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Interval-valued intuitionistic (T, S)-fuzzy implicative filters on lattice implication algebras

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Abstract. The notion of interval valued intuitionistic (T, S)-fuzzy filter (IVI (T, S)-fuzzy filter for short) and interval valued intuitionistic (T, S)-fuzzy implicative filter on lattice implication algebras are introduced by linking the interval valued intuitionistic fuzzy set(IVIFS for short), t-norm, s-norm and filter theory of lattice implication algebras; The properties and equivalent characterizations of interval valued intuitionistic (T, S)-fuzzy filter and and interval valued intuitionistic (T, S)-fuzzy implicative filter are investigated, respectively; The relation between IVI (T, S)-fuzzy filter and filter is further studied.

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1. Introduction

As is known to all, certain information processing is based on the classic logic. However, there are many uncertainties in real world. And non-classical logics[3, 12] consist of these logics handling a wide variety of uncertainties (such as fuzziness, randomness, and so on) and fuzzy reasoning. Therefore, non-classical logic has been proved to be a formal and useful technique for computer science to deal with fuzzy and uncertain information. Many-valued logic, as the extension and development of classical logic, has always been a crucial direction in non-classical logic. Lattice-valued logic, an important many-valued logic, has two prominent roles: One is to extend the chain-type truth -valued field of the current logics to some relatively general lattices. The other is that the incompletely comparable property of truth value characterized by the general lattice can more effectively reflect the uncertainty of human beings thinking, judging and decision. Hence, lattice-valued logic has been becoming a research field and strongly influencing the development of algebraic logic, computer science and artificial intelligent technology. In order to provide a reliable logical foundation for uncertain information processing theory, especially for the fuzziness, the incomparability in uncertain information in the reasoning, and, establish a logical system with truth value in a relatively general lattice, Xu proposed the concept of lattice implication algebras(LIAs for short)[18].

Zadeh[24] introduced the notion of fuzzy set in 1965. Since then this idea has been applied to many algebraic structures such as groups, rings, modules, semigroups, topologies, vector spaces, and filter theory of some logical algebraic structure[6, 7, 8, 9, 10, 11, 14, 17, 18, 19, 20, 21, 25, 26]. With the development of fuzzy set, it is widely used in many fields. Atanassov[1] first introduced the concept of intuitionistic fuzzy sets in 1986, this kind fuzzy set is a generalization of the fuzzy sets. Many authors applied the concept of intuitionistic fuzzy sets to other algebraic structure such as groups, fuzzy ideals of BCK-algebras, filter theory of lattice implication and BL-algebras,etc[6, 13, 16, 22, 23]. A generalization of the notion of intuitionistic fuzzy set is given in the spirit of ordinary interval valued fuzzy sets. The new nation is called interval valued intuitionistic fuzzy set (IVIFS) is introduced by Atanassov[2] in 1989.

In this paper, as an extension of filters theory in lattice implication algebras, we will apply the interval-valued intuitionistic fuzzy subset and t-norm T, s-norm S on $D[0,1]$ to filter theory of LIAs, proposed the concept interval-valued intuitionistic (T,S)-fuzzy implicative filters of LIAs and some equivalent results are obtained. Meanwhile, the relations among them are investigated. We desperately hope that our work would serve as a foundation for enriching corresponding many-valued logical system.

2. Preliminaries

Definition 2.1 ([18])**.** Let $(L,\vee,\wedge,O,I)$ be a bounded lattice with an order-reversing involution $'$, the greatest element I and the smallest element O, and

$$\rightarrow: L\times L\longrightarrow L$$

be a mapping. $\mathcal{L}=(L,\vee,\wedge,',\rightarrow,O,I)$ is called a lattice implication algebra if the following conditions hold for any $x,y,z\in L$:

(I_1) $x\rightarrow(y\rightarrow z)=y\rightarrow(x\rightarrow z)$;
(I_2) $x\rightarrow x=I$;
(I_3) $x\rightarrow y=y^{'}\rightarrow x^{'}$;
(I_4) $x\rightarrow y=y\rightarrow x=I$ implies $x=y$;
(I_5) $(x\rightarrow y)\rightarrow y=(y\rightarrow x)\rightarrow x$;
(l_1) $(x\vee y)\rightarrow z=(x\rightarrow z)\wedge(y\rightarrow z)$;
(l_2) $(x\wedge y)\rightarrow z=(x\rightarrow z)\vee(y\rightarrow z)$.

Theorem 2.2 ([21])**.** *In a LIA $\mathcal{L}$. The following hold, for any $x,y,z\in L$*

(1) $I\rightarrow x=x$ and $x\rightarrow O=x'$;
(2)$(x\rightarrow y)\rightarrow((y\rightarrow z)\rightarrow(x\rightarrow z))=I$;
(3) $x\vee y=(x\rightarrow y)\rightarrow y$;
(4) $x\wedge y=(x^{'}\vee y^{'})^{'}$;

(5) $(x \to y) \vee (y \to x) = I$;
(6) $x \to (y \vee z) = (y \to z) \to (x \to z)$.

In a lattice implication algebra $\mathcal{L}$, we define binary operation $\otimes$ as follows: for any $x, y \in L$,

$$x \otimes y = (x \to y')'.$$

Theorem 2.3 ([21]). *Let $\mathcal{L}$ be a lattice implication algebra, and $a, b \in L$, then*

$$a \otimes b = \wedge\{x | a \leq b \to x\}.$$

Definition 2.4 ([21]). Let $\mathcal{L} = (L, \vee, \wedge, ', \to, O, I)$ be a LIA and $F \subseteq L$. There are following conditions:

(F1) $I \in F$;
(F2) for any $x, y \in L$, if $x \in F$ and $x \to y \in F$, then $y \in F$;
(F3) for any $x, y, z \in L$, if $x \to (y \to z) \in F$ and $x \to y \in F$, then $x \to z \in F$;
(F4) for any $x, y, z \in L$, if $x \to ((y \to z) \to y) \in F$ and $x \in F$, then $y \in F$.
Then
(1)F is called a filter of $\mathcal{L}$, if it satisfies (F1) and (F2);
(2)F is called an implication filter of $\mathcal{L}$, if it satisfies (F1)and (F3);
(3)F is called a positive implicative filter of $\mathcal{L}$, if it satisfies (F1) and (F4).

By an interval $\tilde{a}$ we mean an interval $[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. The set of all intervals is denoted by $D[0, 1]$. The interval $[a, a]$ is identified with the number a.

For interval $\tilde{a_i} = [a_i^-, a_i^+], \tilde{b_i} = [b_i^-, b_i^+]$, where $i \in I$, I is an index set, we define
rmax$\{\tilde{a_i}, \tilde{b_i}\} = [max\{a_i^-, b_i^-\}, max\{a_i^+, b_i^+\}]$,
rmin$\{\tilde{a_i}, \tilde{b_i}\} = [min\{a_i^-, b_i^-\}, min\{a_i^+, b_i^+\}]$,
$\wedge_{i \in I} \tilde{a_i} = [\wedge_{i \in I} a_i^-, \wedge_{i \in I} a_i^+]$,
$\vee_{i \in I} \tilde{a_i} = [\vee_{i \in I} a_i^-, \vee_{i \in I} a_i^+]$;
Furthermore, we have
(i) $\tilde{a_i} \leq \tilde{b_i}$ iff $a_i^- \leq b_i^-$ and $a_i^+ \leq b_i^+$,
(ii) $\tilde{a_i} = \tilde{b_i}$ iff $a_i^- = b_i^-$ and $a_i^+ = b_i^+$,
(iii) $k\tilde{a} = [ka_i^-, ka_i^+]$, where $0 \leq k \leq 1$.

Then, it can be shown that $(D[0, 1], \leq, \wedge, \vee)$ is a complete lattice, $\tilde{0} = [0, 0]$ as its least element and $\tilde{1} = [1, 1]$ as its greatest element.

Let X be an nonempty set, by an interval valued fuzzy set on X we mean the set

$$F = \{(x, [A_F^-(x), A_F^+(x)]) | x \in X\},$$

where A_F^- and A_F^+ are two fuzzy sets of X such that $A_F^-(x) \leq A_F^+(x)$ for all $x \in X$.

Putting $A_F(x) = [A_F^-(x), A_F^+(x)])$, we see that $F = \{(x, A_F(x)) | x \in X\}$, where $A_F : X \to D[0, 1]$.

For any $\tilde{t} \in D[0, 1]$, the set $U(F; \tilde{t}) = \{x \in X | \tilde{A_F}(x) \geq \tilde{t}\}$ is called the **interval-valued level subset** of A.

Definition 2.5 ([4, 5]). Let δ be a mapping from $D[0, 1] \times D[0, 1]$ to $D[0, 1]$. δ is called a interval t-norm (resp. interval s-norm) on $D[0, 1]$, if it satisfies: for any $\tilde{x}, \tilde{y}, \tilde{z} \in D[0, 1]$,

(1) $\delta(\tilde{x}, \tilde{1}) = \tilde{x}$(resp. $\delta(\tilde{x}, \tilde{0}) = \tilde{x}$),

(2) $\delta(\tilde{x},\tilde{y}) = \delta(\tilde{y},\tilde{x})$,
(3) $\delta(\delta(\tilde{x},\tilde{y}),\tilde{z}) = \delta(\tilde{x},\delta(\tilde{y},\tilde{z}))$,
(4) if $\tilde{x} \le \tilde{y}$, then $\delta(\tilde{x},\tilde{z}) \le \delta(\tilde{y},\tilde{z})$.

Theorem 2.6 ([15]). *If S' (T') is a t-conorm (t-norm), then $S : D[0,1]\times D[0,1] \to D[0,1]$($T : D[0,1]\times D[0,1] \to D[0,1]$) is an interval t-conorm (t-norm). Characterizations of S and T are given, respectively, by*

$$\begin{aligned} S(\tilde{x},\tilde{y}) &= [S'(x^-,y^-),T'(x^+,y^+)]; \\ T(\tilde{x},\tilde{y}) &= [T'(x^-,y^-),T'(x^+,y^+)]. \end{aligned}$$

The set of all δ-idempotent elements $D_\delta = \{\tilde{x} \in D[0,1] | \delta(\tilde{x},\tilde{x}) = \tilde{x}\}$.

In [4, 5], give some the most familiar examples of t-norm T on $D[0,1]$ as follows: for any $\tilde{x},\tilde{y} \in D[0,1]$,

(1) $$T_W(\tilde{x},\tilde{y}) = [max\{x^- + y^- - 1, 0\}, max\{x^- + y^+ - 1, x^+ + y^- - 1, 0\}];$$

(2) $$T_P(\tilde{x},\tilde{y}) = [x^-y^-, max\{x^-y^+, x^+y^-\}];$$

(3) $$T_D(\tilde{x},\tilde{y}) = \begin{cases} \tilde{x}\wedge\tilde{y} & \tilde{x}\vee\tilde{y} = \tilde{1}, \\ \tilde{0} & \text{otherwise }; \end{cases}$$

(4) $$T_M(\tilde{x},\tilde{y}) = \text{rmin}\{\tilde{x},\tilde{y}\}.$$

From the above example and Theorem 2.6, it is easy to obtain some t-conorms S.

An IVIFS[1] on X is defined as an object of the form

$$A = \{(x, \tilde{M}_A(x), x, \tilde{N}_A(x)) | x \in X\},$$

where $\tilde{M}_A, \tilde{N}_A$ are interval valued fuzzy sets on X such that $[0,0] \le \tilde{M}_A(x) + \tilde{N}_A(x) \le [1,1]$. For the sake of simplicity, in the following, such interval valued intuitionistic fuzzy sets will be denoted by $A = (\tilde{M}_A, \tilde{N}_A)$.

3. IVI (T,S)-fuzzy filters

Definition 3.1. Let A be an IVIFS on $\mathcal{L}$. A is called an interval valued intuitionistic (T,S)-fuzzy filter (IVI (T,S)-fuzzy filter for short) of $\mathcal{L}$, if, for any $x,y,z \in L$:
(V1) $\tilde{M}_A(I) \ge \tilde{M}_A(x)$ and $\tilde{N}_A(I) \le \tilde{N}_A(x)$;
(V2) $\tilde{M}_A(y) \ge T(\tilde{M}_A(x \to y), \tilde{M}_A(x))$ and $\tilde{N}_A(y) \le S(\tilde{N}_A(x \to y), \tilde{N}_A(x))$.

Lemma 3.2. *Suppose A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$ w.r.t. idempotent interval t-norm T and s-norm S. Then, for any $x,y \in L$:*
(V3) if $x \le y$, then $\tilde{M}_A(x) \le \tilde{M}_A(y)$ and $\tilde{N}_A(y) \le \tilde{N}_A(x)$.

Proof. Since $x \le y$, it follows that $x \to y = I$. By A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$, we have $\tilde{M}_A(y) \ge T(\tilde{M}_A(x \to y), \tilde{M}_A(x))$ and $\tilde{N}_A(y) \le S(\tilde{N}_A(x \to y), \tilde{N}_A(x))$. By(V1), $\tilde{M}_A(I) \ge \tilde{M}_A(x), \tilde{N}_A(I) \le \tilde{N}_A(x)$ for any $x \in L$, therefore,

$$\begin{aligned} \tilde{M}(y) &\ge T(\tilde{M}_A(x \to y), \tilde{M}_A(x)) \\ &= T(\tilde{M}_A(I), \tilde{M}_A(x)) \\ &\ge T(\tilde{M}_A(x), \tilde{M}_A(x)) \\ &= \tilde{M}_A(x), \end{aligned}$$

$$\begin{aligned}\tilde{N}_A(y) &\leq S(\tilde{N}_A(x \to y), \tilde{N}_A(x))\\ &\leq S(\tilde{N}_A(I), \tilde{N}_A(x))\\ &\leq S(\tilde{N}_A(x), \tilde{N}_A(x)) = \tilde{N}_A(x).\end{aligned}$$

as T, S are idempotent interval t-norm and s-norm. And so V(3) is valid. □

Theorem 3.3. *Let A be an IVIFS on $\mathcal{L}$. Then A be an IVI (T, S)-fuzzy filter of $\mathcal{L}$ if and only if for any $x, y, z \in L$, (V1) holds and*

(V4) $\tilde{M}_A(x \to z) \geq T(\tilde{M}_A(y \to (x \to z)), \tilde{M}_A(y))$ and $\tilde{N}_A(x \to z) \leq S(\tilde{N}_A(y \to (x \to z)), \tilde{N}_A(y))$.

Proof. Suppose A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, obviously, (V1)and (V4) hold.

Conversely, suppose (V4) hold, taking $x = I$ in (V4), we have $\tilde{M}_A(z) = \tilde{M}_A(I \to z) \geq T(\tilde{M}_A(y \to (I \to z)), \tilde{M}_A(y)) = T(\tilde{M}_A(y \to z), \tilde{M}_A(y))$ and $\tilde{N}_A(z) = \tilde{N}_A(I \to z) \leq S(\tilde{N}_A(y \to (I \to z)), \tilde{N}_A(y)) = S(\tilde{N}_A(y \to z), \tilde{N}_A(y))$. Since (V1)hold, and so A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$. □

Theorem 3.4. *Let A be an IVIFS on $\mathcal{L}$ and satisfy $(V3)$. Then A be an IVI (T, S)-fuzzy filter of $\mathcal{L}$, iff, for any $x, y, z \in L$, A satisfies*

(V5) $\tilde{M}_A(x \otimes y) \geq T(\tilde{M}_A(x), \tilde{M}_A(y))$ and $\tilde{N}_A(x \otimes y) \leq S(\tilde{N}_A(x), \tilde{N}_A(y))$.

Proof. Suppose A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$. Since $x \leq y \to (x \otimes y)$, we have $\tilde{M}_A(y \to (x \otimes y)) \geq \tilde{M}_A(x)$ and $\tilde{N}_A(y \to (x \otimes y)) \leq \tilde{N}_A(x)$. By (V2), it follows that $\tilde{M}_A(x \otimes y) \geq T(\tilde{M}_A(y), \tilde{M}_A(y \to (x \otimes y))) \geq T(\tilde{M}_A(y), \tilde{M}_A(x))$ and

$$\begin{aligned}\tilde{N}_A(x \otimes y) &\leq S(\tilde{N}_A(y), \tilde{N}_A(y \to (x \otimes y)))\\ &\leq S(\tilde{N}_A(y), \tilde{N}_A(x)).\end{aligned}$$

Conversely, taking $y = I$ in (V3), then (V1)holds. As $x \otimes (x \to y) \leq y$, thus $\tilde{M}_A(y) \geq \tilde{M}_A(x \otimes (x \to y))$ and $\tilde{N}_A(y) \leq \tilde{N}_A(x \otimes (x \to y))$. By (V5), we have $\tilde{M}_A(y) \geq T(\tilde{M}_A(x), \tilde{M}_A(x \to y))$ and $\tilde{N}_A(y) \leq S(\tilde{N}_A(x), \tilde{N}_A(x \to y))$. Therefore (V2) is valid, so A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$. □

Corollary 3.5. *Let A be an IVIFS on $\mathcal{L}$ and satisfy $(V3)$, then A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, iff, for any $x, y, z \in L$:*

(V6) if $x \to (y \to z) = I$, then

$$\tilde{M}_A(z) \geq T(\tilde{M}_A(x), \tilde{M}_A(y)) \text{ and } \tilde{N}_A(z) \leq S(\tilde{N}_A(x), \tilde{N}_A(y)).$$

Corollary 3.6. *Let A be an IVIFS on $\mathcal{L}$ and satisfy $(V3)$, then A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, iff, for any $x, y, z \in L$:*

(V7) If $a_n \to (a_{n-1} \to \cdots \to (a_1 \to x) \cdots) = I$, then

$$\tilde{M}_A(x) \geq T(\tilde{M}_A(a_n), \cdots, \tilde{M}_A(a_1)) \text{ and } \tilde{N}_A(x) \leq S(\tilde{N}_A(a_n), \cdots, \tilde{N}_A(a_1)).$$

Theorem 3.7. *Let A be an IVIFS on $\mathcal{L}$ and satisfies $(V3)$. Then A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, iff, for any $x, y, z \in L$, A satisfies*

(V8) $\tilde{M}_A((x \to (y \to z)) \to z) \geq T(\tilde{M}_A(x), \tilde{M}_A(y))$ and $\tilde{N}_A((x \to (y \to z)) \to z) \leq S(\tilde{N}_A(x), \tilde{N}_A(y))$.

Proof. If A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$, (V1) is obvious. Since $\tilde{M}_A((x\to(y\to z))\to z)\geq T(\tilde{M}_A((x\to(y\to z))\to(y\to z)),\tilde{M}_A(y))$ and $\tilde{N}_A((x\to(y\to z))\to z)\leq S(\tilde{N}_A((x\to(y\to z))\to(y\to z)),\tilde{N}_A(y))$. As $(x\to(y\to z))\to(y\to z)=x\vee(y\to z)\geq x$, by(V3), we have $\tilde{M}_A((x\to(y\to z))\to(y\to z))\geq\tilde{M}_A(x)$ and $\tilde{N}_A((x\to(y\to z))\to(y\to z))\leq\tilde{N}_A(x)$. Therefore, $\tilde{M}_A((x\to(y\to z))\to z)\geq T(\tilde{M}_A(x),\tilde{M}_A(y))$ and $\tilde{N}_A((x\to(y\to z))\to z)\leq S(\tilde{N}_A(x),\tilde{N}_A(y))$.

Conversely, suppose (V8) is valid. Since $\tilde{M}_A(y)=\tilde{M}_A(I\to y)=\tilde{M}_A(((x\to y)\to(x\to y))\to y)\geq T(\tilde{M}_A(x\to y),\tilde{M}_A(x))$ and $\tilde{N}_A(y)=\tilde{N}_A(I\to y)=\tilde{N}_A(((x\to y)\to(x\to y))\to y)\leq S(\tilde{N}_A(x\to y),\tilde{N}_A(x))$, so (V2) holds. We can obtain (V1) from $(V3)$, A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$. □

Theorem 3.8. *Let A be an IVIFS on $\mathcal{L}$ and it satisfies $(V3)$. Then A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$, iff, for any $x,y,z\in L$, A satisfies*

(V9) $\tilde{M}_A(x\to z)\geq T(\tilde{M}_A(x\to y),\tilde{M}_A(y\to z))$ and $\tilde{N}_A(x\to z)\leq S(\tilde{N}_A(x\to y),\tilde{N}_A(y\to z))$.

Proof. Suppose A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$. Since $(x\to y)\leq(y\to z)\to(x\to z)$, it follows that $\tilde{M}_A((y\to z)\to(x\to z))\geq\tilde{M}_A(x\to y)$ and $\tilde{N}_A((y\to z)\to(x\to z))\leq\tilde{N}_A(x\to y)$. As A is an IVI (T,S)-fuzzy filter, so $\tilde{M}_A(x\to z)\geq T(\tilde{M}_A(y\to z),\tilde{M}_A((y\to z)\to(x\to z)))$ and $\tilde{N}_A(x\to z)\leq S(\tilde{N}_A(y\to z),\tilde{N}_A((y\to z)\to(x\to z)))$. We have

$$\tilde{M}_A(x\to z)\geq T(\tilde{M}_A(y\to z),\tilde{M}_A(x\to z))$$

and

$$\tilde{N}_A(x\to z)\leq S(\tilde{N}_A(y\to z),\tilde{N}_A(x\to z)).$$

Conversely, if $\tilde{M}_A(x\to z)\geq T(\tilde{M}_A(x\to y),\tilde{M}_A(y\to z))$ and $\tilde{N}_A(x\to z)\leq S(\tilde{N}_A(x\to y),\tilde{N}_A(y\to z))$ for any $x,y,z\in L$, then

$$\tilde{M}_A(I\to z)\geq T(\tilde{M}_A(I\to y),\tilde{M}_A(y\to z))$$

and

$$\tilde{N}_A(I\to z)\leq S(\tilde{N}_A(I\to y),\tilde{N}_A(y\to z)),$$

that is $\tilde{M}_A(z)\geq T(\tilde{M}_A(y),\tilde{M}_A(y\to z))$ and $\tilde{N}_A(z)\leq S(\tilde{N}_A(y),\tilde{N}_A(y\to z))$. By (V3), (V1) holds, we have A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$. □

Theorem 3.9. *Let A be an IVIFS on $\mathcal{L}$ and T,S be idempotent. Then A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$, iff, for any $\tilde{\alpha},\tilde{\beta}\in D[0,1]$ and $\tilde{\alpha}+\tilde{\beta}\leq\tilde{1}$, the sets $U(\tilde{M}_A;\tilde{\alpha})(\neq\varnothing)$ and $L(\tilde{N}_A;\tilde{\beta})(\neq\varnothing)$ are filters of $\mathcal{L}$, where $U(\tilde{M}_A;\tilde{\alpha})=\{x\in L|\tilde{M}_A(x)\geq\tilde{\alpha}\}$, $L(\tilde{N}_A;\tilde{\beta})=\{x\in L|\tilde{N}_A(x)\leq\tilde{\beta}\}$.*

Proof. Assume A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$, then $\tilde{M}_A(I)\geq\tilde{M}_A(x)$. By the condition $U(\tilde{M}_A,\tilde{\alpha})\neq\varnothing$, it follows that there exists $a\in L$ such that $\tilde{M}_A(a)\geq\tilde{\alpha}$, and so $\tilde{M}_A(I)\geq\tilde{\alpha}$, hence $I\in U(\tilde{M}_A;\tilde{\alpha})$.

Let $x,x\to y\in U(\tilde{M}_A;\tilde{\alpha})$, then $\tilde{M}_A(x)\geq\tilde{\alpha},\tilde{M}_A(x\to y)\geq\tilde{\alpha}$. Since A is an IVI (T,S)-filter of $\mathcal{L}$, it follows that $\tilde{M}_A(y)\geq T(\tilde{M}_A(x),\tilde{M}_A(x\to y))\geq T(\tilde{\alpha},\tilde{\alpha})=\tilde{\alpha}$. Hence $y\in U(\tilde{M}_A;\tilde{\alpha})$.Therefore $U(\tilde{M}_A;\tilde{\alpha})$ is a filter of $\mathcal{L}$.

We will show that $L(\tilde{N}_A;\tilde{\beta})$ is a filter of $\mathcal{L}$.

Since A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$, it follows that $\tilde{N}_A(I) \leq \tilde{N}_A(x)$. By the condition $L(\tilde{N}_A, \tilde{\beta}) \neq \varnothing$, it follows that there exists $a \in L$ such that $\tilde{N}_A(a) \leq \tilde{\beta}$, and so $\tilde{N}_A(a) \leq \tilde{\beta}$, we have $\tilde{N}_A(I) \leq \tilde{N}_A(a) \leq \tilde{\beta}$, hence $I \in L(\tilde{N}_A; \tilde{\beta})$.

Let $x, x \to y \in L(\tilde{N}_A; \tilde{\beta})$, then $\tilde{N}_A(x) \leq \tilde{\beta}; \tilde{N}_A(x \to y) \leq \tilde{\beta}$. Since A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$, then $\tilde{N}_A(y) \leq S(\tilde{N}_A(x), \tilde{N}_A(x \to y)\} \leq S(\tilde{\beta}, \tilde{\beta}) = \tilde{\beta}$. It follows that $\tilde{N}_A(y) \leq \tilde{\beta}$, hence $y \in L(\tilde{N}_A; \tilde{\beta})$. Therefore $L(\tilde{N}_A; \tilde{\beta})$ is a filter of $\mathcal{L}$.

Conversely, suppose that $U(\tilde{M}_A; \tilde{\alpha})(\neq \varnothing)$ and $L(\tilde{N}_A; \tilde{\beta})(\neq \varnothing)$ are filters of $\mathcal{L}$, then, for any $x \in L$, $x \in U(\tilde{M}_A; \tilde{M}_A(x))$ and $x \in L(\tilde{N}_A; \tilde{N}_A(x))$. By $U(\tilde{M}_A, \tilde{M}_A(x))$ $(\neq \varnothing)$ and $L(\tilde{N}_A, \tilde{N}_A(x))(\neq \varnothing)$ are filters of $\mathcal{L}$, it follows that $I \in U(\tilde{M}_A, \tilde{M}_A(x))$ and $I \in L(\tilde{N}_A, \tilde{MN}_A(x))$, and so $\tilde{M}_A(I) \geq \tilde{M}_A(x)$ and $\tilde{N}_A(I) \leq \tilde{N}_A(x)$.

For any $x, y \in L$, let $\tilde{\alpha} = T(\tilde{M}_A(x), \tilde{M}_A(x \to y))$ and $\tilde{\beta} = S(\tilde{N}_A(x), \tilde{N}_A(x \to y)\}$, then $x, x \to y \in U(\tilde{M}_A; \tilde{\alpha})$ and $x, x \to y \in L(\tilde{N}_A; \tilde{\beta})$. And so $y \in U(\tilde{M}_A; \tilde{\alpha})$ and $y \in L(\tilde{N}_A; \tilde{\beta})$. Therefore

$$\tilde{M}_A(y) \geq \tilde{\alpha} = T(\tilde{M}_A(x), \tilde{M}_A(x \to y))$$

and

$$\tilde{N}_A(y) \leq \tilde{\beta} = S(\tilde{N}_A(x), \tilde{N}_A(x \to y)\}.$$

We have A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$. □

Let A, B be two IVIFSs on $\mathcal{L}$, denote C by the intersection of A and B, i.e. $C = A \cap B$, where

$$\begin{aligned} \tilde{M}_C(x) \quad &= T(\tilde{M}_A(x), \tilde{M}_B(x)), \\ \tilde{N}_C(x) \quad &= S(\tilde{N}_A(x), \tilde{N}_B(x)) \end{aligned}$$

for any $x \in L$.

Theorem 3.10. *Let A, B be two IVI (T,S)-fuzzy filters of $\mathcal{L}$, then $A \cap B$ is also an IVI (T,S)-fuzzy filter of $\mathcal{L}$.*

Proof. Since A, B are two IVI (T,S)-fuzzy filters of $\mathcal{L}$, we have

$$\tilde{M}_A(y) \geq T(\tilde{M}_A(x \to y), \tilde{M}_A(x)),\ \tilde{N}_A(y) \leq S(\tilde{N}_A(x \to y), \tilde{N}_A(x))$$

and

$$\tilde{M}_B(y) \geq T(\tilde{M}_B(x \to y), \tilde{M}_B(x)),\ \tilde{N}_B(y) \leq S(\tilde{N}_B(x \to y), \tilde{N}_B(x)).$$

Since

$$\begin{aligned} \tilde{M}_{A\cap B}(y) &= T(\tilde{M}_A(y), \tilde{M}_B(y)\} \\ &\geq T(T(\tilde{M}_A(x \to y), \tilde{M}_A(x)), T(\tilde{M}_B(x \to y), \tilde{M}_B(x))) \\ &= T(T(\tilde{M}_A(x \to y), \tilde{M}_B(x \to y)), T(\tilde{M}_A(x), \tilde{M}_B(x))) \\ &= T(\tilde{M}_{A\cap B}(x \to y), \tilde{M}_{A\cap B}(x)) \end{aligned}$$

and

$$\begin{aligned}\tilde{N}_{A\cap B}(y) &= S(\tilde{N}_A(y),\tilde{N}_B(y))\\ &\leq S(S(\tilde{N}_A(x\to y),\tilde{N}_A(x)),S(\tilde{N}_B(x\to y),\tilde{N}_B(x)))\\ &= S(S(\tilde{N}_A(x\to y),\tilde{N}_B(x\to y)),S(\tilde{N}_A(x),\tilde{N}_B(x)))\\ &= S(\tilde{N}_{A\cap B}(x\to y),\tilde{N}_{A\cap B}(x)).\end{aligned}$$

Since A,B are two IVI (T,S)-fuzzy filters of $\mathcal{L}$, we have $\tilde{M}_A(I)\geq \tilde{M}_A(x)), \tilde{N}_A(I)\leq \tilde{N}_A(x))$ and $\tilde{M}_B(I)\geq \tilde{M}_B(x)$, $\tilde{N}_B(I)\leq \tilde{N}_B(x)$. Hence

$$\begin{aligned}\tilde{M}_{A\cap B}(I) &= T(\tilde{M}_A(I),\tilde{M}_B(I))\\ &\geq T(\tilde{M}_A(x),\tilde{M}_B(x))\\ &= \tilde{M}_{A\cap B}(x).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\tilde{N}_{A\cap B}(I) &= S(\tilde{N}_A(I),\tilde{N}_B(I))\\ &\leq S(\tilde{N}_A(x),\tilde{N}_B(x))=\tilde{N}_{A\cap B}(x).\end{aligned}$$

Then $A\cap B$ is an IVI (T,S)-fuzzy filters of $\mathcal{L}$. □

Let A_i be a family IVIFSs on $\mathcal{L}$, where i is an index set. Denoting C by the intersection of A_i, i.e. $\cap_{i\in I}A_i$, where

$$\begin{aligned}\tilde{M}_C(x) &= T(\tilde{M}_{A_1}(x),\tilde{M}_{A_2}(x),\cdots),\\ \tilde{N}_C(x) &= S(\tilde{N}_{A_1}(x),\tilde{N}_{A_2}(x),\cdots)\end{aligned}$$

for any $x\in L$.

Corollary 3.11. *Let A_i be a family IVI (T,S)-fuzzy filters of $\mathcal{L}$, where $i\in I$, I is an index set. then $\cap_{i\in I}A_i$ is also an IVI (T,S)-fuzzy filter of $\mathcal{L}$.*

Suppose A is an IVIFS on $\mathcal{L}$ and $\tilde{\alpha},\tilde{\beta}\in D[0,1]$. Denoting $A_{(\tilde{\alpha},\tilde{\beta})}$ by the set $\{x\in L|\tilde{M}_A(x)\geq\tilde{\alpha},\tilde{N}_A(x)\leq\tilde{\beta}\}$.

Theorem 3.12. *Let A be an IVIFS on $\mathcal{L}$. Then*

(1) for any $\tilde{\alpha},\tilde{\beta}\in D[0,1]$, if $A_{(\tilde{\alpha},\tilde{\beta})}$ is a filter of $\mathcal{L}$. Then, for any $x,y,z\in L$,
(V10) $\tilde{M}_A(z)\leq T(\tilde{M}_A(x\to y),\tilde{M}_A(x))$ and $\tilde{N}_A(z)\geq S(\tilde{N}_A(x\to y),\tilde{N}_A(x))$ imply $\tilde{M}_A(z)\leq\tilde{M}_A(y)$ and$\tilde{N}_A(z)\geq\tilde{N}_A(y)$.

(2) if A satisfy (V1)and (V10), then, for any $\tilde{\alpha},\tilde{\beta}\in D[0,1]$, $A_{(\tilde{\alpha},\tilde{\beta})}$ is a filter of $\mathcal{L}$.

Proof. (1) Suppose $A_{(\tilde{\alpha},\tilde{\beta})}$ is a filter of $\mathcal{L}$ for any $\tilde{\alpha},\tilde{\beta}\in D[0,1]$. Since $\tilde{M}_A(z)\leq T(\tilde{M}_A(x\to y),\tilde{M}_A(x))$ and $\tilde{N}_A(z)\geq S(\tilde{N}_A(x\to y),\tilde{N}_A(x))$, it follows that

$$\tilde{M}_A(z)\leq\tilde{M}_A(x\to y),\tilde{M}_A(z)\leq\tilde{M}_A(x)$$

and

$$\tilde{N}_A(z)\geq\tilde{N}_A(x\to y),\tilde{N}_A(z)\geq\tilde{N}_A(x).$$

Therefore,

$$x\to y\in A_{(\tilde{M}_A(z),\tilde{N}_A(z))},x\in A_{(\tilde{M}_A(z),\tilde{N}_A(z))}.$$

As $\tilde{M}_A(z), \tilde{N}_A(z) \in D[0,1]$, and $A_{(\tilde{M}_A(z),\tilde{N}_A(z))}$ is a filter of $\mathcal{L}$, so

$$y \in A_{(\tilde{M}_A(z),\tilde{N}_A(z))}.$$

Thus $\tilde{M}_A(z) \leq \tilde{M}_A(y)$ and$\tilde{N}_A(z) \geq \tilde{N}_A(y)$.

(2) Assume A satisfy (V1) and (V10). For any $x, y \in L$, $\tilde{\alpha}, \tilde{\beta} \in D[0,1]$, we have $x \to y \in A_{(\tilde{\alpha},\tilde{\beta})}, x \in A_{(\tilde{\alpha},\tilde{\beta})}$, therefore $\tilde{M}_A(x \to y) \geq \tilde{\alpha}$, $\tilde{N}_A(x \to y) \leq \tilde{\beta}$ and $\tilde{M}_A(x) \geq \tilde{\alpha}$, $\tilde{N}_A(x) \leq \tilde{\beta}$, and so

$$T(\tilde{M}_A(x \to y), \tilde{M}_A(x)) \geq T(\tilde{\alpha}, \tilde{\alpha}) = \tilde{\alpha},$$

$$S(\tilde{N}_A(x \to y), \tilde{N}_A(x)) \leq S(\tilde{\beta}, \tilde{\beta}) = \tilde{\beta}.$$

By (V10), we have $\tilde{M}_A(y) \geq \tilde{\alpha}$ and $\tilde{N}_A(y) \leq \tilde{\beta}$, that is, $y \in A_{(\tilde{\alpha},\tilde{\beta})}$.

Since $\tilde{M}_A(I) \geq \tilde{M}_A(x)$ and $\tilde{N}_A(I) \leq \tilde{N}_A(x)$ for any $x \in L$, it follows that $\tilde{M}_A(I) \geq \tilde{\alpha}$ and $\tilde{N}_A(I) \leq \tilde{\beta}$, that is, $I \in A_{(\tilde{\alpha},\tilde{\beta})}$. Then, for any $\tilde{\alpha}, \tilde{\beta} \in [0,1]$, $A_{(\tilde{\alpha},\tilde{\beta})}$ is a filter of $\mathcal{L}$. □

Theorem 3.13. *Let A be an IVI (T,S)-fuzzy filter of $\mathcal{L}$, then, for any $\tilde{\alpha}, \tilde{\beta} \in D[0,1]$, $A_{(\tilde{\alpha},\tilde{\beta})}(\neq \phi)$ is a filter of $\mathcal{L}$.*

Proof. Since $A_{(\tilde{\alpha},\tilde{\beta})} \neq \phi$, there exist $\tilde{\alpha}, \tilde{\beta} \in D[0,1]$ such that $\tilde{M}_A(x) \geq \tilde{\alpha}, \tilde{N}_A(x) \leq \tilde{\beta}$. And A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$, we have $\tilde{M}_A(I) \geq \tilde{M}_A(x) \geq \tilde{\alpha}, N_A(I) \leq \tilde{N}_A(x) \leq \tilde{\beta}$, therefore $I \in A_{(\tilde{\alpha},\tilde{\beta})}$.

Let $x, y \in L$ and $x \in A_{(\tilde{\alpha},\tilde{\beta})}, x \to y \in A_{(\tilde{\alpha},\tilde{\beta})}$, therefore $\tilde{M}_A(x) \geq \tilde{\alpha}, \tilde{N}_A(x) \leq \tilde{\beta}, \tilde{M}_A(x \to y) \geq \tilde{\alpha}, \tilde{M}_A(x \to y) \leq \tilde{\beta}$. Since A is an IVI (T,S)-fuzzy filter $\mathcal{L}$, thus $\tilde{M}_A(y) \geq T(\tilde{M}_A(x \to y), \tilde{M}_A(x)) \geq \tilde{\alpha}$ and $\tilde{N}_A(y) \leq S(\tilde{N}_A(x \to y), \tilde{N}_A(x)) \leq \tilde{\beta}$, it follows that $y \in A_{(\tilde{\alpha},\tilde{\beta})}$. Therefore, $A_{(\tilde{\alpha},\tilde{\beta})}$ is a filter of $\mathcal{L}$. □

In the Theorem 3.13, the filter $A_{(\tilde{\alpha},\tilde{\beta})}$ is also called **IVI-cut** filter of $\mathcal{L}$.

Theorem 3.14. *Any filter F of $\mathcal{L}$ is a IVI-cut filter of some IVI (T,S)-fuzzy filter of $\mathcal{L}$.*

Proof. Consider the IVIFS A of $\mathcal{L}$: $A = \{(x, \tilde{M_A}(x), x, \tilde{N_A}(x)) | x \in L\}$, where

If $x \in F$,

$$\tilde{M}_A(x) = \tilde{\alpha}, \tilde{N}_A(x) = \tilde{1} - \tilde{\alpha}. \tag{3.1}$$

If $x \notin F$,

$$\tilde{M}_A(x) = \tilde{0}, \tilde{N}_A(x) = \tilde{1}. \tag{3.2}$$

where $\tilde{\alpha} \in D[0,1]$. Since A is a filter of $\mathcal{L}$, we have $I \in F$. Therefore $\tilde{M}_A(I) = \tilde{\alpha} \geq \tilde{M}_A(x)$ and $\tilde{N}_A(I) = \tilde{1} - \tilde{\alpha} \leq \tilde{N}_A(x)$.

For any $x, y \in L$, if $y \in F$, then $\tilde{M}_A(y) = \tilde{\alpha} \geq T(\tilde{\alpha}, \tilde{\alpha}) \geq T(\tilde{M}_A(x \to y), \tilde{M}_A(x))$ and $\tilde{N}_A(y) = \tilde{1} - \tilde{\alpha} \leq S(\tilde{1} - \tilde{\alpha}, \tilde{1} - \tilde{\alpha}) \leq S(\tilde{N}_A(x \to y), \tilde{N}_A(x))$.

If $y \notin F$, then $x \notin F$ or $x \to y \notin F$. And so $\tilde{M}_A(y) = \tilde{0} = T(\tilde{0}, \tilde{0}) = T(\tilde{M}_A(x \to y), \tilde{M}_A(x))$ and $\tilde{N}_A(y) = \tilde{1} = S(\tilde{1}, \tilde{1}) = S(\tilde{N}_A(x \to y), \tilde{N}_A(x))$. Therefore A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$. □

Theorem 3.15. *Let A be IVI (T,S)-fuzzy filter of $\mathcal{L}$. Then $F = \{x \in L | \tilde{M}_A(x) = \tilde{M}_A(I), \tilde{N}_A(x) = \tilde{N}_A(I)\}$ is a filter of $\mathcal{L}$.*

Proof. Since $F = \{x \in L | \tilde{M}_A(x) = \tilde{M}_A(I), \tilde{N}_A(x) = \tilde{N}_A(I)\}$, obviously $I \in F$. Let $x \to y \in F, x \in F$, so $\tilde{M}_A(x \to y) = \tilde{M}_A(x) = \tilde{M}_A(I)$ and $\tilde{N}_A(x \to y) = \tilde{N}_A(x) = \tilde{N}_A(I)$, Therefore

$$\tilde{M}_A(y) \geq T(\tilde{M}_A(x \to y), \tilde{M}_A(x)\} = \tilde{M}_A(I).$$

And $\tilde{M}_A(I) \geq \tilde{M}_A(y)$, then $\tilde{M}_A(y) = \tilde{M}_A(I)$. Similarly, we have $\tilde{N}_A(y) = \tilde{N}_A(I)$. Thus $y \in F$. It follows that A is a filter of $\mathcal{L}$. □

4. IVI (T,S)-fuzzy implicative filters

Definition 4.1. Let A be an IVIFS. A is called an IVI (T,S)-fuzzy implicative filter if it satisfies (V1) and

(V11)$\tilde{M}_A(x \to z) \geq T(\tilde{M}_A(x \to (y \to z), \tilde{M}_A(x \to y))$ and $\tilde{N}_A(x \to z) \leq S(\tilde{N}_A(x \to (y \to z), \tilde{N}_A(x \to y)))$ for any $x, y, z \in L$.

Taking $x = 1$ in (V11), we can obtain the following Theorem.

Theorem 4.2. *Each IVI (T,S)-fuzzy implicative filter is an IVI (T,S)-fuzzy filter.*

Theorem 4.3. *Let A be an IVI (T,S)-fuzzy filter of $\mathcal{L}$ and satisfy $(V3)$. Then the following are equivalent:*

(1) A is an IVI (T,S)-fuzzy implicative filter;

(2) $\tilde{M}_A(x \to y) \geq \tilde{M}_A(x \to (x \to y))$ and $\tilde{N}_A(x \to y) \leq \tilde{N}_A(x \to (x \to y))$ for any $x, y \in L$;

(3) $\tilde{M}_A((x \to y) \to (x \to z)) \geq \tilde{M}_A(x \to (y \to z))$ and $\tilde{N}_A((x \to y) \to (x \to z)) \leq \tilde{N}_A(x \to (y \to z))$ for any $x, y, z \in L$.

Proof. Suppose A is an IVI (T,S)-fuzzy implicative filter, we have $\tilde{M}_A(x \to y) \geq T(\tilde{M}_A(x \to (x \to y)), \tilde{M}_A(x \to x)) = T(\tilde{M}_A(x \to (x \to y)), \tilde{M}_A(I)) \geq \tilde{M}_A(x \to (x \to y))$ and $\tilde{N}_A(x \to y) \leq S(\tilde{N}_A(x \to (x \to y)), \tilde{N}_A(x \to x)) = S(\tilde{N}_A(x \to (x \to y)), \tilde{N}_A(I)) \leq \tilde{N}_A(x \to (x \to y))$. Thus (2) is valid.

Suppose that (2) holds. That is, Suppose A is an IVI (T,S)-fuzzy filter of $\mathcal{L}$ and $\tilde{M}_A(x \to y) \geq \tilde{M}_A(x \to (x \to y)), \tilde{N}_A(x \to y) \leq \tilde{N}_A(x \to (x \to y))$. Note that $x \to (y \to z) \leq x \to ((x \to y) \to (x \to z))$, it follows that

$$\begin{aligned} \tilde{M}_A((x \to y) \to (x \to z)) &= \tilde{M}_A(x \to ((x \to y) \to z)) \\ &\geq \tilde{M}_A(x \to (x \to ((x \to y) \to z))) \\ &= \tilde{M}_A(x \to ((x \to y) \to (x \to z))) \\ &\geq \tilde{M}_A(x \to (y \to z)). \end{aligned}$$

and

$$\begin{aligned} \tilde{N}_A((x \to y) \to (x \to z)) &= \tilde{N}_A(x \to ((x \to y) \to z)) \\ &\leq \tilde{N}_A(x \to (x \to ((x \to y) \to z))) \\ &= \tilde{N}_A(x \to ((x \to y) \to (x \to z))) \\ &\leq \tilde{N}_A(x \to (y \to z)). \end{aligned}$$

Suppose (3) holds, since A ia an IVI (T,S)-fuzzy filter, we have $\tilde{M}_A(x \to z) \geq T(\tilde{M}_A(x \to y), \tilde{M}_A((x \to y) \to (x \to z))) \geq T(\tilde{M}_A(x \to y), \tilde{M}_A(x \to (y \to z)))$ and $\tilde{N}_A(x \to z) \leq S(\tilde{N}_A(x \to y), \tilde{N}_A((x \to y) \to (x \to z))) \leq TS(\tilde{N}_A(x \to y), \tilde{N}_A(x \to (y \to z)))$. Therefore, A is an IVI (T,S)-fuzzy implicative filter of $\mathcal{L}$. □

Theorem 4.4. *Let A be an IVI (T,S)-fuzzy filter of $\mathcal{L}$ and satisfy $(V3)$. Then A is an IVI (T,S)-fuzzy implicative filter iff $\tilde{M}_A(y \to x) \geq T(\tilde{M}_A(z \to (y \to (y \to x))), \tilde{M}_A(z))$ and $\tilde{N}_A(y \to x) \leq S(\tilde{N}_A(z \to (y \to (y \to x))), \tilde{N}_A(z))$ for any $x, y, z \in L$.*

Proof. Since A is an IVI (T,S)-fuzzy implicative filter, we have $\tilde{M}_A(y \to (y \to x)) \geq T(\tilde{M}_A(z \to (y \to (y \to x))), \tilde{M}_A(z))$ and $\tilde{N}_A(y \to (y \to x)) \leq S(\tilde{N}_A(z \to (y \to (y \to x))), \tilde{N}_A(z))$. Since $y \to x = I \to (y \to x) = (y \to y) \to (y \to x)$, it follows from that $\tilde{M}_A(y \to x) = \tilde{M}_A((y \to y) \to (y \to x)) \geq \tilde{M}_A(y \to (y \to x)) \geq T(\tilde{M}_A(z \to (y \to (y \to x))), \tilde{M}_A(z))$ and $\tilde{N}_A(y \to x) = \tilde{N}_A((y \to y) \to (y \to x)) \leq \tilde{N}_A(y \to (y \to x)) \leq S(\tilde{N}_A(z \to (y \to (y \to x))), \tilde{N}_A(z))$.

Conversely, let A be an IVI (T,S)-fuzzy filter and satisfy the condition: $\tilde{M}_A(y \to x) \geq T(\tilde{M}_A(z \to (y \to (y \to x))), \tilde{M}_A(z))$ and $\tilde{N}_A(y \to x) \leq S(\tilde{N}_A(z \to (y \to (y \to x))), \tilde{N}_A(z))$ for any $x, y, z \in L$. And so

$$\tilde{M}_A(z \to x) \geq T(\tilde{M}_A((z \to y) \to (z \to (z \to x))), \tilde{M}_A(z \to y)).$$

and

$$\tilde{N}_A(z \to x) \leq S(\tilde{N}_A((z \to y) \to (z \to (z \to x))), \tilde{N}_A(z \to y)).$$

Since $z \to (y \to x) = y \to (z \to x) \leq (z \to y) \to (z \to (z \to x))$, we have $\tilde{M}_A((z \to y) \to (z \to (z \to x))) \geq \tilde{M}_A(z \to (y \to x))$ and $\tilde{N}_A((z \to y) \to (z \to (z \to x))) \leq \tilde{N}_A(z \to (y \to x))$. Together with (2), we have

$$\begin{aligned}\tilde{M}_A(z \to x) &\geq T(\tilde{M}_A((z \to y) \to (z \to (z \to x))), \tilde{M}_A(z \to y)) \\ &\geq T(\tilde{M}_A(z \to (y \to x)), \tilde{M}_A(z \to y))\end{aligned}$$

and

$$\begin{aligned}\tilde{N}_A(z \to x) &\leq S(\tilde{N}_A((z \to y) \to (z \to (z \to x))), \tilde{N}_A(z \to y)) \\ &\leq S(\tilde{N}_A(z \to (y \to x)), \tilde{N}_A(z \to y)).\end{aligned}$$

Hence A is an IVI (T,S)-fuzzy implicative filter. □

Theorem 4.5. *Let A be an IVI (T,S)-fuzzy filter and T,S be idempotent. Then A is an IVI (T,S)-fuzzy implicative filter iff $\tilde{M}_A(x \to x^2) = \tilde{M}_A(I)$ and $\tilde{N}_A(x \to x^2) = \tilde{N}_A(I)$ for any $x \in L$.*

Proof. Suppose that A is an IVI (T,S)-fuzzy implicative filter. Since $x \to (x \to x^2) = I$, we have $\tilde{M}_A(x \to (x \to x^2)) = \tilde{M}_A(I)$ and $\tilde{N}_A(x \to (x \to x^2)) = \tilde{N}_A(I)$. Therefore $\tilde{M}_A(x \to x^2) \geq T(\tilde{M}_A(x \to (x \to x^2)), \tilde{M}_A(x \to x)) = T(\tilde{M}_A(I), \tilde{M}_A(I)) = \tilde{M}_A(I)$ and $\tilde{N}_A(x \to x^2) \leq S(\tilde{N}_A(x \to (x \to x^2)), \tilde{N}_A(x \to x)) = S(\tilde{N}_A(I), \tilde{N}_A(I)) = \tilde{N}_A(1)$. Therefore $\tilde{M}_A(x \to x^2) = \tilde{M}_A(I)$ and $\tilde{N}_A(x \to x^2) = \tilde{N}_A(I)$ for any $x \in L$.

Conversely, suppose that $\tilde{M}_A(x \to x^2) = \tilde{M}_A(I)$ and $\tilde{N}_A(x \to x^2) = \tilde{N}_A(I)$ for any $x \in L$. Since $(x \to x^2) \to (x \to y) \geq x^2 \to y = x \to (x \to y)$ and A is an IVI

(T,S)-fuzzy filter of $\mathcal{L}$, we have $\tilde{M}_A((x \to x^2) \to (x \to y)) \geq \tilde{M}_A(x \to (x \to y))$ and $\tilde{N}_A((x \to x^2) \to (x \to y)) \leq \tilde{N}_A(x \to (x \to y))$. Hence $\tilde{M}_A(x \to y) \geq T(\tilde{M}_A((x \to x^2) \to (x \to y)), \tilde{M}_A(x \to x^2)) \geq T(\tilde{M}_A(x \to (x \to y)), \tilde{M}_A(x \to x^2)) = T(\tilde{M}_A(x \to (x \to y)), \tilde{M}_A(I)) \geq \tilde{M}_A(x \to (x \to y))$ and $\tilde{N}_A(x \to y) \leq S(\tilde{N}_A((x \to x^2) \to (x \to y)), \tilde{N}_A(x \to x^2)) \leq S(\tilde{N}_A(x \to (x \to y)), \tilde{N}_A(x \to x^2)) = S(\tilde{N}_A(x \to (x \to y)), \tilde{N}_A(I)) \leq \tilde{N}_A(x \to (x \to y))$. It follows that A is an IVI (T,S)-fuzzy implicative filter of $\mathcal{L}$. □

Theorem 4.6. *Let A be an IVIFS on $\mathcal{L}$ and T,S be idempotent. Then A is an IVI (T,S)-fuzzy implicative (G) filter of $\mathcal{L}$, if and only if, for any $\tilde{\alpha}, \tilde{\beta} \in D[0,1]$ and $\tilde{\alpha}+\tilde{\beta} \leq \tilde{1}$, the sets $U(\tilde{M}_A;\tilde{\alpha})(\neq \varnothing)$ and $L(\tilde{N}_A;\tilde{\beta})(\neq \varnothing)$ are positive implicative filters of $\mathcal{L}$, where $U(\tilde{M}_A;\tilde{\alpha}) = \{x \in L | \tilde{M}_A(x) \geq \tilde{\alpha}\}$, $L(\tilde{N}_A;\tilde{\beta}) = \{x \in L | \tilde{N}_A(x) \leq \tilde{\beta}\}$.*

Proof. It similar to Theorem 3.9, the details is omitted. □

5. Conclusions

Filter theory plays an very important role in studying logical systems and the related algebraic structures. In this paper, Combining the interval-valued intuitionistic fuzzy set, t-norm, s-norm, and the filter theory, investigating the interval-valued intuitionistic (T,S)-fuzzy implicative filters theory of LIAs. Mainly, we give some new characterizations of interval-valued valued intuitionistic (T,S)-fuzzy (implicative) filters in LIAs. We desperately hope that this work would serve as a foundation for enriching corresponding many-valued logical system.

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