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Fuzzy 2-normed spaces and its fuzzy I-topology

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Abstract. In this paper the Felbin's type fuzzy 2-norm on a vector space is introduced, when $L = \min$ and $R = \max$ and then one of its fuzzy I-topologies is constructed. After making our elementary observations on this fuzzy I-topology, continuity of the vector space operations is discussed and it is proved that the addition is continuous and the scalar product of an open set is open although the scalar multiplication is not continuous. Next fuzzy continuity of functions and fuzzy convergence of sequences on this spaces are studied.

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1. Introduction

In 1999, Das and Das [3] constructed a fuzzy topology on the fuzzy normed linear space of Felbin-type [10]. According to the standardized terminology in [12], more precisely, such fuzzy topology should be called I-topology now. Fang [6] pointed out that the Das's I-topological structure on the fuzzy normed linear space is incompatible with the linear structure, that is, the fuzzy normed linear space is not an I-topological vector space with respect to the Das's I-topology. In order to overcome this incompatibility, Fang [6] constructed another I-topology on the fuzzy normed linear space by changing the definition of open fuzzy subset given by Das [3], and proved that the fuzzy normed linear space is a Hausdorff locally convex I-topological vector space with respect to this I-topology under certain conditions. In [5], by using an approach different from [6], a new I-vector topology is constructed on the fuzzy normed linear space. Some other works on this area can be seen in [7], [5].

In [11], S. Gähler introduced a crisp 2-normed spaces. Using this concept in [13], some locally convex topologies were constructed on the underlying linear space.

For more and some recent works on this area one can see [1], [2], [4], [9], [16].

In this paper we introduce fuzzy 2-norm on a vector space and one of its related I-topologies is studied. Compatibility of this I-topology with the linear structure, fuzzy convergence of sequences and fuzzy continuity of function under this I-topology are also discussed.

2. Preliminaries

Let X and Y be any two sets, $f : X \to Y$ be a mapping and μ be a fuzzy subset of X. Then $f(\mu)$ is a fuzzy subset of Y defined by

$$f(\mu)(y) = \begin{cases} \sup_{x\in f^{-1}(y)} \mu(x), & f^{-1}(y) \neq \varnothing; \\ 0, & \text{otherwise}, \end{cases}$$

for all $y \in Y$, where $f^{-1}(y) = \{x : \ f(x) = y\}$.
If η is a fuzzy subset of Y, then the fuzzy subset $f^{-1}(\eta)$ of X is defined by $f^{-1}(\eta)(x) = \eta(f(x))$, for any $x \in X$.
A fuzzy point x_α, $0 < \alpha \le 1$, $x \in X$, is a fuzzy set defined by

$$x_\alpha(t) = \begin{cases} \alpha, & t = x; \\ 0, & \text{otherwise.} \end{cases}$$

The set of all fuzzy points on X is denoted by $Pt(I^X)$. A fuzzy point x_α is said to be contained in a fuzzy set μ if $\alpha \le \mu(x)$. For $x \in X$, we apply x for the fuzzy point x_1. Also for every $r \in (0, 1]$, let r^* be the fuzzy set on X, which takes constant value r on X.
A fuzzy subset μ of a vector space X is said to be convex if

$$\mu(kx + (1 - k)y) \ge \min(\mu(x), \mu(y)),$$

for all $x, y \in X$ and $k \in [0, 1]$. Equivalently, for each $\alpha \in (0, 1]$, the α-level set $[\mu]_\alpha = \{x \in X : \ \mu(x) \ge \alpha\}$ is convex.

A stratified fuzzy I-topology on a set X is a family τ of fuzzy subsets of X satisfying the following conditions:
(1) The fuzzy subsets r^*, $r \in (0, 1]$, and 0 are in τ.
(2) τ is closed under finite intersection and arbitrary union of fuzzy subsets. In this case the pair (X, τ) is called a stratified fuzzy topological space.

A fuzzy topological space (X, τ) is said to be fuzzy Hausdorff if for every $x, y \in X$, $x \neq y$, there exist $\eta, \beta \in \tau$ with $\eta(x) = \beta(y) = 1$ and $\eta \cap \beta = \overline{0}$, where $\overline{0}$ is the fuzzy set which takes 1 at 0 and zero otherwise.
A fuzzy subset μ in a fuzzy topological space (X, τ) is called a neighborhood of a point $x \in X$, if $\mu(x) > 0$ and there is a ρ in τ such that $\rho \subseteq \mu$ and $\mu(x) = \rho(x)$.

For fuzzy I-topological spaces (X, τ_1) and (Y, τ_2), a mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is called fuzzy continuous at some point $x \in X$, if $f^{-1}(\mu)$ is a neighborhood of x for each neighborhood μ of $f(x)$. f is called fuzzy continuous if f is fuzzy continuous at every point $x \in X$. This means that the inverse of every fuzzy open subset of Y is a fuzzy open set in X.

If μ_1 and μ_2 are two fuzzy subsets of a vector space X, then the sum $\mu_1 + \mu_2$ is defined by

$$(\mu_1 + \mu_2)(x) = \sup_{x=x_1+x_2} (\mu_1(x_1) \wedge \mu_2(x_2)).$$

For a fuzzy point x_α in X, it can be easily seen that $(x_\alpha + \mu_1)(y) = \alpha \wedge \mu_1(y - x)$ and in particular, $(x + \mu_1)(y) = \mu_1(y - x)$. Let μ be a fuzzy subset of a vector space X and t be a scalar. Then the fuzzy set $t\mu$ is defined as follows,

(a) for $t \neq 0$, $(t\mu)(x) = \mu(t^{-1}x)$, for all $x \in X$.

(b) for $t = 0$,

$$(t\mu)(x) = \begin{cases} 0, & x \neq 0; \\ \sup_y \mu(y), & x = 0. \end{cases}$$

According to Mizumoto and Tanaka [14], a fuzzy number is a mapping $x : \mathbb{R} \to [0, 1]$ over the set $\mathbb{R}$ of all reals, so x is a fuzzy set in $\mathbb{R}$.
If there exists a $t_0 \in \mathbb{R}$ such that $x(t_0) = 1$, then x is called normal. For $0 < \alpha \leq 1$, α-level set of an upper semi-continuous convex normal fuzzy set x of $\mathbb{R}$ (i.e. $[x]_\alpha := \{t : x(t) \geq \alpha\}$) is a closed interval $[a_\alpha, b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible. When $a_\alpha = -\infty$, for instance, then $[a_\alpha, b_\alpha]$ means the interval $(-\infty, b_\alpha]$. Similar is the case when $b_\alpha = +\infty$.

In this paper, we consider the concept of fuzzy real numbers (fuzzy intervals) in the sense of Xiao and Zhu [15], which is defined below:
A mapping $\eta : \mathbb{R} \to [0, 1]$, with the α-level sets $[\eta]_\alpha$, $\alpha \in (0, 1]$, is called a fuzzy real number (or fuzzy interval) if it satisfies two axioms:

(N_1) There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.

(N_2) For each $\alpha \in (0, 1]$; $[\eta]_\alpha = [\eta^1_\alpha, \eta^2_\alpha]$, where $-\infty < \eta^1_\alpha \leq \eta^2_\alpha < +\infty$.

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $\mathcal{F}$. For each $r \in \mathbb{R}$, let $\overline{r} \in \mathcal{F}$ be defined by $\overline{r}(t) = 1$, if $t = r$ and $\overline{r}(t) = 0$, if $t \neq r$, so $\overline{r}$ is a fuzzy interval and $\mathbb{R}$ can be embedded in $\mathcal{F}$.

Let $\eta \in \mathcal{F}$. η is called positive fuzzy real number if for all $t < 0$, $\eta(t) = 0$. The set of all positive fuzzy real numbers is denoted by $\mathcal{F}^+$.

A partial order $\preceq$ in $\mathcal{F}$ is defined as follows, $\eta \preceq \delta$ if and only if for all $\alpha \in (0, 1]$, $\eta^1_\alpha \leq \delta^1_\alpha$ and $\eta^2_\alpha \leq \delta^2_\alpha$ where, $[\eta]_\alpha = [\eta^1_\alpha, \eta^2_\alpha]$ and $[\delta]_\alpha = [\delta^1_\alpha, \delta^2_\alpha]$. The strict inequality in $\mathcal{F}$ is defined by $\eta \prec \delta$ if and only if for all $\alpha \in (0, 1]$, $\eta^1_\alpha < \delta^1_\alpha$ and $\eta^2_\alpha < \delta^2_\alpha$.

According to Mizamoto and Tanaka [14], the arithmetic operations $\oplus$, $\ominus$, $\odot$ on $\mathcal{F} \times \mathcal{F}$ are defined by

$(x \oplus y)(t) = \sup_{s \in R} \min\{x(s), y(t - s)\}$, $t \in \mathbb{R}$,

$(x \ominus y)(t) = \sup_{s \in R} \min\{x(s), y(s - t)\}$, $t \in \mathbb{R}$,

$(x \odot y)(t) = \sup_{0 \neq s \in R} \min\{x(s), y(\frac{t}{s})\}$, $t \in \mathbb{R}$.

We also consider an operation $\oslash$ on $\eta \in \mathcal{F}$ and $\delta(\succ \overline{0}) \in \mathcal{F}^+$ as follows

$(\eta \oslash \delta)(t) = \sup_{s \in R} \min\{\eta(st), \delta(s)\}, t \in \mathbb{R}$.

It is well known that(see [10]) for $\eta, \delta \in \mathcal{F}$, if $[\eta]_\alpha = [\eta^1_\alpha, \eta^2_\alpha]$, $[\delta]_\alpha = [\delta^1_\alpha, \delta^2_\alpha]$, $\alpha \in (0, 1]$, then $[\eta \oplus \delta]_\alpha = [\eta^1_\alpha + \delta^1_\alpha, \eta^2_\alpha + \delta^2_\alpha]$, $[\eta \ominus \delta]_\alpha = [\eta^1_\alpha - \delta^2_\alpha, \eta^2_\alpha - \delta^1_\alpha]$. Furthermore if η , $\delta \in \mathcal{F}^+$, then $[\eta \odot \delta]_\alpha = [\eta^1_\alpha.\delta^1_\alpha, \eta^2_\alpha.\delta^2_\alpha]$, and when $\delta \succ 0$, $[\overline{1} \oslash \delta]_\alpha = [\frac{1}{\delta^2_\alpha}, \frac{1}{\delta^1_\alpha}]$.

3. Felbin's type fuzzy 2-norm and I-topology

In this section, first we introduce a fuzzy 2-norm which is similar to Felbin-type fuzzy norm, when $L = \min$ and $R = \max$ and then we generate a fuzzy I-topology with this 2-norm.

Definition 3.1. Let X be a vector space over $\mathbb{R}$. Consider a mapping $\|.,.\|$: $X \times X \to \mathcal{F}^+$. For x, y in X and $\alpha \in (0,1]$ let

$$[\|x,y\|]_\alpha = [\|x,y\|_\alpha^1, \|x,y\|_\alpha^2].$$

Suppose that there exists $\alpha_0 \in (0,1]$, independent of linearly independent vectors $x, y \in X$, such that for all $\alpha \leq \alpha_0$,

(1) $\|x,y\|_\alpha^2 < \infty$,
(2) $\inf_\alpha \|x,y\|_\alpha^2 > 0$.

Now the mapping $\|.,.\|$ is called a fuzzy 2-norm, if for any $x, y, z \in X$,

(1) $\|x,y\| = \overline{0}$ if and only if x, y are linearly dependent,
(2) $\|x,y\| = \|y,x\|$,
(3) $\|\alpha x, y\| = |\alpha|\|x,y\|$,
(4) $\|x+y,z\| \preceq \|x,z\| \oplus \|y,z\|$.

In this case $(X, \|.,.\|)$ is called a fuzzy 2-normed space.

One can easily see that with a fuzzy 2-norm $\|.,.\|$, for all $\alpha \in (0,1]$, $\|.,.\|_\alpha^1$ and $\|.,.\|_\alpha^2$ are two crisp 2-norm on X.

Example 3.2. Suppose that $\|.,.\|$ is a crisp 2-norm on a real vector space X. Define $\||.,.\||$ on $X \times X$ into $\mathcal{F}^+$ as follows

$$\||x,y\||(t) = \begin{cases} \frac{\|x,y\|}{t}, & t \geq \|x,y\|, \\ 0, & \text{otherwise.} \end{cases}$$

So for every $\alpha \in (0,1]$, $[\||x,y\||]_\alpha = [\|x,y\|, \frac{\|x,y\|}{\alpha}]$. One can easily see that $\||.,.\||$ is a fuzzy 2-norm on X.

Now we are going to define an I-topology corresponding to a fuzzy 2-norm.

Definition 3.3. Suppose that $\|.,.\|$ is a 2-norm on a real vector space X. For $\alpha \in (0,1]$, $\epsilon > 0$ and $x, y \in X$, a fuzzy set $\mu_\alpha(x,y,\epsilon)$ in X which is defined by

$$\mu_\alpha(x,y,\epsilon)(z) = \begin{cases} \alpha, & \|x-z,y\|_\alpha^2 < \epsilon; \\ 0, & \text{otherwise,} \end{cases}$$

is called an α-open sphere in a fuzzy 2-normed space.
A fuzzy set $\mu \in I^X$ is said to be $\|.,.\|$-open if for every $x \in X$ with $\mu(x) > 0$, there exist $n \in \mathbb{N}$, $\epsilon_1, \epsilon_2, ..., \epsilon_n > 0$, $\alpha_1, \alpha_2, ...\alpha_n \in (0,1]$ and $y_1, y_2, ...y_n \in X$, such that $\cap_{i=1}^n \mu_{\alpha_i}(x, y_i, \epsilon_i) \subseteq \mu$.

Theorem 3.4. *Every α-open sphere is a $\|.,.\|$-open and convex fuzzy set.*

Proof. For fixed $\alpha \in (0,1]$, $\epsilon > 0$ and $x, y \in X$, we are going to show that $\mu_\alpha(x,y,\epsilon)$ is $\|.,.\|$-open.
Let $\mu_\alpha(x,y,\epsilon)(z) > 0$, which means that $\|x-z,y\|_\alpha^2 < \epsilon$. Put $\epsilon_1 = \epsilon - \|x-z,y\|_\alpha^2$. If for $k \in X$, $\|z-k,y\|_\alpha^2 < \epsilon_1$, then

$$\|x-k,y\|_\alpha^2 \leq \|z-k,y\|_\alpha^2 + \|x-z,y\|_\alpha^2 < \epsilon_1 + \|x-z,y\|_\alpha^2 = \epsilon$$

i.e. $\mu_\alpha(x,y,\epsilon)(k) = \alpha$. So $\mu_\alpha(z,y,\epsilon_1)(k) = \alpha = \mu_\alpha(x,y,\epsilon)(k)$.
Also if $\|z-k,y\|_\alpha^2 \geq \epsilon_1$, then

$$\mu_\alpha(z,y,\epsilon_1)(k) = 0 \leq \mu_\alpha(x,y,\epsilon)(k).$$

Therefore $\mu_\alpha(z,y,\epsilon_1) \subseteq \mu_\alpha(x,y,\epsilon)$.
To prove its convexity, it is enough to show that for any $\beta \in (0,1]$, $(\beta \leq \alpha)$, the β-level set of $\mu_\alpha(x,y,\epsilon)$ is convex. For the points z_1, z_2 of the β-level set $[\mu_\alpha(x,y,\epsilon)]_\beta = \{z \in X : \ \mu_\alpha(x,y,\epsilon)(z) \geq \beta > 0\} = \{z \in X : \ \|x-z,y\|^2_\alpha < \epsilon\}$, and $k \in [0,1]$, we have

$$\begin{aligned} \|x-kz_1-(1-k)z_2,y\|^2_\alpha &= \|kz_1+(1-k)z_2-kx-(1-k)x,y\|^2_\alpha \\ &\leq \|k(z_1-x),y\|^2_\alpha + \|(1-k)(z_2-x),y\|^2_\alpha \\ &= |\,k\,|\,\|(z_1-x),y\|^2_\alpha + |\,1-k\,|\,\|z_2-x,y\|^2_\alpha \\ &< k\epsilon+(1-k)\epsilon = \epsilon. \end{aligned}$$

Therefore,

$$kz_1+(1-k)z_2 \in [\mu_\alpha(x,y,\epsilon)]_\beta = \{z \in X : \ \|x-z,y\|^2_\alpha < \epsilon\},$$

which completes the proof. □

Theorem 3.5. *In a fuzzy 2-normed space $(X, \|.,.\|)$, the collection*

$$\tau_{\|.,.\|} = \{\mu : \mu \textit{ is } \|.,.\| \textit{ -open }\},$$

is a fuzzy I-topology on X.

Proof. For any $r \in (0,1]$, if $\alpha \leq r$, then trivially $\mu_\alpha(x,y,\epsilon) \subseteq r^*$, for every $\epsilon > 0$ and $y \in X$. So $r^* \in \tau_{\|.,.\|}$. Also for each $x \in X$, $0(x) = 0$ which is not positive, so 0 belongs to $\tau_{\|.,.\|}$.
Now let $\mu_1, \mu_2 \in \tau_{\|.,.\|}$. For $x \in X$, if $(\mu_1 \wedge \mu_2)(x) > 0$, then we have $\mu_1(x) > 0$ and $\mu_2(x) > 0$. Hence there exist $\epsilon_1, ...\epsilon_n > 0$, $\alpha_1, ..., \alpha_n \in (0,1]$ and $y_1, ...y_n \in X$, such that $\cap^n_{i=1}\mu_{\alpha_i}(x,y_i,\epsilon_i) \subset \mu_1$ and there is $\eta_1, ...\eta_m > 0$, $\beta_1, ...\beta_m \in (0,1]$ and $z_1, ...z_m \in X$ for which $\cap^m_{j=1}\mu_{\beta_j}(x,z_j,\eta_j) \subset \mu_2$. So

$$(\cap^m_{j=1}\mu_{\beta_j}(x,z_j,\eta_j)) \cap (\cap^n_{i=1}\mu_{\alpha_i}(x,y_i,\epsilon_i)) \subseteq \mu_1 \cap \mu_2.$$

Finally if $\{\mu_j\}_{j\in J} \in \tau_{\|.,.\|}$ and for $x \in X$, $\cup_j \mu_j(x) > 0$, then there exists $j_0 \in J$ such that $\mu_{j_0}(x) > 0$. So for some $\epsilon_1, ..., \epsilon_n > 0$, $\alpha_1, ..., \alpha_n \in (0,1]$, and $y_1, ..., y_n \in X$, we have $\cap^n_{i=1}\mu_{\alpha_i}(x,y_i,\epsilon_i) \subset \mu_{j_0}$. Therefore $\cap^n_{i=1}\mu_{\alpha_i}(x,y_i,\epsilon_i) \subset \cup_{j\in J}\mu_j$. □

Theorem 3.6. *The fuzzy I-topological space $(X, \tau_{\|.,.\|})$ is fuzzy Hausdorff.*

Proof. Let $a, b \in X$ and $a \neq b$. So for any $y \in X$ which is linearly independent of $a-b$, we have $\inf_\alpha \|a-b,y\|^1_\alpha > 0$. Let $\inf_\alpha \|a-b,y\|^1_\alpha = \delta$. Consider the fuzzy open spheres $\mu_1(a,y,\delta/2)$ and $\mu_1(b,y,\delta/2)$ in $(X, \tau_{\|.,.\|})$. Trivially $\mu_1(a,y,\delta/2)(a) = 1$ and $\mu_1(b,y,\delta/2)(b) = 1$. We claim that $\mu_1(a,y,\delta/2) \cap \mu_1(b,y,\delta/2) = \overline{0}$.
Suppose not, so there exists $x_0 \in X$ such that,

$$\mu_1(a,y,\delta/2)(x_0) > 0 \text{ and } \mu_1(b,y,\delta/2)(x_0) > 0.$$

This implies that $\|x_0-a,y\|^2_1 < \delta/2$ and $\|x_0-b,y\|^2_1 < \delta/2$. Hence,

$$\begin{aligned} \delta &\leq \|a-b,y\|^1_1 \leq \|a-b,y\|^2_1 \leq \|x_0-a,y\|^2_1 + \|x_0-b,y\|^2_1 \\ &< \delta/2+\delta/2 = \delta, \end{aligned}$$

which is a contradiction. Therefore $(X, \tau_{\|.,.\|})$ is fuzzy Hausdorff. □

In the sequel we study the relations of this fuzzy topology with the operations of the vector space X

Theorem 3.7. *If μ is open in $(X, \tau_{\|.,.\|})$, then for any $x_\alpha \in Pt(I^X)$, $x_\alpha + \mu$ is also open.*

Proof. Let $(x_\alpha + \mu)(y) > 0$. So $\mu(y-x) \wedge \alpha > 0$, and $\mu(y-x) > 0$. But μ is $\|.,.\|$-open, so there exist $\alpha_i \in (0,1], \epsilon_i > 0$, and $z_i \in X$, $i = 1, ..., n$, such that $\cap_{i=1}^n \mu_{\alpha_i}(y-x, z_i, \epsilon_i) \subseteq \mu$. For $i = 1, ..., n$, let $\eta_i = \min\{\alpha, \alpha_i\}$. We have

$$\cap_{i=1}^n \mu_{\eta_i}(y-x, z_i, \epsilon_i) \subseteq \cap_{i=1}^n \mu_{\alpha_i}(y-x, z_i, \epsilon_i) \subseteq \mu.$$

If for $k \in X$, $\|y-k, z_i\|^2_{\eta_i} < \epsilon_i$, then

$$\|y - x - (k-x), z_i\|^2_{\eta_i} = \|y-k, z_i\|^2_{\eta_i} < \epsilon_i.$$

So

$$\mu_{\eta_i}(y, z_i, \epsilon_i)(k) = \eta_i = \mu_{\eta_i}(y-x, z_i, \epsilon_i)(k-x),$$

which implies that

$$\cap_{i=1}^n \mu_{\eta_i}(y, z_i, \epsilon_i)(k) = \cap_{i=1}^n \eta_i \mu_{\eta_i}(y-x, z_i, \epsilon_i)(k-x) \leq \mu(k-x).$$

Also in this case, from $\|y-k, z_i\|^2_{\eta_i} < \epsilon_i$, $i = 1, 2, ..., n$, we have $\mu_{\eta_i}(y, z_i, \epsilon_i)(k) = \eta_i \leq \alpha$. Therefore

$$\cap_{i=1}^n \mu_{\eta_i}(y, z_i, \epsilon_i)(k) \leq \alpha \wedge \mu(k-x) = (\mu + x_\alpha)(k).$$

If $\|y-k, z_{i_0}\|^2_{\eta_{i_0}} \geq \epsilon_{i_0}$, for some i_0, we have

$$0 = \cap_{i=1}^n \mu_{\eta_i}(y, z_i, \epsilon_i)(k) \leq \alpha \wedge \mu(k-x) = (\mu + x_\alpha)(k).$$

Thus $\cap_{i=1}^n \mu_{\eta_i}(y, z_i, \epsilon_i) \subseteq \mu + x_\alpha$. □

Theorem 3.8. *If a fuzzy subset μ of X is open in $(X, \tau_{\|.,.\|})$, then $t\mu$ is also open for $t \neq 0$.*

Proof. Let $t\mu(x) > 0$. This implies that $\mu(x/t) > 0$. Therefore there exists $\alpha_i \in (0,1], \epsilon_i > 0$ and $y_i \in X$, $i = 1, ..., n$, such that

$$\cap_{i=1}^n \mu_{\alpha_i}(x/t, y_i, \epsilon_i) \subseteq \mu.$$

Now for $i = 1, ..., n$, put $\eta_i = \mid t \mid \epsilon_i$. If for every $i = 1, ..., n$, $\|x-z, y_i\|^2_{\alpha_i} < \eta_i = \mid t \mid \epsilon_i$, then $\|x/t - z/t, , y_i\|^2_{\alpha_i} < \epsilon_i$. Hence $\mu_{\alpha_i}(x/t, y_i, \epsilon_i)(z/t) = \alpha_i$. This implies that

$$\mu_{\alpha_i}(x, y_i, \eta_i)(z) = \mu_{\alpha_i}(x/t, y_i, \epsilon_i)(z/t),$$

and so

$$\cap_{i=1}^n \mu_{\alpha_i}(x, y_i, \eta_i)(z) = \cap_{i=1}^n \mu_{\alpha_i}(x/t, y_i, \epsilon_i)(z/t) \leq \mu(z/t) = t\mu(z).$$

Now if there exists i_0 such that $\|x-z, y_{i_0}\|^2_{\alpha_{i_0}} \geq \eta_{i_0}$, then we have

$$\overline{0} = \cap_{i=1}^n \mu_{\alpha_i}(x, y_i, \eta_i)(z) \leq \mu(z/t) = t\mu(z).$$

So $\cap_{i=1}^n \mu_{\beta_i}(x, t_i, \eta_i)(z) \subseteq t\mu$. □

In the sequel we are going to show that $(X, +, ., \tau_{\|.,.\|})$ is not a stratified I-topological vector space. For this, we need some preliminaries from [5].

Definition 3.9. Let $\mu \in I^X$ and x_α be a fuzzy point. We say x_α quasi-coincides with μ, denoted by $x_\alpha \tilde{\in} \mu$, if $\mu(x) > 1 - \alpha$.

Definition 3.10. A stratified I-topology τ on a real vector space X is said to be an I-vector topology, if the following two mapping

$$f: \ X \times X \to X, \ \ (x, y) \mapsto x + y,$$

$$g: \ \mathbb{R} \times X \to, \ \ (t, x) \mapsto tx,$$

are fuzzy continuous, where $\mathbb{R}$ is equipped with the I-topology induced by its usual topology, and $X \times X$ and $\mathbb{R} \times X$ are equipped with the corresponding product I-topologies.
A vector space X with an I-vector topology τ, is called an I-topological vector space.

Definition 3.11. Let (X, τ) be an I-topology space and $x_\lambda \tilde{\in} Pt(I^X)$.
(a) A fuzzy set μ on X is called Q-neighborhood of x_λ if there exists $G \in \tau$ such that $x_\lambda \tilde{\in} G \subset \mu$.
(b). A family U_{x_λ} of Q-neighborhoods of x_λ, is called Q-neighborhood base of x_λ if for every Q-neighborhood A of x_λ, there exists $\mu \in U_{x_\lambda}$ such that $\mu \subset A$.

Lemma 3.12. *(Fang [8]) Let τ be a stratified I-topology on a vector space X. Then*
i) The mapping f (addition) is continuous if and only if for every fuzzy point $(x+y)_\lambda$ in $X \times X$ and every Q-neighborhood w of $(x + y)_\lambda$, there exist Q-neighborhoods u and v of x_λ, y_λ, respectively, such that $u + v \subset w$;
ii) The mapping g (scalar multiplication) is continuous if and only if for every fuzzy point $(t, x)_\lambda$ in $\mathbb{R} \times X$ and every Q-neighborhood w of $(tx)_\lambda$, there exists a Q-neighborhood u of x_λ and a $\delta > 0$, such that $su \subset w$, for all $s \in \mathbb{R}$ with $| s - t | < \delta$.

Theorem 3.13. *Let $(X, \|., .\|)$ be a fuzzy 2-norm space. Then*
i) The addition mapping f is continuous.
ii) The scalar multiplication g is not continuous.

Proof. Suppose that w is a Q-neighborhood of $(x + y)_\lambda$, so by definition, there exists a $\mu \in \tau_{\|.,.\|}$, such that $(x + y)_\lambda \tilde{\in} \mu \subset w$. But μ is $\tau_{\|.,.\|}$-open, so there exist $\alpha_i \in (0, 1], \epsilon_i > 0$ and $z_i \in X$, $i = 1, ..., n$, such that $\cap_i \mu_{\alpha_i}(x + y, z_i, \epsilon_i) \subset \mu$. Let $\epsilon = \min\{\epsilon_1, ..., \epsilon_n\}, \alpha = \min\{\alpha_1, ..., \alpha_n\}$ and

$$U_{\epsilon,\alpha,z_i} = \{x \in X \ ; \|x, z_i\|_\alpha^2 < \epsilon\}.$$

By definition of fuzzy 2-norm, we have $U_{\epsilon/2,\alpha,z_i} + U_{\epsilon/2,\alpha,z_i} \subset U_{\epsilon,\alpha,z_i}$. Hence

$$(x + U_{\epsilon/2,\alpha,z_i}) + (y + U_{\epsilon/2,\alpha,z_i}) \subset [(x + y) + U_{\epsilon,\alpha,z_i}],$$

which implies that,

$$(x + U_{\epsilon/2,\alpha,z_i}) \cap \alpha^* + (y + U_{\epsilon/2,\alpha,z_i}) \cap \alpha^* \subset [(x + y) + U_{\epsilon,\alpha,z_i}] \cap \alpha^*.$$

It is easy to verify that $(x + U_{\epsilon,\alpha,z_i}) \cap \alpha^* = \mu_\alpha(x, z_i, \epsilon)$. From this we have,

$$\mu_\alpha(x, z_i, \epsilon/2) + \mu_\alpha(y, z_i, \epsilon/2) \subset \mu_\alpha(x + y, z_i, \epsilon).$$

Also for $i = 1, ..., n$,

$$\cap_i \mu_\alpha(x, z_i, \epsilon/2) + \cap_i \mu_\alpha(y, z_i, \epsilon/2) \subset \cap_i \mu_\alpha(x + y, z_i, \epsilon).$$

If $\alpha > 1-\lambda$, then one may see that $\cap_i \mu_{\alpha_i}(x, z_i, \epsilon/2)$ and $\cap_i \mu_\alpha(y, z_i, \epsilon/2)$ are Q-neighborhoods of x_λ and y_λ, respectively.
If $\alpha \leq 1-\lambda$, we define the fuzzy sets A and B on X as follows,

$$A(z) = \begin{cases} \beta, & z = x; \\ \cap_i \mu_\alpha(x, z_i, \epsilon/2), & \text{otherwise}, \end{cases}$$

and

$$B(z) = \begin{cases} \beta, & z = y; \\ \cap_i \mu_\alpha(y, z_i, \epsilon/2), & \text{otherwise}, \end{cases}$$

where $\beta = \mu(x+y) > 1-\lambda$. We are going to show that $A + B \subset w$.
If $z = x+y$, then $(A+B)(z) = \beta = \mu(z)$ and if $z \neq x+y$, then

$$\begin{aligned} (A+B)(z) &= [A(x) \wedge B(z-x)] \vee [A(z-y) \wedge B(y)] \\ &\vee \sup_{z=u+v, u\neq x, v\neq y} [A(u) \wedge B(v)] \\ &= \cap_i \mu_\alpha(y, z_i, \epsilon/2)(z-x) \vee \cap_i \mu_\alpha(x, z_i, \epsilon/2)(z-y) \\ &\vee \sup_{z=u+v, u\neq x, v\neq y} [A(u) \wedge B(v)], \end{aligned}$$

and so if for every $i = 1, ..., n$, $\|x+y-z, z_i\|_\alpha^2 < \epsilon$ then

$$(A+B)(z) \leq \cap_i \mu_\alpha(x+y, z_i, \epsilon_i)(z) \leq \mu(z).$$

Also if there exists i_0 such that $\|x+y-z, z_{i_0}\|_\alpha^2 \geq \epsilon$, then we have

$$\|u-x, z_{i_0}\|_\alpha^2 \geq \epsilon/2 \text{ or } \|v-y, z_{i_0}\|_\alpha^2 \geq \epsilon/2,$$

where $z = u+v$. Hence$(A+B)(z) = 0 \leq \mu(z)$. This implies that $A+B \subset \mu \subset w$.
It is easy to see that A and B are $\tau_{\|.,.\|}$-open Q-neighborhoods of x_λ and y_λ, respectively. Therefore by Lemma 3.12, the addition mapping f is continuous.

To prove ii), let $(t_0, x) \in \mathbb{R} \times X, x \neq 0$ and $\lambda \in (0,1]$. Let $\beta, \alpha \in (0,1]$ be such that $\alpha < 1-\lambda < \beta$. Now define a fuzzy set μ of X by

$$\mu(z) = \begin{cases} \beta, & z = t_0 y; \\ \alpha, & \text{otherwise}. \end{cases}$$

Suppose that y is in *supp* μ. Then for every $\epsilon > 0$ and $z \in X$, we have

$$\mu_\alpha(y, z, \epsilon) \subset \mu \text{ and } \mu(t_0 x) = \beta > 1-\lambda.$$

Hence μ is a $\tau_{\|.,.\|}$-open Q-neighborhood of $t_0 x_\lambda$.
With $z = (t_0+\delta)x$, for every Q-neighborhood u of x_λ and every $\delta > 0$, we have $[(t_0+\delta)u](z) = u(x) > 1-\lambda$; but $\mu(z) = \alpha < 1-\lambda$. This together with Lemma 3.12, imply that the scalar multiplication is not continuous. □

4. Sequences and continuity in fuzzy 2-normed space

Let (X, τ) be a fuzzy topological space. A sequence $\{x_n\}$ in X is said to be converges to a point x and is denoted by $\lim_{n\to\infty} x_n = x$ if for every open neighborhood μ of x (i. e. $\mu(x) > 0$), there exists $n_0 \in N$ such that $\mu(x_n) > 0$ for all $n \geq n_0$.
In this section we introduce a more concrete convergence for a sequence in a fuzzy 2-nomed space and their relations will be studied. Also the fuzzy continuity of a function on fuzzy 2-normed spaces will be discussed.

Definition 4.1. Let $(X, \|.,.\|)$ be a fuzzy 2-norm space. A sequence $\{x_n\}$ in X is said to be weakly α-convergent to x, if for any $y \in X$, $\epsilon > 0$, and $\alpha \in (0,1]$, there exists $n_0 \in N$ such that $\|x_n - x, y\|^2_\alpha < \epsilon$ for all $n > n_0$.

Our next result shows the relationship between weakly α-convergence and convergence of a sequence in $(X, \tau_{\|.,.\|})$.

Theorem 4.2. *In a fuzzy 2-norm space $(X, \|.,\|)$, a sequence $\{x_n\}$ in X is weakly α-convergent, if and only if $x_n \to x$ with respect to the fuzzy topology $\tau_{\|.,.\|}$.*

Proof. Let $\{x_n\}$ weakly α-converges to x, μ be a fuzzy open subset of $(X, \tau_{\|.,.\|})$ and $\mu(x) > 0$. Then there exist $\epsilon_i > 0, \alpha_i \in (0,1]$ and $y_i \in X$, $i = 1, ..., m$, such that $\cap_{i=1}^m \mu_{\alpha_i}(x, y_i, \epsilon_i) \subseteq \mu$. Now by the hypothesis, for any $\epsilon > 0, y \in X$ and $\alpha \in (0,1]$, there exists $n_0 \in N$ such that for $n > n_0$, $\|x_n - x, y\|^2_\alpha < \epsilon$. So for $\epsilon_i > 0, \alpha_i \in (0,1]$ and $y_i \in X$, there exists $n_i \in N$ such that $\|x_n - x, y_i\|^2_{\alpha_i} < \epsilon_i$, for $n \geq n_i$ and $i = 1, ..., m$, i.e. $\mu_{\alpha_i}(x, y_i, \epsilon_i)(x_n) = \alpha_i > 0$. Let $n_* = \max\{n_1, ..., n_m\}$. Then

$$\cap_{i=1}^m \mu_{\alpha_i}(x, y_i, \epsilon_i)(x_n) = \min\{\alpha_1 ..., \alpha_n\} > \alpha > 0$$

for all $n \geq n_*$. Thus $\mu(x_n) > 0$, which implies that $x_n \to x$ in $\tau_{\|.,.\|}$.
Conversely, suppose that $x_n \to x$ in $(X, \tau_{\|.,.\|})$. So for any $\mu \in \tau_{\|.,.\|}$, there exists $n_0 \in \mathbb{N}$, such that $\mu(x_n) > 0$, for all $n > n_0$. Let $\epsilon > 0$, $\alpha_0 \in (0,1]$ and $y_0 \in X$ be given. For the fuzzy open set $\mu_{\alpha_0}(x, y_0, \epsilon)$ the fact that $\mu_{\alpha_0}(x, y_0, \epsilon)(x) > 0$, implies that there exists $n_0 \in \mathbb{N}$, such that $\mu_{\alpha_0}(x, y_0, \epsilon)(x_n) > 0$, for all $n > n_0$. This implies that $\|x_n - x, y\|^2_\alpha < \epsilon$, for any $n > n_*$, and so $\{x_n\}$ converges to x, weakly for $\alpha \in (0,1]$. □

Definition 4.3. Let $(X, \|.,.\|)$ and $(Y, \|.,.\|^*)$ be two fuzzy 2-norm spaces. A function $f : X \to Y$ is said to be weakly sequentially continuous if for any weakly α-convergent sequence $\{x_n\}$ in X which weakly α-converges to x, we have $f(x_n)$ weakly α-converges to $f(x_0)$ in Y.

Theorem 4.4. *Let $(X, \tau_{\|.,.\|})$ and $(Y, \tau_{\|.,.\|^*})$ be the fuzzy topological spaces generated by the fuzzy norms $\|.,.\|$ and $\|.,.\|^*$, respectively. Then $f : (X, \tau_{\|.,.\|}) \to (Y, \tau_{\|.,.\|^*})$ is weakly sequentially continuous at the point $x \in X$ if and only if f is fuzzy continuous at x.*

Proof. First, let $f : (X, \tau_{\|.,.\|}) \to (Y, \tau_{\|.,.\|^*})$ be fuzzy continuous at $x \in X$ and $\{x_n\}$ be a sequence in X such that $x_n \to x$ with respect to $\tau_{\|.,.\|}$. To show that $f(x_n) \to f(x)$, let $\mu \in \tau_{\|.,.\|}$, $\mu(f(x)) > 0$ and $\rho = f^{-1}(\mu)$. By continuity of f, ρ is an open neighborhood of x, i.e. $\rho \in \tau_{\|.,.\|}$ and $\rho(x) > 0$. Now,

$$\begin{aligned} f(\rho)(y) &= \sup_{x \in f^{-1}(y)} \rho(x) \\ &= \sup_{x \in f^{-1}(y)} f^{-1}(\mu)(x) \\ &= \sup_{x \in f^{-1}(y)} \mu(f(x)) = \begin{cases} \mu(y), & y = f(x); \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore $f(\rho)(y) \leq \mu(y)$, for all $y \in Y$, i.e. $f(\rho) \subseteq \mu$. Also $\rho(x) > 0$ and $x_n \to x$ implies there exists $n_0 \in N$ such that $\rho(x_n) > 0$, for all $n > n_0$. Also we have

$$f(\rho)(y) = \sup_{x \in f^{-1}(y)} \rho(x).$$

So $f(\rho)(f(x_n)) = \sup_{u \in f^{-1}f(f(x_n))} \rho(u)$.
This implies that $f(\rho)(f(x_n)) > 0$, i.e. $\mu(f(x_n)) > 0$. Thus $f(x_n) \in \mu$, for all $n > n_0$. Therefore $f(x_n) \to f(x)$ in $(Y, \tau_{\|.,.\|^*})$ which means that f is weakly sequentially continuous.
Conversely, let f be sequentially continuous at $x_0 \in X$, and in contrary suppose that f is not fuzzy continuous at x_0. So there exists an open neighborhood μ of $f(x_0)$ in $\tau_{\|.,.\|^*}$ such that $f^{-1}(\mu)$ is not an open neighborhood of x_0 in $\tau_{\|.,.\|}$, i.e. $f^{-1}(\mu)(x_0) > 0$ and for every $y \in X, \epsilon > 0$ and $\alpha \in (0, 1]$, $\mu_\alpha(x_0, y, \epsilon)$ is not subset of $f^{-1}(\mu)$. So for every $\alpha \in (0, 1]$ and $n \in N$, there exists x_n such that $\|x_n - x, y\|_\alpha^2 < 1/n$, but x_n is not in $f^{-1}(\mu)$. Thus $\mu(f(x_0)) > 0$ but $\mu(f(x_n))$ is not positive, i.e. $x_n \to x$ in $\tau_{\|.,.\|}$, but $f(x_n)$ is not convergent to $f(x_0)$ in $\tau_{\|.,.\|^*}$, which is a contradiction. □

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