\cap}G_E))))\\&\tilde{\subseteq}cl(int(F_E\tilde{\cap}G_E))\end{aligned}$$

from Theorem 4.8 (2). Hence $F_E\tilde{\cap}G_E\in SOS(X)$. □

Theorem 4.21. *Let (X,τ,E) be a soft topological space and $F_E\in SS(X)_E$. Then $F_E\in\beta C(X)$ if and only if $int(cl(int(F_E)))\tilde{\subseteq}F_E$.*

Proof. Let $F_E\in\beta C(X)$, then F'_E is a β-open soft set, this means that

$$F'_E\tilde{\subseteq}cl(int(cl(\tilde{X}-F_E)))=\tilde{X}-int(cl(int(F_E))).$$

Therefore $int(cl(int(F_E)))\tilde{\subseteq}F_E$.

Conversely, let $int(cl(int(F_E)))\tilde{\subseteq}F_E$. Then $\tilde{X}-F_E\tilde{\subseteq}cl(int(cl(\tilde{X}-F_E)))$, hence $\tilde{X}-F_E$ is β-open soft set. Therefore F_E is a β-closed soft set. □

5. Decompositions of some forms of soft continuity

Definition 5.1. Let (X,τ,A) and (Y,τ^*,B) be soft topological spaces. Let $u:X\to Y$ and $p:A\to B$ be a mappings. Let $f_{pu}:SS(X)_A\to SS(Y)_B$ be a function.Then

- **(1):** The function f_{pu} is called a pre-soft continuous function (Pre-cts soft) if $f^{-1}_{pu}(G,B)\in POS(X)\ \forall\ (G,B)\in OS(Y)$.
- **(2):** The function f_{pu} is called an α-soft continuous function (α-cts soft) if $f^{-1}_{pu}(G,B)\in\alpha OS(X)\forall\ (G,B)\in OS(Y)$.
- **(3):** The function f_{pu} is called a semi-soft continuous function (semi-cts soft) if $f^{-1}_{pu}(G,B)\in SOS(X)\forall\ (G,B)\in OS(Y)$.
- **(4):** The function f_{pu} is called a β-soft continuous function (β-cts soft) if $f^{-1}_{pu}(G,B)\in\beta OS(X)\forall\ (G,B)\in OS(Y)$.

Theorem 5.2. *Let (X,τ,A) and (Y,τ^*,B) be soft topological spaces. Let $u : X \to Y$ and $p : A \to B$ be a mappings. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a function.Then for the classes, pre-continuous (resp. α-continuous, semi-continuous and β-continuous) soft functions the following are equivalent (we give an example for the the class of pre-soft continuous functions).*

(1): *f_{pu} is pre-soft continuous function.*
(2): $f_{pu}^{-1}(H,B) \in PCS(X)\ \forall\ (H,B) \in CS(Y)$.
(3): $f_{pu}(PScl(G,A) \subseteq cl_{\tau^*}(f_{pu}(G,A))\ \forall\ (G,A) \in SS(X)_A$.
(4): $PScl(f_{pu}^{-1}(H,B)) \subseteq f_{pu}^{-1}(cl_{\tau^*}(H,B))\ \forall\ (H,B) \in SS(Y)_B$.
(5): $f_{pu}^{-1}(int_{\tau^*}(H,B)) \subseteq PSint(f_{pu}^{-1}(H,B))\ \forall\ (H,B) \in SS(Y)_B$.

Proof. (1) $\Rightarrow$ (2) Let (H,B) be a closed soft set over Y. Then $(H,B)' \in OS(Y)$ and $f_{pu}^{-1}(H,B)' \in POS(X)$ from Definition 5.1. Since $f_{pu}^{-1}(H,B)' = (f_{pu}^{-1}(H,B))'$ from [[21], Theorem 3.14]. Thus $f_{pu}^{-1}(H,B) \in PCS(X)$.

(2) $\Rightarrow$ (3) Let $(G,A) \in SS(X)_A$. Since

$$(G,A)\tilde{\subseteq} f_{pu}^{-1}(f_{pu}(G,A))\tilde{\subseteq} f_{pu}^{-1}(cl_{\tau^*}(f_{pu}(G,A))) \in PCS(X)$$

from (2) and [21, Theorem 3.14] Then $(G,A)\tilde{\subseteq} PScl(G,A)\tilde{\subseteq} f_{pu}^{-1}(cl_{\tau^*}(f_{pu}(G,A)))$. Hence $f_{pu}(PScl(G,A))\tilde{\subseteq} f_{pu}(f_{pu}^{-1}(cl_{\tau^*}(f_{pu}(G,A))))\tilde{\subseteq} cl_{\tau^*}(f_{pu}(G,A)))$ from [21, Theorem 3.14]. Thus $f_{pu}(PScl(G,A))\tilde{\subseteq} cl_{\tau^*}(f_{pu}(G,A))$.

(3) $\Rightarrow$ (4) Let $(H,B) \in SS(Y)_B$ and $(G,A) = f_{pu}^{-1}(H,B)$. Then

$$f_{pu}(PScl f_{pu}^{-1}(H,B))\tilde{\subseteq} cl_{\tau^*}(f_{pu}(f_{pu}^{-1}(H,B)))$$

from (3). Hence

$$\begin{aligned} PScl(f_{pu}^{-1}(H,B))&\tilde{\subseteq} f_{pu}^{-1}(f_{pu}(PScl(f_{pu}^{-1}(H,B)))) \\ &\tilde{\subseteq} f_{pu}^{-1}(cl_{\tau^*}(f_{pu}(f_{pu}^{-1}(H,B)))) \\ &\tilde{\subseteq} f_{pu}^{-1}(cl_{\tau^*}(H,B)) \end{aligned}$$

from [21, Theorem 3.14]. Thus $PScl(f_{pu}^{-1}(H,B))\tilde{\subseteq} f_{pu}^{-1}(cl_{\tau^*}(H,B))$.

(4) $\Rightarrow$ (2) Let (H,B) be a closed soft set over Y. Then

$$PScl(f_{pu}^{-1}(H,B))\tilde{\subseteq} f_{pu}^{-1}(cl_{\tau^*}(H,B)) = f_{pu}^{-1}(H,B) = f_{pu}^{-1}(H,B)$$

for all $(H,B) \in SS(Y)_B$ from (4), but clearly $f_{pu}^{-1}(H,B)\tilde{\subseteq} PScl(f_{pu}^{-1}(H,B))$. This means that $f_{pu}^{-1}(H,B) = PScl(f_{pu}^{-1}(H,B)) \in PCS(X)$.

(1) $\Rightarrow$ (5) Let $(H,B) \in SS(Y)_B$. Then $f_{pu}^{-1}(int_{\tau^*}(H,B)) \in POS(X)$ from (1). Hence $f_{pu}^{-1}(int_{\tau^*}(H,B)) = PSint(f_{pu}^{-1} int_{\tau^*}(H,B))\tilde{\subseteq} PSint(f_{pu}^{-1}(H,B))$. Thus $f_{pu}^{-1}(int_{\tau^*}(H,B))\tilde{\subseteq} PSint(f_{pu}^{-1}(H,B))$.

(5) $\Rightarrow$ (1) Let (H,B) be an open soft set over Y. Then $int_{\tau^*}(H,B) = (H,B)$ and $f_{pu}^{-1}(int_{\tau^*}(H,B)) = f_{pu}^{-1}((H,B))\tilde{\subseteq} PSint(f_{pu}^{-1}(H,B))$ from (5). But we have $PSint(f_{pu}^{-1}(H,B))\tilde{\subseteq} f_{pu}^{-1}(H,B)$. This means that $PSint(f_{pu}^{-1}(H,B)) = f_{pu}^{-1}(H,B) \in POS(X)$. Thus f_{pu} is soft continuous function. □

Theorem 5.3. *Let (X,τ,A), (Y,τ^*,B) be soft topological spaces and $f_{pu} : SS(X)_A \to SS(Y)_B$ be a function. Then*

(1): *every soft continuous function is pre-soft continuous function.*
(2): *every soft continuous function is semi-soft continuous function.*
(3): *every soft continuous function is α-soft continuous function.*
(4): *every soft continuous function is β-soft continuous function.*

Proof. Immediate from Theorem 4.1. □

Theorem 5.4. *Let (X, τ, A), (Y, τ^*, B) be soft topological spaces and $f_{pu} : SS(X)_A \to SS(Y)_B$ be a function. Then*

(1): *Every α-soft continuous function is semi-soft continuous function.*
(2): *Every semi-soft continuous function is β-soft continuous function.*
(3): *Every pre-soft continuous function is β-soft continuous function.*
(4): *Every α-soft continuous function is pre-soft continuous function.*

Proof. Immediate from Theorem 4.4. □

Theorem 5.5. *Let (X, τ, A), (Y, τ^*, B) be soft topological spaces and $f_{pu} : SS(X)_A \to SS(Y)_B$ be a function. Then f_{pu} is an α-soft continuous function if and only if it is a pre-continuous and semi-soft continuous function .*

Proof. Immediate from Theorem 4.13. □

On accounting of Theorem 5.3 and Theorem 5.4 we have the following corollary.

Corollary 5.6. *For a soft topological space (X, τ, E) we have the following implications.*

$$\begin{tikzcd}
soft \quad cts \arrow[r] \arrow[d] & \alpha - soft\, cts \arrow[r] \arrow[dl] & semi - soft\, cts \arrow[d] \\
Pre - soft\, cts \arrow[rr] & & \beta - soft\, cts
\end{tikzcd}$$

6. Conclusion

Topology is an important and major area of mathematics and it can give many relationships between other scientific areas and mathematical models. Recently, many scientists have studied and improved the soft set theory, which is initiated by Molodtsov [14] and easily applied to many problems having uncertainties from social life. In this paper, we introduce the new classes namely, classes of pre-open (resp. α-open, semi-open and β-open) soft sets. The properties of each class has obtained. We unified the study of these classes by using the γ-operator. Also, the notion of pre-continuous (resp. α-continuous, semi-continuous and β-continuous) soft functions have given. In the next study, we extend the notion of soft sets to supra soft topological spaces and and other topological properties. Also, we will use some topological tools in soft set application, like rough sets.

Acknowledgements. The authors express their sincere thanks to the reviewers for their valuable suggestions. The authors are also thankful to the editors-in-chief and managing editors for their important comments which helped to improve the presentation of the paper.

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