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Prime and semiprime fuzzy bi-ideals in Γ-semigroups-revisited

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Abstract. In this paper some properties of prime and semiprime fuzzy bi-ideals in Γ-semigroups have been obtained. We show that in some restricted types of Γ-semigroups the notions of prime fuzzy ideal and prime fuzzy bi-ideal coincide. We also revisit, by providing new proofs, some characterization theorems of regular and intra regular Γ-semigroups in terms of prime, semiprime fuzzy bi-ideals.

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1. Introduction

The concept of fuzzy set was introduced by Lofti A. Zadeh[16] in 1965. Since its inception, the notion of fuzzy set has been applied in many directions of mathematical sciences like logic, set theory, group theory, semigroup theory, real analysis, measure theory, topology etc. In 1974, Wong[14] introduced the notion of a fuzzy point belonging to a fuzzy subset. Kuroki[5] developed fuzzy set in terms of different types of ideals of semigroups. Fuzzification of different concepts of semigroups, Γ-semigroups, groups[6] motivated us to do this work. Sen[11] defined the concept of a Γ-semigroup as a generalization of a semigroup. Γ-semigroups turned out to be a generalization of ternary semigroups as well. Γ-semigroups and its fuzzification have been studied by a lot of researchers such as Dutta et al[4], Kar et al[2], Sardar,Davvaz et al[7, 8, 9, 10]. In [15], Williams, Latha and Chandrasekeran introduced fuzzy bi-ideals in Γ-semigroups. Bashir et al[1] defined prime fuzzy bi-ideal, strongly prime fuzzy bi-ideal, semiprime fuzzy bi-ideal, strongly irreducible fuzzy bi-ideal and irreducible fuzzy bi-ideal of a Γ-semigroup and used them to characterize

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regular and intra-regular Γ-semigroups. It was noted there[1] that a strongly prime fuzzy bi-ideal is a prime fuzzy bi-ideal and a prime fuzzy bi-ideal is a semiprime fuzzy bi-ideal. In this paper we give some counter examples to illustrate that the converse of the above cases are not true in general. We also show by Wong's notion of fuzzy point that notions of prime, strongly prime, semiprime fuzzy bi-ideals are compatible with the characteristic function criterion, among them only semiprime fuzzy bi-ideals satisfy level subset criterion. This property of semiprime fuzzy bi-ideals has been very effective in revisiting some characterization results(*cf.* Theorem 4.22) of Bashir et al[1]. It is well known that[10] in a regular duo Γ-semigroup the notions of fuzzy ideal and fuzzy bi-ideal coincide. To conclude this paper we observe that similar results hold for prime fuzzy ideals and prime fuzzy bi-ideals of duo Γ-semigroups as well as of normal Γ-semigroups.

2. Preliminaries

In this section we recall some preliminary notions and results of Γ-semigroups as well as of fuzzy subsets for their use in the sequel.

Let $S = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. Then S is called a Γ-*semigroup*[11, 12] if there exists a mapping $S \times \Gamma \times S \to S$(images to be denoted by $a\alpha b$) satisfying

(1) $x\gamma y \in S$,
(2) $(x\beta y)\gamma z = x\beta(y\gamma z)$, for all $x, y, z \in S$ and $\beta, \gamma \in \Gamma$.

A nonempty subset A of a Γ-semigroup S is called a *subsemigroup* of S if $A\Gamma A \subseteq A$, where $A\Gamma A = \{a\alpha b : a, b \in A \text{ and } \alpha \in \Gamma\}$.

A subsemigroup A of a Γ-semigroup S is called a *bi-ideal* of S if $A\Gamma S\Gamma A \subseteq A$.

A bi-ideal $A(\neq S)$ of a Γ-semigroup S is called a *prime bi-ideal* of S if for any two bi-ideals A_1 and A_2 of S, $A_1\Gamma A_2 \subseteq A$ implies $A_1 \subseteq A$ or $A_2 \subseteq A$.

A bi-ideal $A(\neq S)$ of a Γ-semigroup S is called *semiprime bi-ideal* of S if for any bi-ideal A_1 of S, $A_1\Gamma A_1 \subseteq A$ implies $A_1 \subseteq A$.

A bi-ideal A of a Γ-semigroup S is called an *irreducible*(*strongly irreducible*) bi-ideal if for any two bi-ideals A_1 and A_2 of S, $A_1 \cap A_2 = A$ (resp. $A_1 \cap A_2 \subseteq A$) implies $A_1 = A$ or $A_2 = A$(resp. $A_1 \subseteq A$ or $A_2 \subseteq A$).

Definition 2.1 ([9]). For any two fuzzy subsets μ and ν of a Γ-semigroup S, we define a fuzzy subset $\mu \circ \nu$ of S by

$$(\mu \circ \nu)(x) = \begin{cases} \sup\limits_{x=y\gamma z} \min\{\mu(y), \nu(z)\} & \text{if } x = y\gamma z \text{ for some } y, z \in S, \gamma \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

We call $\mu \circ \nu$ to be the product of μ and ν.

Note 2.2. In Bashir et al[1], $\mu \circ \nu$ is written as $\mu\Gamma\nu$.

Definition 2.3 ([14]). A fuzzy subset μ of a set X of the form

$$\mu(y) = \begin{cases} t(\neq 0) & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a *fuzzy point* with *support* x and *value* t and is denoted by x_t.

The following result is immediate from the preceding definition.

Proposition 2.4. *For any fuzzy subset* μ *of a nonempty set* S, $\mu = \bigcup_{a_t \subseteq \mu} a_t$.

We recall the following proposition for its use in the sequel.

Proposition 2.5 ([10])**.** *Let* A *and* B *be two nonempty subsets of a set* S. *Then*
(*i*) $A \subseteq B$ *if and only if* $C_A \subseteq C_B$
(*ii*) $C_A \cup C_B = C_{A\cup B}$
(*iii*) $C_A \cap C_B = C_{A\cap B}$
(*iv*) $C_A \circ C_B = C_{A\Gamma B}$, *provided* S *is a* Γ*-semigroup.*
Here C_U *denotes the characteristic function of* U.

Throughout the paper the notation for characteristic function is as above.

Definition 2.6. Let A be a nonempty subset of a set S. Then for any $t \in (0, 1]$, the fuzzy subset tC_A is defined on S by

$$tC_A(x) = \begin{cases} t & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Now the following result is an obvious generalization of the preceding proposition.

Proposition 2.7. *For any two nonempty subsets* A *and* B *of a* Γ*-semigroup* S *and* $t, r \in (0, 1]$,
(*i*) $A \subseteq B$ *if and only if* $tC_A \subseteq tC_B$
(*ii*) $tC_A \cup tC_B = tC_{A\cup B}$
(*iii*) $tC_A \cap rC_B = \min\{t, r\}C_{A\cap B}$
(*iv*) $tC_A \circ rC_B = \min\{t, r\}C_{A\Gamma B}$.

Definition 2.8 ([3])**.** Let μ be a nonempty fuzzy subset of a Γ-semigroup S. Then for any $t \in [0, 1]$, the *level subset* μ_t of S is defined by $\mu_t = \{x \in S : \mu(x) \geq t\}$.

The following result, regarding some properties of level subsets, is frequently used in the development of the paper.

Proposition 2.9. *Let* μ *and* ν *be two fuzzy subsets of a* Γ*-semigroup* S. *Then for* $t \in (0, 1]$,
(*i*) $\mu_t \cap \nu_t = (\mu \cap \nu)_t$.
(*ii*) $\mu_t \Gamma \nu_t = (\mu \circ \nu)_t$.
(*iii*) $\mu = \nu$ *if and only if* $\mu_t = \nu_t$ *for all* $t \in (0, 1]$.

Proof. (*i*) If $x \in \mu_t \cap \nu_t$ then $x \in \mu_t$ and $x \in \nu_t$ which implies $\mu(x) \geq t$ and $\nu(x) \geq t$, whence $\min\{\mu(x), \nu(x)\} \geq t$ *i.e.*, $(\mu \cap \nu)(x) \geq t$. Hence $x \in (\mu \cap \nu)_t$. Reversing the above argument we deduce that if $x \in (\mu\cap\nu)_t$ then $x \in \mu_t \cap \nu_t$. Hence $\mu_t \cap \nu_t = (\mu \cap \nu)_t$.

(*ii*) Let $x \in \mu_t\Gamma\nu_t$. Then $\exists\ y \in \mu_t$, $z \in \nu_t$ and $\alpha \in \Gamma$ such that $x = y\alpha z$. Now $(\mu\circ\nu)(x) = \bigvee_{x=a\gamma b} \min\{\mu(a), \nu(b)\} \geq \min\{\mu(y), \nu(z)\} \geq t$, whence $x \in (\mu\circ\nu)_t$. Hence $\mu_t\Gamma\nu_t \subseteq (\mu \circ \nu)_t$.

Let $x \notin \mu_t\Gamma\nu_t$. Then for $x = y\gamma z$, $y \notin \mu_t$ or $z \notin \nu_t$ *i.e.*, $\mu(y) < t$ or $\nu(z) < t$. So $(\mu \circ \nu)(x) = \bigvee_{x=a\gamma b} \min\{\mu(a), \nu(b)\} < t$. If $x \neq y\gamma z$ for any $y, z \in S$, for any $\gamma \in \Gamma$ then $(\mu\circ\nu)(x) = 0 < t$. Hence if $x \notin \mu_t\Gamma\nu_t$ then $x \notin (\mu\circ\nu)_t$. Hence $(\mu\circ\nu)_t \subseteq \mu_t\Gamma\nu_t$. Consequently, $\mu_t\Gamma\nu_t = (\mu \circ \nu)_t$.

(iii) If $\mu = \nu$, then $\mu_t = \nu_t$ for all $t \in (0, 1]$.

Let the converse be true. If possible let for some $x \in S$, $\mu(x) \neq \nu(x)$ and without loss of generality let us assume that $\mu(x) > \nu(x)$. Then $x \in \mu_{\mu(x)}$ but $x \notin \nu_{\mu(x)}$ which is a contradiction. Hence $\mu = \nu$. □

For other elementary definitions, examples and results on Γ-semigroups, semigroups and fuzzy sets we refer to the references.

3. Fuzzy bi-ideals

In this section we discuss some properties of fuzzy bi-ideals of a Γ-semigroup.

Definition 3.1 ([10]). A non-empty fuzzy subset μ of a Γ-semigroup S is called a *fuzzy subsemigroup* of S if $\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in S, \gamma \in \Gamma$.

A fuzzy subsemigroup μ of a Γ-semigroup S is called a *fuzzy bi-ideal* of S if $\mu(x\alpha y\beta z) \geq \min\{\mu(x), \mu(z)\}$ for all $x, y, z \in S, \alpha, \beta \in \Gamma$.

Proposition 3.2. *For any two fuzzy bi-ideals μ and ν of a Γ-semigroup S, $\mu \circ \nu$ is also a fuzzy bi-ideal of S.*

Proof. Let μ and ν be two fuzzy bi-ideals of a Γ-semigroup S. Then using properties of fuzzy bi-ideals(*cf.* Theorem 3.8[10]), $(\mu\circ\nu)\circ(\mu\circ\nu) = \mu\circ(\nu\circ\mu\circ\nu) \subseteq \mu\circ(\nu\circ C_S\circ\nu) \subseteq \mu \circ \nu$ and $(\mu \circ \nu) \circ C_S \circ (\mu \circ \nu) = \mu \circ \nu \circ (C_S \circ \mu) \circ \nu \subseteq \mu \circ (\nu \circ C_S \circ \nu) \subseteq \mu \circ \nu$. Then by Theorem 3.8[10], $\mu \circ \nu$ is a fuzzy bi-ideal. □

Theorem 3.3. *Let B be a bi-ideal of a Γ-semigroup S. Then for any $t \in (0, 1]$, ∃ a fuzzy bi-ideal μ of S such that $\mu_t = B$.*

Proof. Let B be a bi-ideal of a Γ-semigroup S. Let $t \in (0, 1]$. Let us define a mapping $\mu : S \to [0, 1]$ by

$$\mu(x) = \begin{cases} t & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

i.e., $\mu = tC_B$ (*cf.* Definition 2.6). Then μ is a nonempty fuzzy subset of S. Let $x, y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$. If $x \notin B$ or $y \notin B$, then $\mu(x) = 0$ or $\mu(y) = 0$. Hence $\mu(x\gamma y) \geq 0 = \min\{\mu(x), \mu(y)\}$. If $x, y \in B$, then $\mu(x) = t = \mu(y)$ and $x\gamma y \in B$. Hence $\mu(x\gamma y) = t = \min\{\mu(x), \mu(y)\}$. Therefore μ is a fuzzy subsemigroup of S. Also if $x \notin B$ or $z \notin B$, then $\mu(x) = 0$ or $\mu(z) = 0$. So $\mu(x\alpha y\beta z) \geq 0 = \min\{\mu(x), \mu(z)\}$. If $x, z \in B$, then $\mu(x) = t = \mu(z)$ and $x\alpha y\beta z \in B$. So $\mu(x\alpha y\beta z) = t = \min\{\mu(x), \mu(z)\}$. Therefore μ is a fuzzy bi-ideal of S. Also $\mu_t = \{x \in S : \mu(x) \geq t\}$. If $x \in B$, then $\mu(x) = t$ which implies that $x \in \mu_t$. So $B \subseteq \mu_t$. If $a \in \mu_t$, then $\mu(a) \geq t$. Then by definition of μ, $\mu(a) = t$, whence $a \in B$. Hence $\mu_t \subseteq B$. Consequently, $\mu_t = B$. □

Note 3.4. In above theorem if we take $t = 1$, then $\mu = C_B$. From the above proof it also follows that for any bi-ideal B of S and $t \in (0, 1]$, tC_B is a fuzzy bi-ideal of S.

Definition 3.5. Let A be a subset of a Γ-semigroup S and $B(A)$ denote the intersection of all bi-ideals of S containing A. Then $B(A)$ is a bi-ideal of S and is said to be the bi-ideal generated by A.

By routine verification we obtain the following result.

Lemma 3.6. *Let S be a Γ-semigroup and $a \in S$. Then the bi-ideal generated by a is $B(a) = \{a\} \cup a\Gamma a \cup a\Gamma S\Gamma a$.*

The fuzzy analogue of the preceding definition is as follows.

Definition 3.7. Let μ be a nonempty fuzzy subset of a Γ-semigroup S and $B(\mu)$ denotes the intersection of all fuzzy bi-ideals of S containing μ. Then $B(\mu)$ is a fuzzy bi-ideal of S and is said to be the fuzzy bi-ideal generated by μ.

The following result characterizes a fuzzy bi-ideal generated by a fuzzy point.

Proposition 3.8. *Let a_t be a fuzzy point of a Γ-semigroup S. Then the fuzzy bi-ideal generated by a_t is given by*

$$B(a_t)(x) = \begin{cases} t & \text{if } x \in B(a) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let μ be a fuzzy subset of the Γ-semigroup S defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in B(a) \\ 0 & \text{otherwise} \end{cases}$$

Let $x, y \in S$ and $\gamma \in \Gamma$. If $x, y \in B(a)$ then $x\gamma y \in B(a)$ and so $\mu(x\gamma y) = t = \min\{\mu(x), \mu(y)\}$. If $x \notin B(a)$ or $y \notin B(a)$ then $\min\{\mu(x), \mu(y)\} = 0 \leq \mu(x\gamma y)$. Hence $\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$ and $\gamma \in \Gamma$. Thus μ is a fuzzy subsemigroup of S.
Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. If $x, z \in B(a)$, $x\alpha y\beta z \in B(a)$. So $\mu(x\alpha y\beta z) = t = \min\{\mu(x), \mu(z)\}$. If $x \notin B(a)$ or $z \notin B(a)$ then $\min\{\mu(x), \mu(z)\} = 0 \leq \mu(x\alpha y\beta z)$. Hence $\mu(x\alpha y\beta z) \geq \min\{\mu(x), \mu(z)\}$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Hence μ is a fuzzy bi-ideal of S.
Also $a_t(x) \leq \mu(x)$ for all $x \in S$. So $B(a_t) \subseteq \mu$. Let ν be a fuzzy bi-ideal of S such that $a_t \subseteq \nu$. Then $\nu(a) \geq t = \mu(a)$. Let $x \in S$ be such that $x \neq a$. If $x \notin B(a)$, $\mu(x) = 0 \leq \nu(x)$. If $x \in B(a)$, $x = a\gamma a$ or $x = a\alpha y\beta a$ for some $y \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Then $\nu(x) \geq \min\{\nu(a), \nu(a)\} = \nu(a) \geq t = \mu(x)$. So for all $x \in S$, $\nu(x) \geq \mu(x)$. Hence $\mu \subseteq \nu$. Since ν is an arbitrary fuzzy bi-ideal of S containing a_t, $\mu \subseteq B(a_t)$. Hence $\mu = B(a_t)$. □

The above proposition and the following theorem play vital roles in obtaining the important results such as Theorems 4.2, 4.8, 4.16, 4.17, 4.22.

Theorem 3.9. *Let S be a Γ-semigroup. Then following are equivalent:*
(i) μ is a fuzzy bi-ideal of S.
(ii) μ_t is a bi-ideal of S for all $t (\neq 0) \in Im(\mu)$, where $Im(\mu) = \{\mu(x) : x \in S\}$.
(iii) μ_t is a bi-ideal of S for all $t \in (0, 1]$, provided μ_t is nonempty.

Proof. $(i) \Leftrightarrow (ii)$ follow from Theorem 3.2[10].
$(ii) \Rightarrow (iii)$ Let (ii) be true and μ_t is nonempty for some $t \in (0, 1]$. Let $t_0 = \min\{\mu(a) : a \in \mu_t\}$. Then $\mu_{t_0} = \mu_t$ and so μ_t is a bi-ideal of S.
$(iii) \Rightarrow (ii)$ Straightforward. □

4. Prime fuzzy bi-ideal

In this section we obtain main results such as characterization of regular and intra regular Γ-semigroups in terms of prime and semiprime fuzzy bi-ideals.

Definition 4.1 ([1])**.** A fuzzy bi-ideal μ of a Γ-semigroup S is called a *prime fuzzy bi-ideal* of S if for any two fuzzy bi-ideals ν and δ of S, $\nu \circ \delta \subseteq \mu$ implies that $\nu \subseteq \mu$ or $\delta \subseteq \mu$.

Theorem 4.2. *Let I be a nonempty subset of a Γ-semigroup S. Then I is a prime bi-ideal of S if and only if the characteristic function C_I of I is a prime fuzzy bi-ideal of S.*

Proof. Let I be a prime bi-ideal of S. Then by Theorem 3.1[10], C_I is a fuzzy bi-ideal of S. Let μ and ν be two fuzzy bi-ideals of S such that $\mu \circ \nu \subseteq C_I$ and $\mu \nsubseteq C_I$. Then there exists $a_t \subseteq \mu$ such that $a_t \nsubseteq C_I$ which implies that $a \notin I$. Let $b_r \subseteq \nu$. Then in view of Definition 3.7 and Definition 2.1, $B(a_t) \circ B(b_r) \subseteq \mu \circ \nu$. Hence $B(a_t) \circ B(b_r) \subseteq C_I$. Now for all $x \in S$, in view of Proposition 2.7 and Proposition 3.8,

$$(B(a_t) \circ B(b_r))(x) = \begin{cases} \min\{t, r\} & \text{if } x \in B(a)\Gamma B(b) \\ 0 & \text{otherwise} \end{cases}$$

Hence for $x \in B(a)\Gamma B(b)$, $C_I(x) \geq \min\{t, r\}$ whence $x \in I$. Consequently, $B(a)\Gamma B(b) \subseteq I$. This, I being a prime bi-ideal of S, implies that $B(a) \subseteq I$ or $B(b) \subseteq I$. Since $a \notin I$, $B(a) \nsubseteq I$. Hence $B(b) \subseteq I$ whence $b_r \subseteq C_I$. Hence $\nu \subseteq C_I$. Consequently, C_I is a prime fuzzy bi-ideal of S.

Conversely, let C_I be a prime fuzzy bi-ideal of S and A, B be two bi-ideals of S such that $A\Gamma B \subseteq I$.Then by Theorem 3.1[10], C_A and C_B are fuzzy bi-ideals of S. Hence by Proposition 2.5, $C_{A\Gamma B} \subseteq C_I$ whence $C_A \circ C_B \subseteq C_I$. Hence by hypothesis $C_A \subseteq C_I$ or $C_B \subseteq C_I$ which implies that $A \subseteq I$ or $B \subseteq I$. Hence I is a prime bi-ideal of S. □

Theorem 4.3. *Let μ be a nonempty fuzzy subset of a Γ-semigroup S. If μ is a prime fuzzy bi-ideal of S then μ_t is a prime bi-ideal of S for all nonzero $t \in Im(\mu)$.*

Proof. Let μ be a prime fuzzy bi-ideal of S and $(0 \neq)t \in Im(\mu)$. Then by Theorem 3.2[10], μ_t is a bi-ideal of S. Let B_1 and B_2 be two bi-ideals of S such that $B_1\Gamma B_2 \subseteq \mu_t$. Then by Theorem 3.3 and Note 3.4, $tC_{B_1} = \nu(\text{say})$ and $tC_{B_2} = \delta(\text{say})$ are fuzzy bi-ideals of S. Now for all $x \in S$, by a similar argument as applied in the proof of preceding theorem,

$$(\nu \circ \delta)(x) = \begin{cases} t & \text{if } x \in B_1\Gamma B_2 \\ 0 & \text{otherwise} \end{cases}$$

Hence $\nu \circ \delta \subseteq \mu$. So μ being a prime fuzzy bi-ideal, $\nu \subseteq \mu$ or $\delta \subseteq \mu$.

Let $\nu \subseteq \mu$. Then for $x \in B_1$, $\nu(x) = t \leq \mu(x)$ which implies $x \in \mu_t$. Therefore $B_1 \subseteq \mu_t$. Similarly if $\delta \subseteq \mu$ then $B_2 \subseteq \mu_t$. Hence μ_t is a prime bi-ideal of S. □

Note 4.4. The following example illustrates that the converse of the above theorem is not true in general.

Example 4.5. Let $S = \{a, b, c\}$ and $\Gamma = \{\gamma, \delta\}$, where γ, δ are defined on S with the following Cayley tables:

γ	a	b	c
a	b	b	b
b	b	b	b
c	b	b	b

and

δ	a	b	c
a	b	b	b
b	b	b	b
c	b	b	c

Then S is a Γ-semigroup. Let $B = \{a, b\}$ and $t \in (0, 1)$. Then B is a prime bi-ideal of S and so tC_B is a fuzzy bi-ideal of S. Now $(tC_B)_t = B$ is a prime bi-ideal of S but tC_B is not a prime fuzzy bi-ideal of S.

Definition 4.6 ([1])**.** A fuzzy bi-ideal μ of a Γ-semigroup S is called a *strongly prime fuzzy bi-ideal* of S if for any two fuzzy bi-ideals ν and δ of S, $\nu \circ \delta \cap \delta \circ \nu \subseteq \mu$, then $\nu \subseteq \mu$ or $\delta \subseteq \mu$.

Theorem 4.7. *Let I be a nonempty subset of a Γ-semigroup S. Then I is a strongly prime bi-ideal of S if and only if the characteristic function C_I of I is a strongly prime fuzzy bi-ideal of S.*

Proof. Let I be a strongly prime bi-ideal of S. Then by Theorem 3.1[10], C_I is a fuzzy bi-ideal of S. Also let μ and ν be two fuzzy bi-ideals of S such that $\mu \circ \nu \cap \nu \circ \mu \subseteq C_I$ and $\mu \nsubseteq C_I$. Then there exists $a_t \subseteq \mu$ such that $a_t \nsubseteq C_I$ whence $C_I(a) < t$. Hence $C_I(a) = 0$ whence $a \notin I$. Let $b_r \subseteq \nu$. Then in view of Definition 3.7, $(B(a_t) \circ B(b_r)) \cap (B(b_r) \circ B(a_t)) \subseteq \mu \circ \nu \cap \nu \circ \mu \subseteq C_I$. Then for all $x \in S$,

$$((B(a_t) \circ B(b_r)) \cap (B(b_r) \circ B(a_t)))(x) = \begin{cases} \min\{t, r\} & \text{if } x \in B(a)\Gamma B(b) \cap B(b)\Gamma B(a) \\ 0 & \text{otherwise} \end{cases}$$

Hence for $x \in B(a)\Gamma B(b) \cap B(b)\Gamma B(a)$,

$$C_I(x) \geq ((B(a_t) \circ B(b_r)) \cap (B(b_r) \circ B(a_t)))(x) = \min\{t, r\} > 0.$$

Then $C_I(x) = 1$ whence $x \in I$. So $B(a)\Gamma B(b) \cap B(b)\Gamma B(a) \subseteq I$. So by the hypothesis, $B(a) \subseteq I$ or $B(b) \subseteq I$. Since $a \notin I$, $B(a) \nsubseteq I$. Hence $B(b) \subseteq I$ whence $b_r \subseteq C_I$ and so $\nu \subseteq C_I$. Hence C_I is a strongly prime fuzzy bi-ideal of S.

Conversely, let C_I be a strongly prime fuzzy bi-ideal of S. Also let A and B be two bi-ideals of S such that $A\Gamma B \cap B\Gamma A \subseteq I$. Then by Theorem 3.1[10], C_A and C_B are fuzzy bi-ideals of S. Hence by Proposition 2.5, $C_{A\Gamma B \cap B\Gamma A} \subseteq C_I$ whence $(C_A \circ C_B) \cap (C_B \circ C_A) \subseteq C_I$. So by the hypothesis, $C_A \subseteq C_I$ or $C_B \subseteq C_I$. Hence $A \subseteq I$ or $B \subseteq I$. Consequently, I is a strongly prime bi-ideal of S. □

Theorem 4.8. *Let μ be a nonempty fuzzy subset of a Γ-semigroup S. If μ is a strongly prime fuzzy bi-ideal of S then μ_t is a strongly prime bi-ideal of S for all $t(\neq 0) \in Im(\mu)$.*

Proof. Let μ be a strongly prime fuzzy bi-ideal of S and $t(\neq 0) \in Im(\mu)$. Then by Theorem 3.2[10], μ_t is a bi-ideal of S. Let B_1 and B_2 be two bi-ideals of S such that $(B_1\Gamma B_2) \cap (B_2\Gamma B_1) \subseteq \mu_t$. Now $tC_{B_1} = \nu$(say) and $tC_{B_2} = \delta$(say) are two fuzzy bi-ideals of S. Now for $x \in S$,

$$((\nu \circ \delta) \cap (\delta \circ \nu))(x) = \begin{cases} t & \text{if } x \in (B_1\Gamma B_2) \cap (B_1\Gamma B_2) \\ 0 & \text{otherwise} \end{cases}$$

Hence for any $x \in S$, $((\nu \circ \delta) \cap (\delta \circ \nu))(x) \leq \mu(x)$. Consequently, $(\nu \circ \delta) \cap (\delta \circ \nu) \subseteq \mu$. Then by the hypothesis, $\nu \subseteq \mu$ or $\delta \subseteq \mu$.

If $\nu \subseteq \mu$, for $x \in B_1$, $\nu(x) = t \leq \mu(x)$ whence $x \in \mu_t$. Hence $B_1 \subseteq \mu_t$. Similarly if $\delta \subseteq \mu$ then $B_2 \subseteq \mu_t$. Hence μ_t is a strongly prime bi-ideal of S. □

Note 4.9. The converse of the above theorem is not true in general. For instance, in Example 4.5, B is also a strongly prime bi-ideal of S. Then $(tC_B)_t = B$ is a strongly prime bi-ideal of S but tC_B is not a strongly prime fuzzy bi-ideal of S.

Note 4.10. It is easy to see that[1] every strongly prime fuzzy bi-ideal of a Γ-semigroup S is a prime fuzzy bi-ideal of S. The following examples illustrates that the converse is not true.

Example 4.11. Let $S = \{1, 2, 3, 4\}$ and $\Gamma = \{\gamma, \delta\}$, where γ, δ is defined on S with the following Cayley tables:

γ	1	2	3	4
1	1	2	2	2
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4

and

δ	1	2	3	4
1	2	2	2	2
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4

Then S is a Γ-semigroup. Let us take a fuzzy subset μ on S such that $\mu(1) = \mu(2) > \mu(3) > \mu(4)$. Then μ is a prime fuzzy bi-ideal of S, but not a strongly prime fuzzy bi-ideal of S.

Example 4.12. Let $S = \{a, b, c\}$ and $\Gamma = \{\gamma, \delta\}$, where γ, δ is defined on S with the following Cayley tables:

γ	a	b	c
a	a	b	b
b	b	b	b
c	c	c	c

and

δ	a	b	c
a	b	b	b
b	b	b	b
c	c	c	c

Then S is a Γ-semigroup. Let us take a fuzzy subset μ on S such that $\mu(a) = \mu(b) > \mu(c)$. Then μ is a prime fuzzy bi-ideal of S, but not a strongly prime fuzzy bi-ideal of S.

If we take a fuzzy subset ν of S such that $\nu(a) = \nu(b) < \nu(c)$, then ν is prime fuzzy bi-ideal, but not a strongly prime fuzzy bi-ideal of S.

Definition 4.13 ([1])**.** A fuzzy bi-ideal μ of a Γ-semigroup S is called *semiprime fuzzy bi-ideal* of S if for any fuzzy bi-ideal ν of S, $\nu \circ \nu \subseteq \mu$ implies $\nu \subseteq \mu$.

Note 4.14. Clearly every prime fuzzy bi-ideal of a Γ-semigroup S is a semiprime fuzzy bi-ideal of S[1]. The following example illustrates that the converse is not true.

Example 4.15. Let $S = \{a, b, c\}$ and $\Gamma = \{\gamma, \delta\}$, where γ, δ is defined on S with the following Cayley tables:

γ	a	b	c
a	a	b	c
b	b	a	c
c	c	c	c

and

δ	a	b	c
a	b	a	c
b	a	b	c
c	c	c	c

Then S is a Γ-semigroup. Let us take a fuzzy subset μ on S such that $\mu(a) = \mu(b) < \mu(c)$. Then μ is a semiprime fuzzy bi-ideal of S, but not a prime fuzzy bi-ideal of S.

Theorem 4.16. *Let A be a subset of a Γ-semigroup S. Then A is a semiprime bi-ideal of S if and only if the characteristic function C_A of A is a semiprime fuzzy bi-ideal of S.*

Proof. Let A be a semiprime bi-ideal of S. Then by Theorem 3.1[10], C_A is a fuzzy bi-ideal of S. Let μ be a fuzzy bi-ideal of S such that $\mu \circ \mu \subseteq C_A$. If possible let, $\mu \nsubseteq C_A$. Then there exists $a_t \subseteq \mu$ such that $a_t \nsubseteq C_A$. Then $C_A(a) < t$ whence $C_A(a) = 0$ which implies $a \notin A$. Now in view of Definition 3.7 and Definition 2.1, $B(a_t) \circ B(a_t) \subseteq \mu \circ \mu \subseteq C_A$. Also (*cf.* Proposition 3.8 and Proposition 2.7) for all $x \in S$,

$$(B(a_t) \circ B(a_t))(x) = \begin{cases} t & \text{if } x \in B(a)\Gamma B(a) \\ 0 & \text{otherwise} \end{cases}$$

Hence for $x \in B(a)\Gamma B(a)$, $C_A(x) \geq (B(a_t) \circ B(a_t))(x) = t$ whence $C_A(x) = 1$ and so $x \in A$. Hence $B(a)\Gamma B(a) \subseteq A$. So by hypothesis $B(a) \subseteq A$ which implies $a \in A$, which is a contradiction. Hence $\mu \subseteq C_A$. Hence C_A is a semiprime fuzzy bi-ideal of S.

Conversely, let C_A be a semiprime fuzzy bi-ideal of S. Then by Theorem 3.1[10], A is a bi-ideal of S. Let B be a bi-ideal of S such that $B\Gamma B \subseteq A$. Then C_B is a fuzzy bi-ideal(*cf.* Theorem 3.1[10]) and so by Proposition 2.5, $C_{B\Gamma B} \subseteq C_A$ whence $C_B \circ C_B \subseteq C_A$. Then by hypothesis $C_B \subseteq C_A$ which implies $B \subseteq A$. Hence A is a semiprime bi-ideal of S. □

Theorem 4.17. *Let μ be a nonempty fuzzy subset of a Γ-semigroup S. Then μ is a semiprime fuzzy bi-ideal of S if and only if μ_t is a semiprime bi-ideal of S for all $t(\neq 0) \in Im(\mu)$.*

Proof. Let μ be a semiprime fuzzy bi-ideal of S and $t(\neq 0) \in Im(\mu)$. Then by Theorem 3.2[10], μ_t is a bi-ideal of S. Let B be a bi-ideal of S such that $B\Gamma B \subseteq \mu_t$. Now by Theorem 3.3 and Note 3.4, $tC_B = \gamma$ is a fuzzy bi-ideal of S. Now for all $x \in S$,

$$(\nu \circ \nu)(x) = \begin{cases} t & \text{if } x \in B\Gamma B \\ 0 & \text{otherwise} \end{cases}$$

Hence $\nu \circ \nu \subseteq \mu$ whence by hypothesis $\nu \subseteq \mu$. For $x \in B$, $\nu(x) = t \leq \mu(x)$ which implies $x \in \mu_t$. Therefore $B \subseteq \mu_t$. Hence μ_t is a semiprime bi-ideal of S.

Conversely, let μ_t be semiprime bi-ideal of S for all $t(\neq 0) \in Im(\mu)$. Then by Theorem 3.2[10], μ is a fuzzy bi-ideal of S. Let ν be a fuzzy bi-ideal of S such that $\nu \circ \nu \subseteq \mu$. Also let $x_t \subseteq \nu$. Then by Definition 3.7, $B(x_t) \subseteq \nu$. Hence $B(x_t) \circ B(x_t) \subseteq \nu \circ \nu \subseteq \mu$. Now for all $a \in S$,

$$(B(x_t) \circ B(x_t))(a) = \begin{cases} t & \text{if } a \in B(x)\Gamma B(x) \\ 0 & \text{otherwise} \end{cases}$$

Hence for $a \in B(x)\Gamma B(x)$, $\mu(a) \geq (B(x_t) \circ B(x_t))(a) = t$ whence $a \in \mu_t$. So $B(x)\Gamma B(x) \subseteq \mu_t$. Then by hypothesis, $B(x) \subseteq \mu_t$, whence $x \in \mu_t$ *i.e.* $\mu(x) \geq t$. Hence $x_t \subseteq \mu$. Consequently, $\nu \subseteq \mu$ and μ is a semiprime fuzzy bi-ideal of S. □

Definition 4.18. A fuzzy bi-ideal μ of a Γ-semigroup S is called *irreducible* (*strongly irreducible*) fuzzy bi-ideal of S if for any two fuzzy bi-ideals ν and δ of S, $\nu \cap \delta = \mu$(resp. $\nu \cap \delta \subseteq \mu$) implies that $\nu = \mu$ or $\delta = \mu$(resp. $\nu \subseteq \mu$ or $\delta \subseteq \mu$).

Note 4.19. Every strongly irreducible fuzzy bi-ideal of a Γ-semigroup S is an irreducible fuzzy bi-ideal of S.

Proposition 4.20. *Let A be a bi-ideal of a Γ-semigroup S. Then A is irreducible if and only if the characteristic function C_A of A is irreducible.*

Proof. Let A be an irreducible bi-ideal of S. Then by Theorem 3.1[10], C_A is a fuzzy bi-ideal of S. Let $C_A = \mu \cap \nu$ for two fuzzy bi-ideals μ and ν of S. Then $C_A \subseteq \mu$ and $C_A \subseteq \nu$. If possible let $x_t \subseteq \mu$ and $y_r \subseteq \nu$, but $x_t, y_r \nsubseteq C_A$. Without loss of generality let us assume that $t = \min\{t, r\}$. Now by Proposition 2.9, $(C_A)_t = (\mu \cap \nu)_t = \mu_t \cap \nu_t$ whence $A = \mu_t \cap \nu_t$. As μ_t and ν_t are bi-ideals(*cf.* Theorem 3.9), so by the hypothesis $A = \mu_t$ or $A = \nu_t$. If $A = \mu_t$, $C_A(x) \geq t$ as by our assumption $x \in \mu_t$. Then $x_t \subseteq C_A$ which is a contradiction. If $A = \nu_t$, $C_A(y) \geq t$ as by our assumption $y \in \nu_t$. Then $C_A(y) = 1 \geq r$ whence $y_r \subseteq C_A$ which also gives a contradiction. Thus $C_A = \mu$ or $C_A = \nu$. Hence C_A is irreducible fuzzy bi-ideal of S. Conversely, let C_A be irreducible fuzzy bi-ideal of S. Then by Theorem 3.1[10], A is a bi-ideal of S. Let $A = B \cap D$ for two bi-ideals B, D of S. Then C_B and C_D are fuzzy bi-ideals of S(*cf.* Theorem 3.1[10]). Then by Proposition 2.5, $C_A = C_{B\cap D} = C_B \cap C_D$. Then by the hypothesis $C_A = C_B$ or $C_A = C_D$ whence $A = B$ or $A = D$. Hence A is an irreducible bi-ideal of S. □

Theorem 4.21. *Let μ be a fuzzy bi-ideal of a Γ-semigroup S and $x_t \nsubseteq \mu$ for a fuzzy point x_t of S. Then there exists an irreducible fuzzy bi-ideal δ of S such that $\mu \subseteq \delta$ and $x_t \nsubseteq \delta$.*

Proof. Let $\mathcal{F} = \{\mu_i : i \in \Delta\}$ be the collection of all fuzzy bi-ideals μ_i such that $\mu \subseteq \mu_i$ and $x_t \nsubseteq \mu_i$. Then $\mathcal{F}$ is nonempty as $\mu \in \mathcal{F}$. Also $\mathcal{F}$ is partially ordered under inclusion. Let $\mathcal{G} = \{\mu_i : i \in \Lambda\}$ be a totally ordered subcollection of $\mathcal{F}$. Then $\underset{i\in\Lambda}{\cup} \mu_i \in \mathcal{F}$ and is an upper bound for $\mathcal{G}$. Then by Zorn's lemma $\mathcal{F}$ has a maximal element μ_0(*say*). Then μ_0 is a fuzzy bi-ideal of S such that $x_t \nsubseteq \mu_0$ and $\mu \subseteq \mu_0$. Now we prove that μ_0 is an irreducible fuzzy bi-ideal of S. Let ν and η be two fuzzy bi-ideals of S such that $\mu_0 = \nu \cap \eta$. If possible let $\nu \neq \mu_0$ and $\eta \neq \mu_0$. Then $\mu_0 \subset \nu$ and $\mu_0 \subset \eta$. Since μ_0 is the maximal element of $\mathcal{F}$, $x_t \subseteq \nu$ and $x_t \subseteq \eta$. But then $x_t \subseteq \nu \cap \eta = \mu_0$, which is a contradiction. Hence $\nu = \mu_0$ or $\eta = \mu_0$. Consequently, μ_0 is an irreducible fuzzy bi-ideal of S such that $\mu \subseteq \mu_0$ and $x_t \nsubseteq \mu_o$. □

Theorem 4.22. *Let S be a Γ-semigroup. Then the following are equivalent:*

(i) *S is both regular and intra-regular.*

(ii) *$\mu \circ \mu = \mu$ for every fuzzy bi-ideal μ of S.*

(iii) *$\nu \cap \delta = \nu \circ \delta \cap \delta \circ \nu$ for any fuzzy bi-ideals ν and δ of S such that the intersection is nonempty.*

(iv) *Each fuzzy bi-ideal of S is semiprime fuzzy bi-ideal.*

(v) *Each proper fuzzy bi-ideal μ of S is the intersection of all irreducible semiprime fuzzy bi-ideals of S which contains μ.*

Proof. $(i) \Rightarrow (ii)$ Let (i) be true and μ be a fuzzy bi-ideal of S. Then μ_t is a bi-ideal of S for $t \in (0,1]$(*cf.* Theorem 3.9). Then by Theorem 3.9[13], $\mu_t \Gamma \mu_t = \mu_t$. Then in view of Proposition 2.9, $(\mu \circ \mu)_t = \mu_t$ for any $t \in (0,1]$ whence $\mu \circ \mu = \mu$.

$(ii) \Rightarrow (iii)$ Let (ii) be true. Let B be a bi-ideal of S. Then C_B is a fuzzy bi-ideal of S (*cf.* Theorem 3.1[8]). Then by (ii), $C_B \circ C_B = C_B$. Hence in view of Proposition 2.5, $B\Gamma B = B$. Let ν and δ be two fuzzy bi-ideals of S and $t \in (0,1]$. Then ν_t and δ_t are two bi-ideals of S(*cf.* Theorem 3.9). Then by Theorem 3.9[13], $\nu_t \cap \delta_t = \nu_t \Gamma \delta_t \cap \delta_t \Gamma \nu_t$. Hence for all $t \in (0,1]$, $(\nu \cap \delta)_t = (\nu \circ \delta)_t \cap (\delta \circ \nu)_t = (\nu \circ \delta \cap \delta \circ \nu)_t$, whence $\nu \cap \delta = \nu \circ \delta \cap \delta \circ \nu$(*cf.* Proposition 2.9).

$(iii) \Rightarrow (iv)$ Let (iii) be true. Let B_1 and B_2 be any two bi-ideals of S. Then C_{B_1} and C_{B_2} are fuzzy bi-ideals of S(*cf.* Theorem 3.1[8]). Then by (iii), $C_{B_1} \cap C_{B_2} = C_{B_1} \circ C_{B_2} \cap C_{B_2} \circ C_{B_1}$. Then by repeated application of Proposition 2.5, $C_{B_1 \cap B_2} = C_{B_1 \Gamma B_2} \cap C_{B_2 \Gamma B_1} = C_{B_1 \Gamma B_2 \cap B_2 \Gamma B_1}$, whence $B_1 \cap B_2 = B_1 \Gamma B_2 \cap B_2 \Gamma B_1$. Hence by Theorem 3.9[13], each bi-ideal is a semiprime bi-ideal. This together with Theorem 3.9 implies that if μ is a fuzzy bi-ideal of S, then every level subset μ_t is a semiprime bi-ideal of S. Hence by Theorems 3.9 and 4.17, μ is a semiprime fuzzy bi-ideal of S.

$(iv) \Rightarrow (v)$ Let (iv) be true. Let μ be a proper fuzzy bi-ideal of S. Then there exists a fuzzy point $x_t \nsubseteq \mu$. Also let $\mathcal{F} = \{\mu_i : i \in \Delta\}$ be the collection of all irreducible semiprime fuzzy bi-ideals of S containing μ. Then by Theorem 4.21, $\exists$ an irreducible fuzzy bi-ideal μ_0(say) containing μ. Also by hypothesis μ_0 is semiprime. So $\mathcal{F}$ is nonempty. Then $\mu \subseteq \bigcap\limits_{i \in \Delta} \mu_i$. If possible, let $x_t \subseteq \bigcap\limits_{i \in \Delta} \mu_i$ but $x_t \nsubseteq \mu$. Then by Theorem 4.21, there is an irreducible fuzzy bi-ideal δ containing μ but not containing x_t. Also by the hypothesis δ is semiprime. So a contradiction arises as $\delta \in \mathcal{F}$ and $x_t \subseteq \bigcap\limits_{i \in \Delta} \mu_i$. Hence $\bigcap\limits_{i \in \Delta} \mu_i = \mu$.

$(v) \Rightarrow (i)$ Let (v) be true and B be a proper bi-ideal of S. Then C_B is a proper fuzzy bi-ideal of S(*cf.* Theorem 3.1[8]). Hence $C_B \circ C_B$ is a fuzzy bi-ideal(*cf.* Proposition 3.2) of S. Then by the hypothesis $C_B \circ C_B = \bigcap\limits_{i \in \Delta} \mu_i$, where $\mathcal{F} = \{\mu_i : i \in \Delta\}$ is the collection of all irreducible semiprime fuzzy bi-ideals of S containing $C_B \circ C_B$. Then $C_B \circ C_B \subseteq \mu_i$ for all $i \in \Delta$. Since each μ_i is semiprime, $C_B \subseteq \mu_i$ for all $i \in \Delta$ whence $C_B \subseteq \bigcap\limits_{i \in \Delta} \mu_i = C_B \circ C_B$. Also C_B being a bi-ideal, we have $C_B \circ C_B \subseteq C_B$. Hence $C_B \circ C_B = C_B$ whence by Proposition 2.5, $B\Gamma B = B$. Consequently, by Theorem 3.9[13], (i) follows. $\square$

Remark 4.23. In fact the above result is due to Bashir et al(Theorem 5.8[1]). We have revisited it by providing a new proof by using the characteristic function criterion(Theorem 4.16), level subset criterion(Theorem 4.17) and the notion of fuzzy point.

Theorem 4.24. [9] *For any two fuzzy ideals μ and ν of a Γ-semigroup S, $\mu \circ \nu \subseteq \mu \cap \nu$. Moreover, if S is regular then $\mu \circ \nu = \mu \cap \nu$.*

Definition 4.25. A fuzzy ideal μ of a Γ-semigroup S is called a *prime fuzzy ideal* of S if for any two fuzzy ideals ν and δ of S, $\nu \circ \delta \subseteq \mu$, then $\nu \subseteq \mu$ or $\delta \subseteq \mu$.

Definition 4.26. A fuzzy ideal μ of a Γ-semigroup S is called an *irreducible* (*strongly irreducible*) of S if for any two fuzzy ideals ν and δ of S, $\nu \cap \delta = \mu$(resp. $\nu \cap \delta \subseteq \mu$), then $\nu = \mu$ or $\delta = \mu$(resp. $\nu \subseteq \mu$ or $\delta \subseteq \mu$).

Note 4.27. Every strongly irreducible fuzzy ideal of a Γ-semigroup S is an irreducible fuzzy ideal of S.

Note 4.28. From Definitions 4.25 and 4.26 and by Theorem 4.24 it is clear that every prime fuzzy ideal of a Γ-semigroup is strongly irreducible fuzzy ideal. For a regular Γ-semigroup they coincide.

Definition 4.29 ([10])**.** A Γ-semigroup S is called *left(right) duo* if every left(resp. right) ideal of S is an ideal of S. A Γ-semigroup S is called *duo* if it is a left as well as right duo.

In [10], we have seen that in a regular duo Γ-semigroup the notions of fuzzy ideal and fuzzy bi-ideal coincide. The following theorem shows that the situation is similar for prime fuzzy ideal and prime fuzzy bi-ideal.

Theorem 4.30. *Let S be a regular duo Γ-semigroup. Then for any nonempty fuzzy subset μ of S the following are equivalent.*
(i) *μ is a prime fuzzy ideal of S.*
(ii) *μ is a prime fuzzy bi-ideal of S.*
(iii) *μ is a strongly irreducible fuzzy ideal of S.*

Proof. $(i) \Rightarrow (ii)$ Let μ be a prime fuzzy ideal of S. Then μ is a fuzzy ideal of S. Hence μ is a fuzzy bi-ideal of S(*cf.* Theorem 4.5[10]). Let ν and δ be two fuzzy bi-ideals of S such that $\nu \circ \delta \subseteq \mu$. Then by Theorem 4.5[10], ν and δ are both fuzzy ideals of S. Therefore by (i) and Definition 4.25, $\nu \subseteq \mu$ or $\delta \subseteq \mu$. Hence by Definition 4.1, μ is a prime fuzzy bi-ideal of S.

The proof of $(ii) \Rightarrow (i)$ follows in a similar manner as above by repeated application of Theorem 4.5[10].

Equivalence of (i) and (iii) follows in view of Note 4.28. □

In view of Theorem 5.11[1], we obtain the following result as an easy consequence of the above Theorem.

Corollary 4.31. *Suppose S is a duo Γ-semigroup such that the set of all fuzzy bi-ideals is totally ordered by inclusion. Then S is both regular and intra regular if and only if any two of the following sets coincide.*
(i) *The set of all fuzzy bi-ideals of S.*
(ii) *The set of all prime fuzzy bi-ideals of S.*
(iii) *The set of all prime fuzzy ideals of S.*
(iv) *The set of all strongly irreducible fuzzy ideals of S.*

Definition 4.32. A subsemigroup A of a Γ-semigroup S is called *normal* if $a\Gamma A = A\Gamma a$ for all $a \in S$.

Proposition 4.33. *Every normal Γ-semigroup is duo.*

Proof. Let S be a normal Γ-semigroup. Then $a\Gamma S = S\Gamma a \ \ \forall\, a \in S$. Then for any left ideal A of S, $A\Gamma S = S\Gamma A$ whence A is a right ideal of S. Thus A is an ideal. Similarly, we can prove that any right ideal is an ideal in S. Hence S is duo. □

Remark 4.34. In view of Proposition 4.33, we see that the conclusions of Theorem 4.30 and Corollary 4.31 are also true for regular normal Γ-semigroups.

5. Conclusions

Many results of Bashir et al[1] such as Theorems 5.9, 5.10, 5.11 are obtained mainly using their Theorem 5.8 which we have revisited here (*cf.* Theorem 4.22 and Remark 4.23) by using characteristic function criterion and level subset criterion (*cf.* Theorems 4.16, 4.17) of semiprime fuzzy bi-ideal. This indicates that had the level subset criterion been true for other notions such as prime, strongly prime fuzzy bi-ideal (*cf.* Theorems 4.3, 4.8, Example 4.5 and Note 4.9) then more results would have been obtained. So it will be nice if the notion of prime, strongly prime fuzzy bi-ideal can be generalized so as to satisfy the level subset criterion which in turn would develop the theory further.

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