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On the existence and uniqueness of solutions to fuzzy boundary value problems

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Abstract. In this study, we provide some sufficient conditions which guarantee the existence and uniqueness of solutions to boundary value problems for first-order fuzzy nonlinear differential equations by using generalized differentiability.

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1. Introduction

The study of fuzzy differential equations (FDEs) is rapidly growing as a new area in fuzzy analysis. Due to the applicability of the FDEs for the analysis of phenomena which imprecision is inherent, this class of differential equations is a field of increasing interest (see [12, 19, 24, 26, 31, 35, 39]). Toady, FDE plays a prominent role in a range of application areas, including population models [21, 20], civil engineering [33], particle systems [14, 15, 16, 37], medicine [1, 5, 22, 32], bioinformatics and computational biology [4, 8, 11].

There are many approaches to define the concept of solution to a fuzzy differential equation and to study the existence of such solutions. Historically, differentiability in the sense of Hukuhara is one of the earliest. Under this setting, mainly the existence and uniqueness of the solution of a fuzzy differential equation have been studied (see for example[7, 24]). This approach produces the nondecreasing length of the diameter of the level sets of the solution and therefore, the fuzzy solution behaves quite differently from the crisp solution [6, 13, 27]. This drawback was resolved by interpreting the FDE as a family of differential inclusions [23]. However, this approach has a disadvantage, too: we do not have an adequate definition for derivative of a fuzzy-valued function [2, 3]. Bede and Gal [6] have introduced a more general definition of derivative for fuzzy-valued functions called weakly and strongly generalized differentials. The strongly generalized differentiability allows us to resolve the above-mentioned shortcomings [9, 10].

In this paper, following the idea in [17] and using a homotopy principle as well as the Leray-Schauder degrees, we establish some sufficient conditions for the existence and uniqueness of solution to a fuzzy differential equation subject to boundary value conditions under the generalized differentiability.

2. Preliminaries

2.1. **Basic concepts.**

Let $\mathbb{R}_{\mathcal{F}}$ be the set of all real fuzzy numbers which are normal, upper semicontinuous, convex and compactly supported fuzzy sets.
The parametric form of a fuzzy number is shown by $v = (\underline{v}(r), \overline{v}(r))$, where functions $\underline{v}(r)$ and $\overline{v}(r)$; $0 \leq r \leq 1$ satisfy the following requirements [29]:

(1) $\underline{v}(r)$ is monotonically increasing left continuous function.
(2) $\overline{v}(r)$ is monotonically decreasing left continuous function.
(3) $\underline{v}(r) \leq \overline{v}(r), \quad 0 \leq r \leq 1$.

For $0 \leq r \leq 1$, denote $[v]^r = \{x \in \mathbb{R}; v(x) \geq r\}$ and $[v]^0 = \overline{\{x \in \mathbb{R}; v(x) > 0\}}$. Then, it is well-known that for any $r \in [0,1]$, $[v]^r = [\underline{v}^r, \overline{v}^r]$ is a bounded closed interval. For $u,\ v \in \mathbb{R}_{\mathcal{F}}$, and $\lambda \in \mathbb{R}$, the sum $u \oplus v$ and the product $\lambda \odot u$ are defined by $[u \oplus v]^r = [u]^r + [v]^r$, $[\lambda \odot v]^r = \lambda.[v]^r$, $\forall r \in [0,1]$, where $[u]^r + [v]^r$ means the usual addition of two intervals of $\mathbb{R}$ and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$.

Noting that a crisp number α is simply represented by $\underline{v}(r) = \overline{v}(r) = \alpha,\ 0 \leq r \leq 1$.

Let A, B two nonempty bounded subsets of $\mathbb{R}$. The Hausdorff distance between A and B is

$$d_H(A,B) = max\left[\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\right].$$

The metric d_H on $\mathbb{R}_{\mathcal{F}}$ is as follows,

$$d_\infty(u,v) = \sup\left\{d_H([u]^r,[v]^r),\ r \in [0,1]\right\}, \qquad u,v \in \mathbb{R}_{\mathcal{F}}.$$

Definition 2.1. Let I be a real interval. The mapping $f : I \to \mathbb{R}_{\mathcal{F}}$ is called a fuzzy function and its $r-$level set is denoted by

$$[f(t)]^r = \left[\underline{f}^r(t), \overline{f}^r(t)\right], \qquad t \in I,\ r \in [0,1].$$

Definition 2.2. Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y \oplus z$, then z is called the Hukuhara difference of x and y and it is denoted by $x \ominus y$.

Let us remark that $x \ominus y \neq x \oplus (-1) \odot y$.

Definition 2.3 ([9]). Let be $f : I \to \mathbb{R}_{\mathcal{F}}$ and $t_0 \in I$. We say that f is generalized differentiable at t_0 if:

(1) it exists an element $f'(t_0) \in \mathbb{R}_{\mathcal{F}}$ such that, for all $h > 0$ sufficiently near to 0, there are $f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ and the limits (in the metric d_∞)

$$\lim_{h\to 0^+} \frac{f(t_0+h)\ominus f(t_0)}{h} = \lim_{h\to 0^+} \frac{f(t_0)\ominus f(t_0-h)}{h} = f'(t_0) \tag{2.1}$$

or

(2) it exists an element $f'(t_0) \in \mathbb{R}_{\mathcal{F}}$ such that, for all $h > 0$ sufficiently near to 0, there are $f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ and the limits

$$\lim_{h\to 0^-} \frac{f(t_0+h)\ominus f(t_0)}{h} = \lim_{h\to 0^-} \frac{f(t_0)\ominus f(t_0-h)}{h} = f'(t_0). \tag{2.2}$$

We say a function is (1)-differentiable if it satisfies form (1) and call it (2)-differentiable if it satisfies form (2).

Theorem 2.4 ([9])**.** *Let $f : I \to \mathbb{R}_{\mathcal{F}}$ be a function and denote $[f(t)]^r = [\underline{f}^r(t), \overline{f}^r(t)]$, for each $r \in [0, 1]$. Then*

(i) *if f is differentiable in the first form (1), then $\underline{f}^r(t)$ and $\overline{f}^r(t)$ are differentiable functions and*

$$[f'(t)]^r = [(\underline{f}^r)'(t), (\overline{f}^r)'(t)] \tag{2.3}$$

(ii) *if f is differentiable in the second form (2), then $\underline{f}^r(t)$ and $\overline{f}^r(t)$ are differentiable functions and*

$$[f'(t)]^r = [(\overline{f}^r)'(t), (\underline{f}^r)'(t)] \tag{2.4}$$

Let $(\underline{f}^r)'(t)$ and $(\overline{f}^r)'(t)$ also be continuous functions with respect to both t and r. This property is called *continuity condition* (see also [36]). The continuity condition is assumed to hold for all fuzzy functions in the rest of the paper.

Definition 2.5. Let $f : [0, T] \to \mathbb{R}_{\mathcal{F}}$. The integral of f in $[0, T]$, (denoted by $\int_{[0,T]} f(t)dt$ or $\int_0^T f(t)dt$) is defined levelwise as the set of integrals of the (real) measurable selections for $[f]^r$, for each $r \in (0, 1]$. We say that f is integrable over $[0, T]$ if $\int_{[0,T]} f(t)dt \in \mathbb{R}_{\mathcal{F}}$ and we have

$$\left[\int_0^T f(t)dt\right]^r = \left[\int_0^T \underline{f}^r(t)dt, \int_0^T \overline{f}^r(t)dt\right],$$

for each $r \in (0, 1]$.

Remark 2.6. It is obviously satisfied that a continuous function is integrable.

Lemma 2.7 ([30])**.** *The fuzzy differential equation $x'(t) = f(t, x)$, $x(0) = x_0$, where $f : I \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is supposed to be continuous, is equivalent to one of the integral equations:*

$$x(t) = x(0) \oplus \int_0^t f(s, x(s))ds, \qquad \forall t \in I$$

or

$$x(0) = x(t) \oplus (-1) \odot \int_0^t f(s, x(s))ds, \qquad \forall t \in I$$

depending on the generalized differentiability considered, (1)-differentiability or (2)-differentiability, respectively.

2.2. **Fuzzy boundary value problem.**

Let us consider the following fuzzy differential equation subject to boundary conditions

$$\begin{cases} y' = f(t, y), \quad t \in I = [0, T], \\ \lambda y(0) = y(T), \end{cases} \tag{2.5}$$

where $T > 0, \lambda > 0, f : I \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$, and the derivative of y is considered in the sense of generalized differentiable.

Set

$$C(I, \mathbb{R}_{\mathcal{F}}) = \{y : I \to \mathbb{R}_{\mathcal{F}} : \; y \text{ is continuous}\}$$

and

$$C^1(I, \mathbb{R}_{\mathcal{F}}) = \{y : I \to \mathbb{R}_{\mathcal{F}} : \; y, \; y' \text{ are continuous}\},$$

equipped with usual supremum norms.

A solution to problem (2.5) is a continuously differentiable function $y : I \to \mathbb{R}_{\mathcal{F}}$ (i.e., $y \in C^1(I, \mathbb{R}_{\mathcal{F}})$) for which conditions in (2.5) are fulfilled.

To study problem (2.5), we use a topological tool which is the Leray-Schauder degrees are defined and a homotopy principle [28, 34].

3. Existence and Uniqueness

In this section, the solvability of problem (2.5) shall be investigated. We consider the following three cases: $\lambda > 1$, $\lambda = 1$, and $0 < \lambda < 1$.

3.1. **Case $\lambda > 1$.**

The following lemma gives us the integral form of (2.5).

Lemma 3.1. *Suppose $\lambda > 1$. The boundary value problem (2.5) for (1)-differentiability is equivalent to the integral equation*

$$y(t) = \frac{1}{\lambda - 1} \int_0^T f(s, y(s))ds \oplus \int_0^t f(s, y(s)) \, \mathrm{d}s. \tag{3.1}$$

Proof. See [17]. $\square$

Remark 3.2. For (2)-differentiability, from Lemma 2.7, we have

$$y(0) = y(t) \oplus (-1) \odot \int_0^t f(s, y(s)) \, \mathrm{d}s, \tag{3.2}$$

for $t \in [0, T]$ satisfying $\lambda y(0) = y(T)$. So, the boundary condition produces

$$y(0) = y(T) \oplus (-1) \odot \int_0^T f(s, y(s)) \, \mathrm{d}s$$

or

$$y(0) = \lambda y(0) \oplus (-1) \odot \int_0^T f(s, y(s))\, \mathrm{d}s. \tag{3.3}$$

Passing to the level sets, we get

$$\left[\underline{y}^r(0), \overline{y}^r(0)\right] = \lambda.\left[\underline{y}^r(0), \overline{y}^r(0)\right] + (-1).\left[\int_0^T f(s, y(s))\, \mathrm{d}s\right]^r,$$

or

$$\left[(1-\lambda).\underline{y}^r(0), (1-\lambda).\overline{y}^r(0)\right] = (-1).\left[\int_0^T f(s, y(s))\, \mathrm{d}s\right]^r.$$

Consequently,

$$\underline{y}^r(0) = \frac{-1}{1-\lambda}.\left(\int_0^T \overline{f}^r(s, y(s))\, \mathrm{d}s\right)$$

and

$$\overline{y}^r(0) = \frac{-1}{1-\lambda}.\left(\int_0^T \underline{f}^r(s, y(s))\, \mathrm{d}s\right).$$

Since $\lambda > 1$, then $\frac{-1}{1-\lambda} > 0$ and therefore we have

$$(\lambda - 1).[\underline{y}^r(0), \overline{y}^r(0)] = \left[\int_0^T \overline{f}^r(s, y(s))\, \mathrm{d}s, \int_0^T \underline{f}^r(s, y(s))\, \mathrm{d}s\right], \quad \forall\ r \in [0, 1]. \tag{3.4}$$

Generally, the right side of Eq. (3.4) is not a fuzzy number unless for all $r \in [0, 1]$,

$$\int_0^T \overline{f}^r(s, y(s))\, \mathrm{d}s = \int_0^T \underline{f}^r(s, y(s))\, \mathrm{d}s,$$

and this means $\int_0^T f(s, y(s))ds$ or $y(0)$ are crisp (real number). Hence, for (2)-differentiability, the solution is crisp.

Theorem 3.3. **(Existence)**
Suppose $\lambda > 1$ holds and $f \in C([0, T] \times \mathbb{R}_{\mathcal{F}}; \mathbb{R}_{\mathcal{F}})$. If there exist functions $p, r \in C([0, T]; [0, \infty))$ such that

$$d_\infty(f(t, y), \widetilde{0}) \leq p(t) d_\infty(y, \widetilde{0}) + r(t), \qquad \text{for all } t \in [0, T], y \in \mathbb{R}_{\mathcal{F}}, \tag{3.5}$$

and

$$\frac{\lambda}{\lambda - 1}.\int_0^T p(s)\, \mathrm{d}s < 1, \tag{3.6}$$

then the boundary value problem (2.5) in the sense of (1)-differentiability has at least one solution in $C([0, T] \times \mathbb{R}_{\mathcal{F}})$.

Proof. In the view of Lemma 3.1, we are going to show that there exists at least one solution to (3.1), which is equivalent to show that (2.5) has at least one solution. Let us consider the operator $\mathcal{A} : C([0, T]; \mathbb{R}_{\mathcal{F}}) \to C([0, T]; \mathbb{R}_{\mathcal{F}})$ defined for all $t \in [0, T]$ by

$$\mathcal{A}y(t) = \frac{1}{\lambda - 1} \odot \int_0^T f(s, y(s))\, \mathrm{d}s \oplus \int_0^t f(s, y(s))\, \mathrm{d}s. \tag{3.7}$$

Thus our problem is reduced to proving the existence of at least one ν such that

$$\nu = \mathcal{A}\nu. \tag{3.8}$$

Now consider the following family of problems associated with (3.8), namely

$$y = \gamma\mathcal{A}y, \qquad \gamma \in [0,1]. \tag{3.9}$$

Let

$$H_0 = \frac{\lambda}{\lambda - 1}.$$

From (3.9) see that for all $t \in [0,T]$,

$$\begin{aligned}
d_\infty(y(t),\widetilde{0}) = d_\infty(\gamma\mathcal{A}y,\widetilde{0}) &= d_\infty(\frac{\gamma}{\lambda-1}\odot\int_0^T f(s,y(s))\mathrm{d}s \oplus \gamma\odot\int_0^t f(s,y(s))\mathrm{d}s,\widetilde{0}) \\
&\le d_\infty(\frac{\gamma}{\lambda-1}\odot\int_0^T f(s,y(s))\mathrm{d}s,\widetilde{0}) + d_\infty(\gamma\odot\int_0^t f(s,y(s))\mathrm{d}s,\widetilde{0}) \\
&\le \frac{\gamma}{\lambda-1}.\int_0^T d_\infty(f(s,y(s)),\widetilde{0})\mathrm{d}s + \gamma.\int_0^t d_\infty(f(s,y(s)),\widetilde{0})\mathrm{d}s \\
&\le \frac{\gamma}{\lambda-1}.\int_0^T d_\infty(f(s,y(s)),\widetilde{0})\mathrm{d}s + \gamma.\int_0^T d_\infty(f(s,y(s)),\widetilde{0})\mathrm{d}s \\
&\le \gamma H_0.\int_0^T d_\infty(f(s,y(s)),\widetilde{0})\mathrm{d}s \\
&< H_0.\int_0^T d_\infty(f(s,y(s)),\widetilde{0})\mathrm{d}s \\
&\le H_0.\int_0^T \Big(p(s)d_\infty(y(s),\widetilde{0}) + r(s)\Big)\,\mathrm{d}s \\
&\le H_0.\left(\max_{s\in[0,T]} d_\infty(y(s),\widetilde{0})\int_0^T p(s)\mathrm{d}s + \int_0^T r(s)\mathrm{d}s\right).
\end{aligned}$$

By rearranging and taking the maximum, we obtain

$$\max_{s\in[0,T]} d_\infty(y(s),\widetilde{0}) \le \frac{H_0.\int_0^T r(s)\mathrm{d}s}{1 - H_0.\int_0^T p(s)\mathrm{d}s} =: L. \tag{3.10}$$

Define the open ball with center $\widetilde{0}$ by

$$B_{L+1} = \left\{y \in C([0,T];\mathbb{R}_\mathcal{F}) : d_\infty(y(t),\widetilde{0}) < L+1\right\}.$$

We can see from (3.10) that all possible solutions to (3.9) satisfy $d_\infty(y(t),\widetilde{0}) < L+1$ for all $t \in [0,T]$. Thus, the following Leray-Schauder degrees are defined and a homotopy principle is applicable (see [28, 34]); and therefore we have from $\widetilde{0} \in B_{L+1}$ that

$$\begin{aligned}
deg_{LS}(\mathcal{I}-\mathcal{A}, B_{L+1},\widetilde{0}) &= deg_{LS}(\mathcal{I}-\gamma\mathcal{A}, B_{L+1},\widetilde{0}), \\
&= deg_{LS}(\mathcal{I}, B_{L+1},\widetilde{0}) = 1,
\end{aligned}$$

where $\mathcal{I}$ is identity map. Thus the non-zero property of the Leray-Schauder degree [28, 34] ensures the existence of at least one solution in B_{L+1} to (3.1) and hence to (2.5). □

The following corollary is a direct consequence of Theorem 3.3.

Corollary 3.4. *Suppose that* $\lambda > 1$ *and* $f \in C([0,T]\times\mathbb{R}_{\mathcal{F}};\mathbb{R}_{\mathcal{F}})$*. If* $f(t,y)$ *is bounded on* $[0,T]\times\mathbb{R}_{\mathcal{F}}$*, then the fuzzy BVP* (2.5) *has at least one solution in* $C([0,T];\mathbb{R}_{\mathcal{F}})$*.*

Theorem 3.5 (Uniqueness)**.** *Suppose* $\lambda > 1$ *and* $f \in C(I\times\mathbb{R}_{\mathcal{F}},\mathbb{R}_{\mathcal{F}})$*. If there exists function* $p \in C(I;[0,\infty))$ *such that*

$$d_\infty(f(t,y_1),f(t,y_2)) \le p(t)\, d_\infty(y_1,y_2), \qquad \text{for all } t\in[0,T], y\in\mathbb{R}_{\mathcal{F}}, \tag{3.11}$$

and

$$\frac{\lambda}{\lambda-1}.\int_0^T p(s)\mathrm{d}s < 1, \tag{3.12}$$

then, the boundary value problem (2.5) has a unique solution in $C(I,\mathbb{R}_{\mathcal{F}})$ *in the sense of (1)-differentiability.*

Proof. The proof is similar to the proof of Theorem 3.3 in [17]. Let $C(J,\mathbb{R}_{\mathcal{F}})$ denote the set of all continuous functions from $J\subset\mathbb{R}$ to $\mathbb{R}_{\mathcal{F}}$. Define the metric

$$D(u,v)=\sup_J d_\infty(u(t),v(t)),$$

for $u,v\in C(J,\mathbb{R}_{\mathcal{F}})$. Since $(\mathbb{R}_{\mathcal{F}},d_\infty)$ is a complete metric space [24, 25], a standard argument shows that space $(C(J,\mathbb{R}_{\mathcal{F}}),D)$ is also complete.

Now suppose that there are two solutions y_1,y_2 to (2.5). Then

$$\begin{aligned}
d_\infty(y_1,y_2) &= d_\infty(\mathcal{A}y_1,\mathcal{A}y_2)\\
&= d_\infty\Big(\tfrac{1}{\lambda-1}\odot\textstyle\int_0^T f(s,y_1(s))\mathrm{d}s\oplus\int_0^t f(s,y_1(s))\mathrm{d}s,\\
&\qquad \tfrac{1}{\lambda-1}\odot\textstyle\int_0^T f(s,y_2(s))ds\oplus\int_0^t f(s,y_2(s))\mathrm{d}s\Big)\\
&\le \tfrac{1}{\lambda-1}.d_\infty\Big(\textstyle\int_0^T f(s,y_1(s))\mathrm{d}s,\int_0^T f(s,y_2(s))\mathrm{d}s\Big)\\
&\qquad +d_\infty(\textstyle\int_0^t f(s,y_1(s))\mathrm{d}s,\int_0^t f(s,y_2(s))\mathrm{d}s)\\
&\le \tfrac{1}{\lambda-1}.\Big(\textstyle\int_0^T d_\infty(f(s,y_1(s)),f(s,y_2(s)))\mathrm{d}s\Big)\\
&\qquad +\textstyle\int_0^t d_\infty(f(s,y_1(s)),f(s,y_2(s)))\mathrm{d}s\\
&\le \tfrac{1}{\lambda-1}.\Big(\textstyle\int_0^T p(s)d_\infty(y_1(s),y_2(s))\mathrm{d}s\Big)+\int_0^t p(s)d_\infty(y_1(s),y_2(s))\mathrm{d}s\\
&\le \tfrac{1}{\lambda-1}.\Big(\textstyle\int_0^T p(s)d_\infty(y_1(s),y_2(s))\mathrm{d}s\Big)+\int_0^T p(s)d_\infty(y_1(s),y_2(s))\mathrm{d}s\\
&\le \tfrac{1}{\lambda-1}.D(y_1,y_2)\textstyle\int_0^T p(s)\mathrm{d}s+D(y_1,y_2)\int_0^T p(s)\mathrm{d}s\\
&= \tfrac{\lambda D(y_1,y_2)}{\lambda-1}.\textstyle\int_0^T p(s)\mathrm{d}s.
\end{aligned}$$

Consequently it is deduced that

$$D(y_1,y_2)\left(1-\frac{\lambda}{\lambda-1}.\int_0^T p(s)\mathrm{d}s\right)\le 0.$$

So, we have $D(y_1, y_2) = 0$, for all $t \in [0, T]$, since $\frac{\lambda}{\lambda-1}.\int_0^T p(s)\mathrm{d}s < 1$. Hence, the solution is unique. $\square$

3.2. **Case** $0 < \lambda < 1$.

In this case, the following lemma gives us the integral form of (2.5) for (2)-differentiability.

Lemma 3.6. *Suppose* $0 < \lambda < 1$. *The boundary value problem (2.5) for (2)-differentiability is equivalent to the integral equation*

$$y(t) = \frac{1}{\lambda - 1} \odot \int_0^T f(s, y(s))\mathrm{d}s \ominus (-1) \odot \int_0^t f(s, y(s))\mathrm{d}s. \tag{3.13}$$

Proof. Let $y : I \to \mathbb{R}_{\mathcal{F}}$ satisfy (2.5). It is easy to see from Lemma 2.7 that

$$y(t) = y(0) \ominus (-1) \odot \int_0^t f(s, y(s))\mathrm{d}s, \tag{3.14}$$

for $t \in [0, T]$, satisfying $\lambda y(0) = y(T)$. The boundary condition produces

$$y(0) = y(T) \ominus (-1) \odot \int_0^T f(s, y(s))\mathrm{d}s;$$

or equivalently

$$y(0) = \lambda y(0) \ominus (-1) \odot \int_0^T f(s, y(s))\mathrm{d}s. \tag{3.15}$$

Passing to the level sets, we get

$$\left[(1-\lambda).\underline{y}^r(0), (1-\lambda).\overline{y}^r(0)\right] = (-1).\left[\int_0^T \underline{f}^r(s, y(s))\mathrm{d}s, \int_0^T \overline{f}^r(s, y(s))\mathrm{d}s\right].$$

Consequently,

$$\underline{y}^r(0) = \frac{-1}{1-\lambda}.\left(\int_0^T \overline{f}^r(s, y(s))\mathrm{d}s\right)$$

and

$$\overline{y}^r(0) = \frac{-1}{1-\lambda}.\left(\int_0^T \underline{f}^r(s, y(s))\mathrm{d}s\right).$$

Since $0 < \lambda < 1$, then by producing the fuzzy number

$$y(0) = \frac{-1}{1-\lambda} \odot \int_0^T f(s, y(s))\mathrm{d}s, \tag{3.16}$$

So substituting (3.16) into (3.14) we obtain, for $t \in T$,

$$y(t) = \frac{-1}{1-\lambda} \odot \int_0^T f(s, y(s))\mathrm{d}s \ominus (-1) \odot \int_0^t f(s, y(s))\mathrm{d}s.$$

$\square$

Remark 3.7 ([17]). In the sense of (1)-differentiability, one can observe for any solution y of (2.5) that $\operatorname{diam}([y(t)]^r)$ is nondecreasing in variable t, for any fixed $r \in [0,1]$. Therefore the boundary conditions

$$\lambda \underline{y}^r(0) = \underline{y}^r(T), \quad \lambda \overline{y}^r(0) = \overline{y}^r(T)$$

imply that

$$\begin{aligned} \operatorname{diam}([y(T)]^r) &= \overline{y}^r(T) - \underline{y}^r(T) = \lambda\left(\overline{y}^r(0) - \underline{y}^r(0)\right) \\ &= \lambda \operatorname{diam}([y(0)]^r) < \overline{y}^r(0) - \underline{y}^r(0) = \operatorname{diam}([y(0)]^r). \end{aligned}$$

If $y(0)$ is not crisp, then for some r,

$$\operatorname{diam}([y(T)]^r) < \operatorname{diam}([y(0)]^r),$$

so that we can not find a solution to (2.5). For the existence of solution, it is necessary that

$$\lambda y(0) = y(T) = y(0) + \int_0^T f(s, y(s)) \mathrm{d}s;$$

hence

$$(\lambda - 1).\underline{y}^r(0) = \int_0^T \underline{f}^r(s, y(s)) \mathrm{d}s$$

and

$$(\lambda - 1).\overline{y}^r(0) = \int_0^T \overline{f}^r(s, y(s)) \mathrm{d}s.$$

Consequently,

$$(\lambda - 1).\operatorname{diam}([y(0)]^r) = \int_0^T \operatorname{diam}([f(s, y(s)]^r)) \mathrm{d}s \geq 0,$$

and $\operatorname{diam}([y(0)]^r) > 0$ leads to contradiction. Therefore, the unique possibility is $\operatorname{diam}([y(0)]^r) = 0$. Summary, for (1)-differentiability, if $\lambda \in (0,1)$ and $y(0)$ is crisp, then the solution is crisp.

Theorem 3.8. (Existence)
Suppose $0 < \lambda < 1$ holds and $f \in C([0,T] \times \mathbb{R}_{\mathcal{F}}; \mathbb{R}_{\mathcal{F}})$. If there exist functions $p, r \in C([0,T]; [0,\infty))$ such that

$$d_\infty(f(t,y), \widetilde{0}) \leq p(t) d_\infty(y, \widetilde{0}) + r(t), \qquad \text{for all } t \in [0,T], y \in \mathbb{R}_{\mathcal{F}}, \tag{3.17}$$

$$\int_0^T p(s) \mathrm{d}s < \frac{1-\lambda}{2-\lambda}, \tag{3.18}$$

then the boundary value problem (2.5) in the sense of (2)-differentiability has at least one solution in $C([0,T] \times \mathbb{R}_{\mathcal{F}})$.

Proof. In the view of Lemma 3.6, we are going to show that there exists at least one solution to (3.13), which is equivalent to show that (2.5) has at least one solution. Let us consider the operator $\mathcal{A} : C([0,T]; \mathbb{R}_{\mathcal{F}}) \to C([0,T]; \mathbb{R}_{\mathcal{F}})$, defined for all $t \in [0,T]$ by

$$\mathcal{A}y(t) = \frac{-1}{1-\lambda} \odot \int_0^T f(s, y(s)) \mathrm{d}s \ominus (-1) \odot \int_0^t f(s, y(s)) \mathrm{d}s. \tag{3.19}$$

Thus our problem is reduced to proving the existence of a solution ν of

$$\nu = \mathcal{A}\nu. \tag{3.20}$$

Now, consider the following family of problems associated with (3.20), namely

$$y = \gamma \mathcal{A} y, \qquad \gamma \in [0,1]. \tag{3.21}$$

Let

$$H_1 = \frac{2-\lambda}{1-\lambda}.$$

It is seen from (3.21) that for all $t \in [0,T]$,

$$\begin{aligned}
d_\infty(y(t), \widetilde{0}) &= d_\infty\left(\gamma \mathcal{A} y, \widetilde{0}\right) \\
&= d_\infty\left(\frac{-\gamma}{1-\lambda} \odot \int_0^T f(s,y(s))\mathrm{d}s \ominus (-1).\gamma \odot \int_0^t f(s,y(s))\mathrm{d}s, \widetilde{0}\right) \\
&\le |\gamma| d_\infty\left(\frac{1}{1-\lambda} \odot \int_0^T f(s,y(s))\mathrm{d}s, \int_0^t f(s,y(s))\mathrm{d}s\right) \\
&\le \gamma . d_\infty\left(\frac{2-\lambda}{1-\lambda} \odot \int_0^T f(s,y(s))\mathrm{d}s, \widetilde{0}\right) \\
&\le \gamma H_1 . d_\infty\left(\int_0^T f(s,y(s))\mathrm{d}s, \widetilde{0}\right) \\
&\le H_1 . \int_0^T \left(p(s) d_\infty(y(s), \widetilde{0}) + r(s)\right) \mathrm{d}s \\
&\le H_1 \left(\max_{s\in[0,T]} d_\infty(y(s), \widetilde{0}) \int_0^T p(s)\mathrm{d}s + \int_0^T r(s)\mathrm{d}s\right).
\end{aligned}$$

So, by rearranging and taking the maximum, we obtain

$$\max_{s\in[0,T]} d_\infty(y(s), \widetilde{0}) \le \frac{H_1 . \int_0^T r(s)\mathrm{d}s}{1 - H_1 . \int_0^T p(s)\mathrm{d}s} =: L. \tag{3.22}$$

Again, define the open ball with center $\widetilde{0}$ by

$$B_{L+1} = \left\{ y \in C([0,T]; \mathbb{R}_\mathcal{F}) : d_\infty(y(t), \widetilde{0}) < L+1 \right\}.$$

We can see from (3.22) that all possible solution to (3.21) satisfy $d_\infty(y(t), \widetilde{0}) < L+1$ for all $t \in [0,T]$. Therefore, similar to the proof of Theorem 3.3, the non-zero property of the Leray-Schauder degree ensures the existence of at least one solution in B_{L+1} to (3.13) and hence to (2.5). □

Theorem 3.9. (**Uniqueness**)
Suppose $0 < \lambda < 1$ *and* $f \in C(I \times \mathbb{R}_\mathcal{F}, \mathbb{R}_\mathcal{F})$. *If there exists function* $p \in C(I; [0,\infty))$ *such that*

$$d_\infty(f(t,y_1), f(t,y_2)) \le p(t)\, d_\infty(y_1, y_2), \qquad \textit{for all } t \in [0,T], y \in \mathbb{R}_\mathcal{F}, \tag{3.23}$$

and

$$\int_0^T p(s)\mathrm{d}s < \frac{1-\lambda}{2-\lambda}, \tag{3.24}$$

then the boundary value problem (2.5) has a unique solution in $C(I,\mathbb{R}_{\mathcal{F}})$ for (2)-differentiable.

Proof. Suppose that there are two solutions y_1, y_2 to (2.5) in the sense of (2)-differentiability. Then

$$\begin{aligned}
d_\infty(y_1,y_2) &= d_\infty(\mathcal{A}y_1,\mathcal{A}y_2)\\
&= d_\infty\Bigg(\frac{-1}{1-\lambda}\odot\int_0^T f(s,y_1(s))\mathrm{d}s \ominus (-1)\odot\int_0^t f(s,y_1(s))\mathrm{d}s\ ,\\
&\qquad\qquad \frac{-1}{1-\lambda}\odot\int_0^T f(s,y_2(s))\mathrm{d}s \ominus (-1)\odot\int_0^t f(s,y_2(s))\mathrm{d}s\Bigg)\\
&\le \frac{1}{1-\lambda}.d_\infty\left(\int_0^T f(s,y_1(s))\mathrm{d}s, \int_0^T f(s,y_2(s))\mathrm{d}s\right)\\
&\quad + d_\infty\left(\int_0^t f(s,y_1(s))\mathrm{d}s, \int_0^t f(s,y_2(s))\mathrm{d}s\right)\\
&\le \frac{1}{1-\lambda}.\left(\int_0^T d_\infty(f(s,y_1(s)),f(s,y_2(s)))\mathrm{d}s\right)\\
&\quad + \int_0^t d_\infty(f(s,y_1(s)),f(s,y_2(s)))ds\\
&\le \frac{1}{1-\lambda}.\left(\int_0^T p(s)d_\infty(y_1(s),y_2(s))\mathrm{d}s\right) + \int_0^t p(s)d_\infty(y_1(s),y_2(s))\mathrm{d}s\\
&\le \frac{1}{1-\lambda}.\left(\int_0^T p(s)d_\infty(y_1(s),y_2(s))\mathrm{d}s\right) + \int_0^T p(s)d_\infty(y_1(s),y_2(s))\mathrm{d}s\\
&\le \frac{1}{1-\lambda}.D(y_1,y_2)\int_0^T p(s)\mathrm{d}s + D(y_1,y_2)\int_0^T p(s)\mathrm{d}s\\
&= \frac{(2-\lambda)}{1-\lambda}.D(y_1,y_2)\int_0^T p(s)\mathrm{d}s.
\end{aligned}$$

By rearranging, we obtain

$$D(y_1,y_2)\left(1-\frac{(2-\lambda)}{1-\lambda}.\int_0^T p(s)\mathrm{d}s\right) \le 0.$$

So, by using (3.24), we have for all $t\in[0,T]$ that $D(y_1,y_2)=0$, and hence the solution is unique. □

3.3. **Case $\lambda = 1$.**

Here, we consider the following fuzzy differential equation

$$y' = f(t, y),\ t \in I = [0, T], \qquad y(0) = y(T). \tag{3.25}$$

For (1)-differentiable, the equivalent integral expression and the boundary condition imply that

$$y(0) = y(T) = y(0) \oplus \int_0^T f(s, y(s))ds,$$

that is

$$\widetilde{0} = y(0) \ominus y(0) = \int_0^T f(s, y(s))ds,$$

where

$$\widetilde{0}(x) = \begin{cases} 1 & x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This expression is equivalent to

$$0 = \int_0^T \underline{f}^r(s, y(s))ds \le \int_0^T \overline{f}^r(s, y(s))ds = 0, \quad \text{for every } r \in [0, 1].$$

Also, for (2)-differentiable have the equivalent integral expression and the boundary condition imply that

$$y(0) = y(T) \oplus (-1) \odot \int_0^T f(s, y(s))ds,$$

$$y(0) \ominus y(0) = (-1) \odot \int_0^T f(s, y(s))ds,$$

that is

$$\tilde{0} = (-1) \odot \int_0^T f(s, y(s))ds.$$

This expressing is equivalent to

$$0 = (-1). \int_0^T \overline{f}^r(s, y(s))ds \le (-1). \int_0^T \underline{f}^r(s, y(s)) = 0, \text{for every} r \in [0, 1].$$

Hence, in both cases, we have

$$\int_0^T \left(\underline{f}^r(s, y(s)) - \overline{f}^r(s, y(s)) \right) ds = 0, \quad \text{for every } r \in [0, 1],$$

and by continuity

$$\underline{f}^r(s, y(s)) = \overline{f}^r(s, y(s)), \quad \text{for every } r \in [0, 1], s \in [0, T]. \tag{3.26}$$

Therefore, (3.26) and $0 = \int_0^T \underline{f}^r(s, y(s))ds$ are two necessary conditions to obtain (periodic) solutions to problem (3.25).

Remark 3.10. [17] In the case (1)-differentiable, for each $r \in [0,1]$,

$$\begin{aligned}
&\operatorname{diam}\left([y(t)]^r\right)\\
&= \operatorname{diam}\left(\left[\underline{y}^r(0) + \int_0^t \underline{f}^r(s,y(s))ds, \overline{y}^r(0) + \int_0^t \overline{f}^r(s,y(s))ds\right]\right)\\
&= \operatorname{diam}\left([y(0)]^r\right) + \int_0^t \operatorname{diam}\left([f(s,y(s))]^r\right) ds.
\end{aligned}$$

For the function $y(t)$ to be constant in the variable t, for every $r \in [0,1]$ fixed, it is necessary that $\operatorname{diam}\left([f(s,y(s))]^r\right) = 0$, for all $r \in [0,1]$ and $s \in [0,T]$. In other words, assuming that f is continuous, the solution y has level sets with constant diameter if, for every $r \in [0,1]$, and every $s \in [0,T]$, $\operatorname{diam}\left([f(s,y(s))]^r\right) = 0$, that is $f(t,y(t))$ is crisp, for every $t \in [0,T]$.

In particular, if $f(t,x)$ is crisp, for every $t \in [0,T]$ and $x \in \mathbb{R}_{\mathcal{F}}$, then the diameter of each level set for the solutions to the initial value problem associated to equation $y'(t) = f(t,y(t)), t \in [0,T]$, is constant. Note that this does not mean that the solutions are crisp, but $\operatorname{diam}\left([y(t)]^r\right) = \operatorname{diam}\left([y(0)]^r\right)$, for every $t \in [0,T]$ and $r \in [0,1]$, that is, the diameter of each level set of the solution is diameter of the corresponding level set of the initial condition. Under this assumption, there could be fuzzy periodic solutions, too.

4. Summary and conclusions

In this work, we studied a class of fuzzy differential equations with boundary value conditions. Indeed, using the approach of the generalized differentiability and a homotopy principle as well as the Leray-Schauder degrees, we achieved to show that problem (2.5) possesses at least one solution under some appropriate conditions. Our results will be used in further works to verify numerical solutions of (2.5) which their results will be appeared somewhere in our future studies.

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