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Classes of generalized difference ideal convergent sequence of fuzzy numbers

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Abstract. An ideal I is a family of subsets of positive integers $\mathbb{N}$ which is closed under finite unions and subsets of its elements. In this article we introduce the classes of ideal convergent sequence of fuzzy numbers using a new generalized difference matrix $B^n_{(m)}$ and Orlicz function and study their basic facts. Also we investigate the different algebraic and topological properties of these classes of sequences .

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1. Introduction

The concept of statistical convergence is a generalization of the usual notion of convergence of real-valued sequences, that parallels the usual theory of convergence. For a subset E of $\mathbb{N}$ the asymptotic density of E, denoted by $\delta(E)$, is given by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n}|\{k \le n : k \in E\}|,$$

if this limit exists, where $|\{k \le n : k \in E\}|$ denotes the cardinality of the set $\{k \le n : k \in E\}$. A sequence (x_k) is statistically convergent to ℓ (see [17]) if

$$\delta(\{k \in \mathbb{N} : |x_k - \ell| \ge \varepsilon\}) = 0,$$

for every $\varepsilon > 0$. In this case ℓ is called the statistical limit of the sequence (x_k). Schoenberg [51] studied some basic properties of statistical convergence and also studied the statistical convergence as a summability method. Fridy [18] gave characterizations of statistical convergence. Caserta et al.,[7] studied statistical convergence in function spaces while Caserta and Kočinac [8] investigated statistical exhaustivness. For more details on statistical convergence we refer to ([4, 43, 47, 52]) and references therein.

Kostyrko et. al [33] introduced the notion of I-convergence with the help of an admissible ideal I denotes the ideal of subsets of $\mathbb{N}$, which is a generalization of statistical convergence. Later on it was further investigated by Šalát et al. ([48, 49]), Tripathy and Hazarika ([55, 56, 57, 58]), Tripathy et al [59], Esi and Hazarika [12], Gowrisankar and Rajesh [19], Hatir and Rajesh [21], Hazarika ([23, 24, 25, 27]), Hazarika and Savas [22], Hazarika et al., [30], Kumar and Kumar [36], Kumar [37, 38], Savaş [50], Subramanian et al [53] and references therein. Hazarika [26] introduced the concept of generalized difference ideal convergent sequence in random 2-normed space and studied some interesting properties. Çakalli and Hazarika [6] introduced the new concept ideal quasi Cauchy sequences and studied some results in real analysis.

The concepts of fuzzy set and fuzzy set operations were first introduced by Zadeh [61] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarly relations and fuzzy orderings, fuzzy measures of fuzzy events. In fact the fuzzy set theory has become an area of active research for the last 45 years. To overcome the limitations induced by vagueness and uncertainty of real life data, neoclassical analysis [5] has been developed. It extends the scope and results of classical mathematical analysis objects; such as functions, sequences and series. On the other hand the concept of ordinary convergence of sequences of fuzzy numbers was firstly introduced by Matloka [42], where he proved some basic theorems for sequences of fuzzy numbers. Nanda [45] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers from a complete metric space. Kumar et al [39] introduced the notion limit points and cluster points of sequences of fuzzy numbers. Kumar and Kumar [35] introduced the ideal convergence of sequences of fuzzy numbers.

Throughout the article $w^F, \ell_\infty^F, c^F, c_0^F$ denote the classes of all, bounded, convergent and null sequence spaces of fuzzy real numbers, respectively.

We denote w, the set of all real sequences $x = (x_k)$. The difference sequence space is introduced by Kizmaz [32] as follows:

$$Z(\Delta) = \{(x_k) \in w : \Delta x_k \in Z\},$$

for $Z = \ell_\infty, c, c_0$ and $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

The idea of difference sequence was generalized by Colak and Et [9], Colak et al. [10], Et and Basarir [14], Et and Colak [15], Et et al. [16]. The operator $\Delta^n : w \to w$ is defined by

$$(\Delta^0 x_k) = x_k, (\Delta^1 x_k) = \Delta^1 x_k = x_k - x_{k+1},$$

$$(\Delta^n x_k) = \Delta^1(\Delta^{n-1} x_k), n \geq 2 \text{ for all } k \in \mathbb{N},$$

which is equivalent to the following binomial representation

$$\Delta^n x_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x_{k+\nu} \text{ for all } k \in \mathbb{N}.$$

Tripathy and Esi [54] introduced and studied the new type of generalized difference sequence spaces

$$Z(\Delta_m) = \{(x_k) \in w : \Delta_m x_k \in Z\},$$

for $Z = \ell_\infty, c, c_0$ where $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$ for all $k, m \in \mathbb{N}$.

Tripathy, et al [60] was further generalized this notion and introduced the following sequence spaces. For $n \geq 1$ and $m \geq 1$,

$$Z(\Delta_m^n) = \{(x_k) \in w : \Delta_m^n x_k \in Z\},$$

for $Z = \ell_\infty, c, c_0$. This generalized difference operator has the following binomial representation

$$\Delta_m^n x_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x_{k+m\nu} \text{ for all } k \in \mathbb{N}.$$

Dutta [11] introduced the following difference sequence spaces

$$Z(\Delta_{(m)}^n) = \{(x_k) \in w : \Delta_{(m)}^n x_k \in Z\} \text{ for all } n, m \in \mathbb{N},$$

for $Z = \ell_\infty, \bar{c}, \bar{c}_0$ where $\bar{c}, \bar{c}_0$ are the sets of statistically convergent and statistically null sequences, respectively, and $\Delta_{(m)}^n x = (\Delta_{(m)}^n x_k) = (\Delta_{(m)}^{n-1} x_k - \Delta_{(m)}^{n-1} x_{k-m})$ and $\Delta_{(m)}^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_{(m)}^n x_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x_{k-m\nu}.$$

Basar and Altay [1] introduced the generalized difference matrix $B(r, s) = (b_{nk}(r, s))$ which is a generalization of $\Delta_{(1)}^1$-difference operator as follows

$$b_{nk}(r, s) = \begin{cases} r, & \text{if } k = n; \\ s, & \text{if } k = n - 1; \\ 0, & \text{if } 0 \leq k < n - 1 \text{ or } k > n. \end{cases}$$

for all $k, n \in \mathbb{N}, r, s \in \mathbb{R} - \{0\}$.

Basarir and Kayikci [2] have defined the generalized difference matrix B^n of order n, which reduced the difference operator $\Delta_{(1)}^n$ in case $r = 1, s = -1$ and the binomial representation of this operator is

$$B^n x_k = \sum_{\nu=0}^{n} \binom{n}{\nu} r^{n-\nu} s^\nu x_{k-\nu},$$

where $r, s \in \mathbb{R} - \{0\}$ and $n \in \mathbb{N}$.

Recently Basarir et al [3] introduced the following generalized difference sequence spaces

$$Z(B_{(m)}^n) = \{(x_k) \in w : B_{(m)}^n x_k \in Z\} \text{ for all } n, m \in \mathbb{N},$$

for $Z = \ell_\infty, \bar{c}, \bar{c}_0$ where $\bar{c}, \bar{c}_0$ are the sets of statistically convergent and statistically null sequences, respectively, and $B_{(m)}^n x = (B_{(m)}^n x_k) = (rB_{(m)}^{n-1} x_k + sB_{(m)}^{n-1} x_{k-m})$

and $B^0_{(m)}x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$B^n_{(m)}x_k = \sum_{\nu=0}^{n} \binom{n}{\nu} r^{n-\nu} s^{\nu} x_{k-m\nu}.$$

Recall that [34] an *Orlicz function* is a function $M : [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(0) > 0$ as $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Remark 1.1. It is well known if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

An Orlicz function M is said to be satisfy Δ_2-*condition* for all values of u, if there exists a constant $K > 0$ such that $M(Lu) \leq KLM(u)$ for all values of $L > 1$(see Krasnoselskii and Rutitsky [34]).

Lindenstrauss and Tzafriri [41] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$ for $1 \leq p < \infty$.

At the initial stage Lindberg [40] was studied Orlicz space in connection with Banach space with symmetric. Nung and Lee [46] were studied different classes of sequence spaces defined by Orlicz function. Later on the notion was studied by Mursaleen et al. [44], Hazarika [28, 29], Savas [50], Esi and Hazarika [13] and references therein.

2. Preliminaries

In this section we recall some definitions related to ideal convergence and fuzzy real numbers.

Definition 2.1 ([33]). Let S be a non-empty set. Then a non empty class $I \subseteq P(S)$ is said to be an *ideal* on S if and only if

(i) $\phi \in I$.
(ii) I is additive (i.e. $A, B \in I \Rightarrow A \cup B \in I$)
(iii) hereditary (i.e.$A \in I, B \subseteq A \Rightarrow B \in I$).

Definition 2.2 ([33]). An ideal $I \subseteq P(S)$ is said to be non trivial if $I \neq \phi$ and $S \notin I$.

Definition 2.3 ([33]). A non-empty family of sets $F \subseteq P(S)$ is said to be a *filter* on S if and only if

(i) $\phi \notin F$,
(ii) for each $A, B \in F$ we have $A \cap B \in F$
(iii) for each $A \in F$ and $B \supset A$, implies $B \in F$.

For each ideal I, there is a filter $F(I)$ corresponding to I i.e. $F(I) = \{K \subseteq S : K^c \in I\}$, where $K^c = S - K$.

Definition 2.4 ([33])**.** A non-trivial ideal $I \subseteq P(S)$ is said to be

(i) an *admissible ideal* on S if and only if it contains all singletons, i.e., if it contains $\{\{x\} : x \in S\}$.
(ii) *maximal*, if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.

Definition 2.5 ([33])**.** A sequence $\mathbf{x} = (x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to the number ℓ if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$. In this case we write I-$\lim x_k = \ell$.

Definition 2.6 ([20])**.** Let I be an ideal in $\mathbb{N}$. If $\{k+1 : k \in A\} \in I$, for any $A \in I$, then I is said to be a *translation invariant ideal*.

Definition 2.7. A sequence space E is said to be

(i) *normal* (or *solid*) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequence (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.
(ii) *symmetric* if $(x_{\pi(k)}) \in E$, whenever $(x_k) \in E$, where π is a permutation of $\mathbb{N}$.

Let $K = \{k_1 < k_2 < ...\} \subseteq \mathbb{N}$ and E be a sequence space. A *K-step space* of E is a sequence space

$$\lambda_K^E = \{(x_{k_n}) \in w : (k_n) \in E\}.$$

A canonical preimage of a sequence $\{(x_{k_n})\} \in \lambda_K^E$ is a sequence $\{y_k\} \in w$ defined as

$$y_k = \begin{cases} x_k, & \text{if } k \in K \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E , i.e. y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x \in \lambda_K^E$.

Definition 2.8. A sequence space E is said to be *monotone* if E contains the cannical pre-image of all its step spaces.

A fuzzy number X is a fuzzy subset of the real line $\mathbb{R}$ i.e. a mapping $X : \mathbb{R} \to J(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

Definition 2.9. A fuzzy number X is said to be

(i) *convex* if $X(t) \geq X(s) \wedge X(r) = \min\{X(s), X(r)\}$, where $s < t < r$.
(ii) *normal* if there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$.
(iii) *upper-semi continuous* if for each $\varepsilon > 0, X^{-1}([0, a+\varepsilon))$ for all $a \in [0,1]$ is open in the usual topology of $\mathbb{R}$.

The α -level set of a fuzzy real number $X, 0 < \alpha \leq 1$ denoted by X^α is defined as $X^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha\}$.

The set of all upper semi-continuous, normal, convex fuzzy number is denoted by $\mathbb{R}(J)$.

Let D denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R}$. For $X = [x_1, x_2]$ and $Y = [y_1, y_2]$ in D, we define

$$X \leq Y \text{ if and only if } x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$

Define a metric d on D by

$$d(X, Y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

It can be easily proved that d is a metric on D and (D, d) is a complete metric space. Also the relation $\leq$ is a partial order on D.

The absolute value $|X|$ of $X \in \mathbb{R}(J)$ is defined as

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t > 0; \\ 0, & \text{if } t < 0. \end{cases}$$

Let $\bar{d} : \mathbb{R}(J) \times \mathbb{R}(J) \to \mathbb{R}$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha).$$

Then $(\mathbb{R}(J), \bar{d})$ is a complete metric space.

We define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$, for all $\alpha \in J$. The additive identity and multiplicative identity in $\mathbb{R}(J)$ are denoted by $\bar{0}$ and $\bar{1}$, respectively.

Definition 2.10 ([42])**.** A sequence $u = (u_k)$ of fuzzy numbers is said to

(i) *bounded* if the set $\{u_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded.

(ii) *convergent* to a fuzzy real number u_0 if for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\bar{d}(u_k, u_0) < \varepsilon$, for all $k \geq n_0$.

Definition 2.11 ([35])**.** A sequence $u = (u_k)$ of fuzzy numbers is said to *I-convergent* if there exists a fuzzy real number u_0 such that for each $\varepsilon > 0$, the set

$$\{k \in \mathbb{N} : \bar{d}(u_k, u_0) \geq \varepsilon\} \in I.$$

We write I-$\lim u_k = u_0$.

The following well-known inequality will be used throughout the article. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = H, D = \max\{1, 2^{H-1}\}$ then

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}) \text{ for all } k \in \mathbb{N} \text{ and } a_k, b_k \in \mathbb{C}.$$

Also $|a|^{p_k} \leq \max\{1, |a|^H\}$ for all $a \in \mathbb{C}$.

The following result will be used for establishing some results in this article.

Lemma 2.12. *Every normal space is monotone.(please refer to Kamthan and Gupta* [31]*, page 53).*

3. $B^n_{(m)}$-IDEAL CONVERGENCE

In this section we intorduce the following definitions.

Definition 3.1. A sequence $u = (u_k)$ of fuzzy real numbers is said to be $B^n_{(m)}$-*I-convergent* or $B^n_{(m)}(I)$-*convergent* to a fuzzy real numbers u_0 if for every $\varepsilon > 0$

$$\{k \in \mathbb{N} : \bar{d}(B^n_{(m)}x_k, u_0) \geq \varepsilon\} \in I \text{ for all } m, n \in \mathbb{N}.$$

or equivalently

$$\{k \in \mathbb{N} : \bar{d}(B^n_{(m)}x_k, u_0) < \varepsilon\} \in F \text{ for all } m, n \in \mathbb{N}.$$

In this case we write $I - \lim_k B^n_{(m)}u_k = u_0$ or $u_k \overset{B^n_{(m)}(I)}{\rightarrow} u_0$.

Definition 3.2. Let I be an admissible ideal. A sequence $u = (u_k)$ of fuzzy real numbers is said to be $B^n_{(m)}$-*I-Cauchy* if for every $\varepsilon > 0$ there exists a positive integer s such that

$$\{k \in \mathbb{N} : \bar{d}(B^n_{(m)}u_k, B^n_{(m)}u_s) \geq \varepsilon\} \in I.$$

or equivalently

$$\{k \in \mathbb{N} : \bar{d}(B^n_{(m)}u_k, B^n_{(m)}u_s) < \varepsilon\} \in F.$$

The proof of the following results are straightforward, so omitted.

Lemma 3.3. *If I is a translation invariant ideal and $I - \lim_{k\to\infty} B^n_{(m)}u_k = u_0$, then $I - \lim_{k\to\infty} B^n_{(m)}u_{k+1} = u_0$.*

Proposition 3.4. *Let $u = (u_k)$ be a sequence of fuzzy real numbers. If I is an admissible translation invariant ideal and $I - \lim_{k\to\infty} B^{n-1}_{(m)}u_k = u_0$, then $I - \lim_{k\to\infty} B^n_{(m)}u_k = u_0$.*

Theorem 3.5. *Let $u = (u_k)$ be a sequence of fuzzy real numbers. If $I - \lim_k B^n_{(m)}u_k = u_0$ exists, then it is unique.*

Theorem 3.6. *If I is an admissible ideal and $\lim_k B^n_{(m)}u_k = u_0$, then*

$$I - \lim_k B^n_{(m)}u_k = u_0.$$

Remark 3.7. If I is not an admissible ideal, then the Theorem 3.6 fails. It follows from the following example.

Example 3.8. Let us consider $(u_k) = (\bar{\frac{1}{2}})$ for all $k \in \mathbb{N}$. Let $S = \{0, 1, 2, 3\}$ and $I = \{\phi, \{0, 1\}, \{0, 1, 2\}\}$ be a non-admissible ideal of S. For $r = 1, s = -1, n = 1, m = 1$, we have $\lim_{k\to\infty} B^n_{(m)}u_k = \bar{0}$ but $I - \lim_{k\to\infty} B^n_{(m)}u_k \neq \bar{0}$.

Theorem 3.9. *If I is an admissible ideal. Then (u_k) is a $B^n_{(m)}(I)$-convergent sequence if and only if for every $\varepsilon > 0$, there exists $s \in \mathbb{N}$ such that*

$$\left\{k \in \mathbb{N} : \bar{d}(B^n_{(m)}u_k, B^n_{(m)}u_s) < \varepsilon\right\} \in F.$$

Theorem 3.10. *Let $u = (u_k)$ be a sequence of fuzzy real numbers and let I be a non-trivial admissible ideal in $\mathbb{N}$. If there is a $B^n_{(m)}(I)$-convergent sequence $v = (v_k)$ of fuzzy real numbers such that $\{k \in \mathbb{N} : B^n_{(m)}v_k \neq B^n_{(m)}u_k\} \in I$, then u is also $B^n_{(m)}(I)$-convergent.*

Proof. Suppose that $\left\{k \in \mathbb{N} : B^n_{(m)} v_k \neq B^n_{(m)} u_k\right\} \in I$ and $I - \lim_k B^n_{(m)} v_k = \ell$. Then for every $\varepsilon > 0$ we get

$$\left\{k \in \mathbb{N} : \bar{d}(B^n_{(m)} v_k, \ell) \geq \varepsilon\right\} \in I.$$

For every $\varepsilon > 0$ we have

$$\begin{aligned} \left\{k \in \mathbb{N} : \bar{d}(B^n_{(m)} u_k, \ell) \geq \varepsilon\right\} \subseteq \{k \in \mathbb{N} : B^n_{(m)} v_k \neq B^n_{(m)} u_k\} \cup \\ \left\{k \in \mathbb{N} : \bar{d}(B^n_{(m)} v_k, \ell) \geq \varepsilon\right\}. \end{aligned} \tag{3.1}$$

As I being an admissible ideal, so both the sets of right-hand side of (3.1) is in I, therefore

$$\left\{k \in \mathbb{N} : \bar{d}(B^n_{(m)} u_k, \ell) \geq \varepsilon\right\} \in I.$$

This completes the proof. □

4. Generalized difference ideal convergent sequence spaces

Throughout this section we assume that I is an admissible ideal of $\mathbb{N}$. In this section we introduce some new sequence spaces using the generalized difference matrix $B^n_{(m)}$ and Orlicz functions. Let M be an Orlicz function, and $p = (p_k)$ be a sequence of strictly positive real numbers and m, n be nonnegative integers. For some $\rho > 0$, we define the following new sequence spaces:

$$c_0^{IF}(M, B^n_{(m)}, p) = \left\{(u_k) \in w^F : \left\{k \in \mathbb{N} : \left[M\left(\frac{\bar{d}(B^n_{(m)} u_k, \bar{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I\right\}$$

$$c^{IF}(M, B^n_{(m)}, p) = \left\{(u_k) \in w^F : \left\{k \in \mathbb{N} : \left[M\left(\frac{\bar{d}(B^n_{(m)} u_k, u_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I, \text{ for } u_0 \in \mathbb{R}(J)\right\}$$

$$\ell_\infty^F(M, B^n_{(m)}, p) = \left\{(u_k) \in w^F : \sup_k \left[M\left(\frac{\bar{d}(B^n_{(m)} u_k, \bar{0})}{\rho}\right)\right]^{p_k} < \infty\right\}.$$

By using these spaces, we can construct the sequence spaces

$$m^{IF}(M, B^n_{(m)}, p) = c^{IF}(M, B^n_{(m)}, p) \cap \ell_\infty^F(M, B^n_{(m)}, p)$$

and

$$m_0^{IF}(M, B^n_{(m)}, p) = c_0^{IF}(M, B^n_{(m)}, p) \cap \ell_\infty^F(M, B^n_{(m)}, p).$$

Now we are ready to give our main results as follows.

Theorem 4.1. *Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers. The spaces $c_0^{IF}(M, B^n_{(m)}, p)$, $c^{IF}(M, B^n_{(m)}, p)$, $\ell_\infty^F(M, B^n_{(m)}, p)$, $m^{IF}(M, B^n_{(m)}, p)$ and $m_0^{IF}(M, B^n_{(m)}, p)$ are linear.*

Proof. We prove the result only for the space $c^{IF}(M, B^n_{(m)}, p)$. The others can be treated similarly. Let $u = (u_k)$ and $v = (v_k)$ be two elements of $c^{IF}(M, B^n_{(m)}, p)$ and α_1, α_2 be scalars. Let $\varepsilon > 0$ be given. Then there exist some positive numbers ρ_1, ρ_2 such that

$$P = \left\{k \in \mathbb{N} : \left[M\left(\frac{\bar{d}(B^n_{(m)} u_k, u_0)}{\rho_1}\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\} \in I$$

and

$$Q = \left\{ k \in \mathbb{N} : \left[M \left(\frac{\overline{d}(B^n_{(m)} v_k, v_0)}{\rho_2} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \in I.$$

Let $\rho_3 = \max(2|\alpha_1|\rho_1, 2|\alpha_2|\rho_2)$. Since M is non-decreasing and convex function, we have

$$\begin{aligned} &\left[M \left(\frac{\overline{d}(B^n_{(m)}(\alpha_1 u_k + \alpha_2 v_k), \alpha_1 u_0 + \alpha_2 v_0)}{\rho_3} \right) \right]^{p_k} \\ &\leq \left[M \left(\frac{\alpha_1 \overline{d}(B^n_{(m)} u_k, u_0)}{\rho_3} \right) \right]^{p_k} + \left[M \left(\frac{\alpha_2 \overline{d}(B^n_{(m)} v_k, v_0)}{\rho_3} \right) \right]^{p_k} \\ &\leq \left[M \left(\frac{\overline{d}(B^n_{(m)} u_k, u_0)}{\rho_1} \right) \right]^{p_k} + \left[M \left(\frac{\overline{d}(B^n_{(m)} v_k, v_0)}{\rho_2} \right) \right]^{p_k}. \end{aligned}$$

Now,

$$\left\{ k \in \mathbb{N} : \left[M \left(\frac{\overline{d}(B^n_{(m)}(\alpha_1 u_k + \alpha_2 v_k), \alpha_1 u_0 + \alpha_2 v_0)}{\rho_3} \right) \right]^{p_k} \geq \varepsilon \right\} \subseteq P \cup Q \in I.$$

Therefore $(\alpha_1 u + \alpha_2 v) \in c^{IF}(M, B^n_{(m)}, p)$. This completes the proof. □

Theorem 4.2. *For an Orlicz function M,*

$$c_0^{IF}(M, B^n_{(m)}, p),\ c^{IF}(M, B^n_{(m)}, p),\ m^{IF}(M, B^n_{(m)}, p),\ m_0^{IF}(M, B^n_{(m)}, p) \text{ and } \ell^F_\infty(M, B^n_{(m)}, p)$$

are complete metric spaces with the metric

$$g_{B^n_{(m)}}(u, v) = \inf \left\{ \rho^{\frac{p_k}{G}} > 0 : \sup_k M \left(\frac{\overline{d}(B^n_{(m)} u_k, B^n_{(m)} v_k)}{\rho} \right) \leq 1 \right\},$$

where $G = \max\{1, \sup_k p_k\}$.

Proof. We shall prove the result only for the space $c^{IF}(M, B^n_{(m)}, p)$. The other can be treated, similarly. It is easy to prove that g_Δ is a metric on $c^{IF}(M, B^n_{(m)}, p)$. Let (u_k) be a Cauchy sequence of $c^{IF}(M, B^n_{(m)}, p)$. Let $\varepsilon > 0$ be given. For a fixed $u_0 > 0$ and choose $t > 0$ such that $M\left(\frac{tu_0}{2}\right) \geq 1$. Then there exists $n_0 \in \mathbb{N}$ such that

$$g_{B^n_{(m)}}(u^i, u^j) < \frac{\varepsilon}{tu_0} \text{ for all } i, j \geq n_0.$$

$$(4.1) \quad \Rightarrow \inf \left\{ \rho^{\frac{p_k}{G}} > 0 : \sup_k M \left(\frac{\overline{d}(B^n_{(m)} u^i_k, B^n_{(m)} u^j_k)}{\rho} \right) \leq 1 \right\} < \varepsilon \text{ for all } i, j \geq n_0.$$

Let

$$(4.2) \quad \lim_{i \to \infty} u^i_k = u_k \ \text{ for } k = 1, 2, 3, ..., mn.$$

Also

$$\sup_k M\left(\frac{\overline{d}(B^n_{(m)}u^i_k, B^n_{(m)}u^j_k)}{\rho}\right) \leq 1 \text{ for all } i,j \geq n_0 \text{ and } k \in \mathbb{N}. \tag{4.3}$$

$$M\left(\frac{\overline{d}(B^n_{(m)}u^i_k, B^n_{(m)}u^j_k)}{g_{B^n_{(m)}}(u^i, u^j)}\right) \leq 1 \leq M\left(\frac{tu_0}{2}\right) \text{ for all } i,j \geq n_0 \text{ and } k \in \mathbb{N}.$$

$$\overline{d}(B^n_{(m)}u^i_k, B^n_{(m)}u^j_k) < \frac{\varepsilon}{2} \text{ for all } i,j \geq n_0 \text{ and } k \in \mathbb{N}.$$

Thus $(B^n_{(m)}u^i_k)$ is Cauchy sequence of fuzzy numbers. Let $\lim_{i\to\infty} B^n_{(m)}u^i_k = u_k$ for each $k \in \mathbb{N}$.
For $k = 1$ we have, from (4.2),

$$\lim_{i\to\infty} u^i_{1-mn} = u_{1-mn} \text{ for } n \geq 1, m \geq 1.$$

Proceeding in this way inductively we conclude that

$$\lim_{i\to\infty} u^i_k = u_k \text{ for each } k \in \mathbb{N}.$$

Also

$$\lim_{i\to\infty} B^n_{(m)}u^i_k = u_k \text{ for each } k \in \mathbb{N}.$$

By the continuity of M, from (4.3) we have

$$\sup_k M\left(\frac{\overline{d}(B^n_{(m)}u^i_k, u_k)}{\rho}\right) \leq 1 \text{ for all } i \geq n_0, j \to \infty.$$

$$\Rightarrow \inf\left\{\rho^{\frac{p_k}{G}} > 0 : \sup_k M\left(\frac{\overline{d}(B^n_{(m)}u^i_k, u_k)}{\rho}\right) \leq 1\right\} < \varepsilon \text{ for all } i \geq n_0.$$

Hence from (4.1) on taking limit as $j \to \infty$, we get

$$\inf\left\{\rho^{\frac{p_k}{G}} > 0 : \sup_k M\left(\frac{\overline{d}(B^n_{(m)}u^i_k, u_k)}{\rho}\right) \leq 1\right\} < \varepsilon \text{ for all } i \geq n_0.$$

i.e. $g_{B^n_{(m)}}(u^i, u) < \varepsilon$ for all $i \geq n_0$.
Then the inequality

$$g_{B^n_{(m)}}(u, \overline{0}) \leq g_{B^n_{(m)}}(u, B^n_{(m)}u^i) + g_{B^n_{(m)}}(B^n_{(m)}u^i, \overline{0}) \text{ for all } i \geq n_0$$

implies that $(u_k) \in c^{IF}(M, B^n_{(m)}, p)$. This completes the proof. □

Theorem 4.3. *Let M_1 and M_2 be two Orlicz functions. Then*

(i) $Z(M_2, B^n_{(m)}, p) \subseteq Z(M_1 M_2, B^n_{(m)}, p)$.
(ii) $Z(M_1, B^n_{(m)}, p) \cap Z(M_2, B^n_{(m)}, p) \subseteq Z(M_1 + M_2, B^n_{(m)}, p)$,

for $Z = c_0^{IF}, c^{IF}, m_0^{IF}, m^{IF}, \ell_\infty^F$.

Proof. (i) Let $u = (u_k) \in c^{IF}(M_2, B^n_{(m)}, p)$. For some $\rho > 0$ we have

$$\left\{k \in \mathbb{N} : \left[M_2\left(\frac{\overline{d}(B^n_{(m)}u_k, u_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I \text{ for every } \varepsilon > 0. \tag{4.4}$$

Let $\varepsilon > 0$ and choose λ with $0 < \lambda < 1$ such that $M_1(t) < \varepsilon$ for $0 \leq t \leq \lambda$. We define

$$v_k = \frac{\overline{d}(B^n_{(m)}u_k, u_0)}{\rho}$$

and consider

$$\lim_{k\in\mathbb{N}; 0\leq v_k\leq\lambda}[M_1(v_k)]^{p_k} = \lim_{k\in\mathbb{N}; v_k\leq\lambda}[M_1(v_k)]^{p_k} + \lim_{k\in\mathbb{N}; v_k>\lambda}[M_1(v_k)]^{p_k}.$$

We have

$$\lim_{k\in\mathbb{N}; v_k\leq\lambda}[M_1(v_k)]^{p_k} \leq [M_1(2)]^H \lim_{k\in\mathbb{N}; v_k\leq\lambda}[v_k]^{p_k}, H = \sup_k p_k. \tag{4.5}$$

For the second summation (i.e.$v_k > \lambda$), we go through the following procedure. We have

$$v_k < \frac{v_k}{\lambda} < 1 + \frac{v_k}{\lambda}.$$

Since M_1 is non-decreasing and convex, it follows that

$$M_1(v_k) < M_1\left(1 + \frac{v_k}{\lambda}\right) \leq \frac{1}{2}M_1(2) + \frac{1}{2}M_1\left(\frac{2v_k}{\lambda}\right).$$

Since M_1 satisfies Δ_2-condition, we can write

$$M_1(v_k) < \frac{1}{2}K\frac{v_k}{\lambda}M_1(2) + \frac{1}{2}K\frac{v_k}{\lambda}M_1(2) = K\frac{v_k}{\lambda}M_1(2).$$

We get the following estimates:

$$\lim_{k\in\mathbb{N}; v_k>\lambda}[M_1(v_k)]^{p_k} \leq \max\left\{1, (K\lambda^{-1}M_1(2))^H\right\} \lim_{k\in\mathbb{N}; v_k>\lambda}[v_k]^{p_k}. \tag{4.6}$$

From (4.4), (4.5) and (4.6), it follows that $(u_k) \in c^{IF}(M_1.M_2, B^n_{(m)}, p)$.
Hence $c^{IF}(M_2, B^n_{(m)}, p) \subseteq c^{IF}(M_1.M_2, B^n_{(m)}, p)$.

(ii) Let $(u_k) \in c^{IF}(M_1, B^n_{(m)}, p) \cap c^{IF}(M_2, B^n_{(m)}, p)$. Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{k \in \mathbb{N} : \left[M_1\left(\frac{\overline{d}(B^n_{(m)}u_k, u_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I$$

and

$$\left\{k \in \mathbb{N} : \left[M_2\left(\frac{\overline{d}(B^n_{(m)}u_k, u_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I$$

The rest of the proof follows from the following relation:

$$\left\{k \in \mathbb{N} : \left[(M_1 + M_2)\left(\frac{\overline{d}(B^n_{(m)}u_k, u_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\}$$

$$\subseteq \left\{k \in \mathbb{N} : \left[M_1\left(\frac{\overline{d}(B^n_{(m)}u_k, u_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \cup \left\{k \in \mathbb{N} : \left[M_2\left(\frac{\overline{d}(B^n_{(m)}u_k, u_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\}.$$

□

Taking $M_2(x) = x$ and $M_1(x) = M(x)$ for all $x \in [0, \infty)$, we have the following result.

Corollary 4.4. *Then* $Z(B^n_{(m)}, p) \subseteq Z(M, B^n_{(m)}, p)$ *for* $Z = c_0^{IF}, c^{IF}, m_0^{IF}, m^{IF}, \ell^F_\infty$.

Following standard techniques, one can easily prove the following result.

Theorem 4.5. (a) *If* $M_1(x) \leq M_2(x)$ *for all* $x \in [0, \infty)$, *then* $Z(M_1, B^n_{(m)}, p) \subseteq Z(M_2, B^n_{(m)}, p)$ *for* $Z = c_0^{IF}, c^{IF}$ *and* ℓ^F_∞.

(b) *If* $n_1 < n_2$ *then* $Z(B^{n_1}_{(m)}, p) \subseteq Z(B^{n_2}_{(m)}, p)$ *for* $Z = c_0^{IF}, c^{IF}$ *and* ℓ^F_∞.

Theorem 4.6. *Let M be an Orlicz function. Then*

$$c_0^{IF}(M, B^n_{(m)}) \subset c^{IF}(M, B^n_{(m)}) \subset \ell^F_\infty(M, B^n_{(m)})$$

and the inclusions are proper.

Proof. Let $(u_k) \in c^{IF}(M, B^n_{(m)})$. Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{k \in \mathbb{N} : \left[M\left(\frac{\overline{d}(B^n_{(m)}u_k, u_0)}{\rho}\right)\right] \geq \varepsilon\right\} \in I.$$

Since

$$M\left(\frac{\overline{d}(B^n_{(m)}u_k, \overline{0})}{\rho}\right) \leq \frac{1}{2}M\left(\frac{\overline{d}(B^n_{(m)}u_k, u_0)}{\rho}\right) + \frac{1}{2}M\left(\frac{\overline{d}(u_0, \overline{0})}{\rho}\right).$$

Taking supremum over k on both sides implies that $(u_k) \in \ell^F_\infty(M, B^n_{(m)})$.

The inclusion $c_0^{IF}(M, B^n_{(m)}) \subset c^{IF}(M, B^n_{(m)})$ is obvious. The inclusion is strict, follows from the following example.

Example 4.7. Let $M(x) = x$ for all $x \in [0, \infty)$ and $r = 1, s = -1, n = 1, m = 2$. Consider the sequence (u_k) of fuzzy numbers be defined as follows:
For $k = 2^i, i = 1, 2, 3, ...$

$$u_k(t) = \begin{cases} \frac{4}{k}t + 1 & , \text{ if } -\frac{k}{4} \leq t \leq 0; \\ -\frac{4}{k}t + 1 & , \text{ if } 0 < t \leq \frac{k}{4}; \\ 0 & , \text{ otherwise.} \end{cases}$$

otherwise, $u_k(t) = \overline{0}$.

For $\alpha \in (0, 1]$, the α-level sets of u_k and $B^1_{(m)}u_k$ are

$$[u_k]^\alpha = \begin{cases} [\frac{k}{4}(\alpha - 1), \frac{k}{4}(1 - \alpha)] & , \text{ if } k = 2^i, i = 1, 2, 3, ... \\ [0, 0] & , \text{ otherwise .} \end{cases}$$

and

$$[B^1_{(2)}u_k]^\alpha = \begin{cases} [\frac{k}{4}(\alpha-1), \frac{k}{4}(1-\alpha)] & , \text{ for } k = 2^i \\ [\frac{k}{4}(\alpha-1), \frac{k}{4}(1-\alpha)] & , \text{ for } k+1 = 2^i (i > 1) \\ [0,0] & , \text{ otherwise .} \end{cases}$$

It is easy to prove that the sequences (u_k) and $(B^1_{(2)}u_k)$ are bounded but these are not I-convergent. □

Theorem 4.8. *The spaces* $m_0^{IF}(M, B^{n-1}_{(m)}, p)$ *and* $m^{IF}(M, B^{n-1}_{(m)}, p)$ *are nowhere dense subsets of* $\ell^F_\infty(M, B^{n-1}_{(m)}, p)$.

Proof. From Theorem 4.1, it follows that $m_0^{IF}(M, B^{n-1}_{(m)}, p)$ and $m^{IF}(M, B^{n-1}_{(m)}, p)$ are closed subspaces of $\ell^F_\infty(M, B^{n-1}_{(m)}, p)$. Since the inclusion relations $m_0^{IF}(M, B^{n-1}_{(m)}, p) \subset \ell^F_\infty(M, B^{n-1}_{(m)}, p)$ and $m^{IF}(M, B^{n-1}_{(m)}, p) \subset \ell^F_\infty(M, B^{n-1}_{(m)}, p)$ are strict, then the spaces $m_0^{IF}(M, B^{n-1}_{(m)}, p)$ and $m^{IF}(M, B^{n-1}_{(m)}, p)$ are nowhere dense subsets of $\ell^F_\infty(M, B^{n-1}_{(m)}, p)$. □

Theorem 4.9. *The inclusions* $Z(M, B^{n-1}_{(m)}, p) \subseteq Z(M, B^n_{(m)}, p)$ *are strict for* $n \geq 1$. *In general* $Z(M, B^i_{(m)}, p) \subseteq Z(M, B^n_{(m)}, p)$ *for* $i = 1, 2, ..., n-1$ *and the inclusion is strict, for* $Z = c_0^{IF}, c^{IF}, m_0^{IF}, m^{IF}, \ell^F_\infty$.

Proof. Let $u = (u_k) \in c_0^{IF}(M, B^{n-1}_{(m)}, p)$. Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{k \in \mathbb{N} : \left[M\left(\frac{\overline{d}(B^{n-1}_{(m)}u_k, \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I.$$

Since M is non-decreasing and convex it follows that

$$\left[M\left(\frac{\overline{d}(B^n_{(m)}u_k, \overline{0})}{2\rho}\right)\right]^{p_k} \leq \left[M\left(\frac{\overline{d}(B^{n-1}_{(m)}u_k, B^{n-1}_{(m)}u_{k+1}, \overline{0})}{2\rho}\right)\right]^{p_k}$$

$$\leq D\left[\frac{1}{2}M\left(\frac{\overline{d}(B^{n-1}_{(m)}u_k, \overline{0})}{\rho}\right)\right]^{p_k} + D\left[\frac{1}{2}M\left(\frac{\overline{d}(B^{n-1}_{(m)}u_{k+1}, \overline{0})}{\rho}\right)\right]^{p_k}$$

$$\leq DK\left[M\left(\frac{\overline{d}(B^{n-1}_{(m)}u_k, \overline{0})}{\rho}\right)\right]^{p_k} + DK\left[M\left(\frac{\overline{d}(B^{n-1}_{(m)}u_{k+1}, \overline{0})}{\rho}\right)\right]^{p_k},$$

where $K = \max\{1, \left(\frac{1}{2}\right)^H\}$.
Therefore we have

$$\left\{k \in \mathbb{N} : \left[M\left(\frac{\overline{d}(B^n_{(m)}u_k, \overline{0})}{2\rho}\right)\right]^{p_k} \geq \varepsilon\right\}$$

$$\subseteq \left\{k \in \mathbb{N} : DK\left[M\left(\frac{\overline{d}(B^{n-1}_{(m)}u_k, \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\}$$

$$\cup \left\{k \in \mathbb{N} : DK\left[M\left(\frac{\overline{d}(B^{n-1}_{(m)}u_{k+1}, \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\}.$$

$$\text{i.e. } \left\{k \in \mathbb{N} : \left[M\left(\frac{\overline{d}(B^n_{(m)}u_k, \overline{0})}{2\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I.$$

Hence $(u_k) \in c_0^{IF}(M, B^n_{(m)}, p)$.

The inclusion is strict follows from the following example.

Example 4.10. Let $M(x) = x$ for all $x \in [0, \infty), r = 1, s = -1, n = 3, m = 2$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence (u_k) of fuzzy numbers as follows:

$$u_k(t) = \begin{cases} -\frac{t}{k^3-1} + 1 & , \text{ if } k^3 - 1 \leq t \leq 0; \\ -\frac{t}{k^3+1} + 1 & , \text{ if } 0 < t \leq k^3 + 1; \\ 0 & , \text{ otherwise.} \end{cases}$$

For $\alpha \in (0, 1]$, the α-level sets of $u_k, B^1_{(2)}u_k, B^2_{(2)}u_k$ and $B^3_{(2)}u_k$ are
$[u_k]^\alpha = [(1-\alpha)(k^3-1), (1-\alpha)(k^3+1)]$
$[B^1_{(2)}u_k]^\alpha = [(1-\alpha)(-3k^2-3k-3), (1-\alpha)(-3k^2-3k+1)]$
$[B^2_{(2)}u_k]^\alpha = [(1-\alpha)(6k+2), (1-\alpha)(6k+10)]$
$[B^3_{(2)}u_k]^\alpha = [-14(1-\alpha), 2(1-\alpha)]$,
respectively. It is easy to check that the sequence $[B^2_{(2)}u_k]^\alpha$ is not I-bounded but $[B^3_{(2)}u_k]^\alpha$ is I-bounded. □

Theorem 4.11. *Let $0 < p_k \leq q_k < \infty$ for each k. Then*

$$Z(M, B^n_{(m)}, p) \subseteq Z(M, B^n_{(m)}, q)$$

for $Z = c_0^{IF}$ and c^{IF}.

Proof. Let $(u_k) \in c_0^{IF}(M, B^n_{(m)}, p)$. Then there exists a number $\rho > 0$ such that

$$\left\{k \in \mathbb{N} : \left[M\left(\frac{\overline{d}(B^n_{(m)}u_k, \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I.$$

For sufficiently large k. Since $p_k \leq q_k$ for each k, therefore we get

$$\left\{k \in \mathbb{N} : \left[M\left(\frac{\overline{d}(B^n_{(m)}u_k, \overline{0})}{\rho}\right)\right]^{q_k} \geq \varepsilon\right\}$$
$$\subseteq \left\{k \in \mathbb{N} : \left[M\left(\frac{\overline{d}(B^n_{(m)}u_k, \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I.$$

i.e. $(u_k) \in c_0^{IF}(M, B^n_{(m)}, q)$. This completes the proof.
Similarly, it can be shown that $c^{IF}(M, B^n_{(m)}, p) \subseteq c^{IF}(M, B^n_{(m)}, q)$. □

Corollary 4.12. (a) *Let $0 < \inf_k p_k \leq p_k \leq 1$. Then*

$$Z(M, B^n_{(m)}, p) \subseteq Z(M, B^n_{(m)})$$

for $Z = c_0^{IF}$ and c^{IF}.

(b) *Let $1 \leq p_k \leq \sup_k p_k < \infty$. Then $Z(M, B^n_{(m)}) \subseteq Z(M, B^n_{(m)}, p)$ for $Z = c_0^{IF}$ and c^{IF}.*

Theorem 4.13. *If I is an admissible ideal and $I \neq I_f$, then the sequence spaces $c_0^{IF}(M, B^n_{(m)}, p)$, $c^{IF}(M, B^n_{(m)}, p)$, $m^{IF}(M, B^n_{(m)}, p)$ and $m_0^{IF}(M, B^n_{(m)}, p)$ are neither normal nor monotone.*

Proof. We prove this result with the help of following example.

Example 4.14. Let $M(x) = x$ for all $x \in [0, \infty)$ and $r = 1, s = -1, n = 1, m = 1$. For $I = I_\delta$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence (u_k) of fuzzy numbers as follows:

$$u_k(t) = \begin{cases} t - 3k + 1 & , \quad \text{if } t \in [3k-1, 3k]; \\ -t + 3k + 1 & , \quad \text{if } t \in [3k, 3k+1]; \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Let

$$\alpha_k = \begin{cases} 1 & , \quad \text{if } k \text{ is odd;} \\ 0 & , \quad \text{if } k \text{ is even.} \end{cases}$$

Thus $(\alpha_k u_k) \notin Z(M, B^n_{(m)})$ for $Z = c_0^{IF}, c^{IF}, m_0^{IF}, m^{IF}$. Therefore $c_0^{IF}(M, B^n_{(m)}, p)$, $c^{IF}(M, B^n_{(m)}, p)$, $m^{IF}(M, B^n_{(m)}, p)$ and $m_0^{IF}(M, B^n_{(m)}, p)$ are not normal. By Lemma 2.12, these spaces are not monotone. □

Theorem 4.15. *If I is an admissible ideal and $I \neq I_f$, then the sequence space $Z(M, B^n_{(m)}, p)$ is not symmetric, where $Z = c_0^{IF}, c^{IF}, m_0^{IF}, m^{IF}$.*

Proof. We shall prove the result only for $c^{IF}(M, B^n_{(m)}, p)$ with the help of the following example. The rest of the results follow similar way.

Example 4.16. Let $M(x) = x$ for all $x \in [0, \infty)$ and $r = 1, s = -1, n = 1, m = 1$. For $I = I_\delta$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence (u_k) of fuzzy numbers as follows:

$$u_k(t) = \begin{cases} t - 2k + 1 & , \quad \text{if } t \in [2k-1, 2k]; \\ -t + 2k + 1 & , \quad \text{if } t \in [2k, 2k+1]; \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Thus we have $(u_k) \in c^{IF}(M, B^n_{(m)}, p)$. But the rearrangement (v_k) of (u_k) defined as

$$v_k = \{u_1, u_4, u_2, u_9, u_3, u_{16}, u_5, u_{25}, u_6, ...\}.$$

This implies that $(v_k) \notin c^{IF}(M, B^n_{(m)}, p)$.
Hence $c^{IF}(M, B^n_{(m)}, p)$ is not symmetric. □

5. Conclusions

In this article we have investigated the notion of ideal convergence of sequences point of view of fuzzy real numbers using a new generalized difference matrix $B^n_{(m)}$ and Orlicz functions. Still there are a lot to be investigated on sequence spaces applying the notion of ideal convergence. The workers will apply the techniques used in this article for further investigations on ideal convergence for different types of sequence spaces.

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