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Solutions of fuzzy heat-like equations by variational iteration method

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Abstract. We employ the variational iteration method (VIM) to obtain approximate analytical solutions to three dimensional fuzzy heat-like equations. We follow the same strategy as in Buckley-Feuring method for solving three dimensional fuzzy heat-like equations. This method does not always produce a solution, if the method fails to give a solution, then we need to check if Seikkala procedure generates a solution. We get the fuzzy solution for fuzzy heat-like equations which are obtained via the variational iteration method and illustrate examples are presented to show the Buckley-Feuring solution and Seikkala solution.

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1. Introduction

The study of fuzzy partial differential equation forms a suitable setting for mathematical modeling of real world problem in which uncertainties or vagueness. The concept of fuzzy set was introduced by Zadeh[19] in 1965. It is describing vagueness in linguistic and also base on the fuzzy set theory. The term fuzzy differential equation(FDE) was introduced in 1978 by Kaendal and Byatt[13]. There have been many suggestions for fuzzy derivative to study fuzzy differential equation. Fuzzy differential equation solution based on different notations of fuzzy derivative such as Seikkala derivative, Buckley-Feuring derivative, Puri-Ralescu derivative, Kaendel-Friedman-Ming derivative, Goetschel-Voxman derivative, or Dubois-Prade derivative. These derivatives are having some relationship. Some of these relationship are presented by Buckley and Feuring in[1]. The fuzzy differential equations and fuzzy initial value problem were regularly treated by O.Kaleva[12] and S.Seikkala[17].

Buckley and Feuring introducing the concept of elementary fuzzy partial differential equations(FPDE) and also discuss the existence of Buckley-Feuring solution(BF-solution)[1]. When a physical problem is transformed into a deterministic heat-like equation, we cannot usually be sure that this modeling is perfect. Also, the initial and boundary value may not be known exactly. If the nature of errors is random, then instead of a deterministic problem, we get a random heat-like equation with random initial and boundary values. But if the underlying structure is not probabilistic, e.g., because of subjective choice, then it may be appropriate to use fuzzy numbers instead of real random variables. Hence, our idea is solving three dimensional heat-like equation with fuzzy parameters via same startgy as Buckley-Feuring using variational iteration method. The variational iteration method (VIM) plays an important role in both mathematics and engineering. This method was proposed by Ji-Huan He as a modification of a general Lagrange multiplier method [11]. It has been shown that this procedure is a powerful tool for solving various kinds of problems (see [2], [6]-[10], [18]). The VIM gives rapidly convergent successive approximations of the exact solution.

The paper is structured as follows: In section 2, we define the problem, which is a three dimensional fuzzy heat-like equation where a analytical solution is the main interest of this work and the VIM are illustrated. Also the same strategy as in Buckley-Feuring is presented for three dimensional fuzzy heat-like equation. Next section, we give the two examples, in the first example BF-solution exists and the second example if BF-solution does not exist but the S-solution can exist. Conclusions are drown in Section 4.

2. Analysis of fuzzy heat-like equations

In this section, we demonstrate the main algorithm of Variational Iteration Method on heat-like equation and fuzzify the equation. Finally, we will present a sufficient condition for the BF-Solution exist.

2.1. **Fuzzy heat-like equations.** In here, we consider the heat-like equation with variable coefficients described by three dimensions which can be written in the form as follows

$$U_t + f_1(x,y,z)U_{xx} + f_2(x,y,z)U_{yy} + f_3(x,y,z)U_{zz} = Q(x,y,z,t,k) \tag{2.1}$$

subject to the certain initial and boundary conditions. These initial and boundary conditions, in state three-dimensional, can come variety of forms such as $U(x,y,z,0) = c_1,\ \ U_t(x,y,z,0) = \phi_1(x,y,z,c_2),\ \ U(M_1,x,y) = \phi_2(x,y,z,c_3,c_4), \ldots$. At this point we will not give any explicit structure to the boundary conditions except to say they depend on constants $c_l, \ldots c_{m_2}$ with the c_r in intervals L_r, $1 < r < m_1$.

In this work the method is illustrated for heat-like Eq. (2.1). In the following lines, the components of Eq. (2.1) are enumerated:

- $I_1 = [M_1, M_2],\ I_2 = [M_3, M_4],\ I_3 = [M_5, M_6]$ and $I_4 = [0, M_7]$ are intervals, which $M_{n_1}(n_1 = 1,2,3,4,5,6)$ is negative or positive and $M_7 > 0$.
- $Q(x,y,z,t,k),\ U(x,y,z,t),\ f_1(x,y,z),\ f_2(x,y,z)$ and $f_3(x,y,z)$ will be continuous functions for $(x,y,z,t) \in \prod_{j=1}^{4} I_j$.

- $f_1(x,y,z)$, $f_2(x,y,z)$ and $f_3(x,y,z)$ have a finite number of roots for each $(x,y,z) \in \prod_{j=1}^{3} I_j$.
- $k = (k_1,\ldots,k_{m_1})$ and $c = (c_1,\ldots,c_{m_2})$ are vector of constants with k_l in interval J_l and c_r in interval L_r.

Assume that Eq.(2.1) has a solution

$$U(x,y,z,t) = G(x,y,z,t,k,c), \tag{2.2}$$

for continuous G with $(x,y,z,t) \in \prod_{j=1}^{4} I_j, k \in J = \prod_{l=1}^{m_1} J_l$ and $c \in L = \prod_{r=1}^{m_2} Lr$.

Suppose the constants k_l and c_r are imprecise in their values. We will model this uncertainty by substitute triangular fuzzy numbers for k_l and c_r. If we fuzzify Eq. (2.1), then we obtain the fuzzy heat-like equation. Using the extension principle we compute $\bar{Q}$ from Q where $\bar{Q}(x,y,z,\bar{K})$ has $\bar{K} = (\bar{K}_1,\ldots,\bar{K}_{m_1})$, for a triangular fuzzy number $\bar{K}_l \in J_l, 1 \le l \le m_1$. The function U become $\bar{U}$, where $\bar{U} : \prod_{j=1}^{4} I_j \to \mathrm{R}_{\mathcal{F}}$. That is $\bar{U}(x,y,z,t)$ is a fuzzy number. The fuzzy heat-like equation is

$$\bar{U}_t + f_1(x,y,z)\bar{U}_{xx} + f_2(x,y,z)\bar{U}_{yy} + f_3(x,y,z)\bar{U}_{zz} = \bar{Q}(x,y,z,t,k) \tag{2.3}$$

subject to the certain initial and boundary conditions. The initial and boundary conditions can be of the form $\bar{U}(x,y,z,0) = \bar{C}_1, \bar{U}_t(x,y,z,0) = \bar{\phi}_1(x,y,z,\bar{C}_2)$, $\bar{U}(M_1,x,y,t) = \bar{\phi}_2(x,y,z,\bar{C}_3,\bar{C}_4),\ldots$

The $\bar{\phi}_j$ is the extension principle of ϕ_j. We wish to solve the problem given in Eq.(2.3). Finally, we fuzzify G in Eq.(2.2). Let $\bar{Z}(x,y,z,t) = \bar{G}(x,y,z,\bar{K},\bar{C})$ where $\bar{Z}$ is computed using the extension principle and is a fuzzy solution. Next Section , we will discuss solution with the same strategy as Buckley-Feuring for fuzzy heat-like equation. Let $\bar{K}[\alpha] = \prod_{l=1}^{m_1} \bar{K}[\alpha]$ and $\bar{C}[\alpha] = \prod_{r=1}^{m_2} \bar{C}[\alpha]$.

2.2. **He's variational iteration method.** To illustrate the basic idea of the VIM, we consider the following general partial differential equation:

$$L_tU + L_xU + L_yU + L_zU + NU = Q(x,y,z,t), \tag{2.4}$$

where L_t, L_x, L_y and L_z are linear operators of x,y,z and t respectively, N is a nonlinear operator and $Q(x,y,z,t)$ is the source non-homogeneous term. According to the VIM ([6]-[9]), we construct a correction functional for Eq.(2.1) in t-direction as follows

$$\begin{aligned} U_{n+1}(x,y,z,t) \;\;&=\;\; U_n(x,y,z,t) + \int_0^t \lambda(s)\Big\{(U_n)_{ss} + f_1(x,y,z)(\tilde{U_n})_{xx} \\ &\quad + f_2(x,y,z)(\tilde{U_n})_{yy} + f_3(x,y,z)(\tilde{U_n})_{zz} - Q\Big\}\,\mathrm{d}s, \end{aligned} \tag{2.5}$$

where $n \ge 0$ and λ is a Lagrange multiplier[11]. We now determine the Lagrange multiplier

$$\begin{aligned} \delta U_{n+1}(x,y,z,t) \;\;&=\;\; \delta U_n(x,y,z,t) + \delta\int_0^t \lambda(s)\Big\{(U_n)_{s} + f_1(x,y,z)(\tilde{U_n})_{xx} \\ &\quad + f_2(x,y,z)(\tilde{U_n})_{yy} + f_3(x,y,z)(\tilde{U_n})_{zz} - Q\Big\}\,\mathrm{d}s, \end{aligned}$$

$$\delta U_{n+1}(x,y,z,t) \quad = \quad \delta U_n(x,y,z,t) + \lambda(s)\delta((U_n)_s)|_{s=t} - \int_0^t \lambda'(s)\delta U_n \, \mathrm{d}s,$$

$\delta U_n \;:\; \lambda'(s) = 0,$
$\delta U_n \;:\; 1 - \lambda(s)|_{s=t} = 0.$

So, the Lagrange multiplier is $\lambda = -1$. Submitting the result into Eq.(2.5) leads to the following iteration formula

$$\begin{aligned} U_{n+1}(x,y,z,t) \quad &= \quad U_n(x,y,z,t) - \int_0^t \{(U_n)_s + f_1(x,y,z)(U_n)_{xx} \\ &\qquad + f_2(x,y,z)(U_n)_{yy} + f_3(x,y,z)(U_n)_{zz} - Q\} \, \mathrm{d}s. \end{aligned} \tag{2.6}$$

Iteration formula start with an initial approximation, for example $U_0(x,y,z,t) = U(x,y,z,0)$. Note that the VIM also has been used for system of linear and nonlinear partial differential equations([6]-[9]).

2.3. **Solutions.** We first present the Buckley and Feuring solution (BF-solution)[1]. Let

$$\bar{Z}(x,y,z,t)[\alpha] = [z_1(x,y,z,t,\alpha), z_2(x,y,z,t,\alpha)],$$
and
$$\bar{Q}(x,y,z,t,\bar{K})[\alpha] = [q_1(x,y,z,t,\alpha), q_2(x,y,z,t,\alpha)],$$

that by the definition

$$z_1(x,y,z,t,\alpha) = \min\{G(x,y,z,t,k,c)|k \in \bar{K}[\alpha], \quad c \in \bar{C}[\alpha]\},$$
$$z_2(x,y,z,t,\alpha) = \max\{G(x,y,z,t,k,c)|k \in \bar{K}[\alpha], \quad c \in \bar{C}[\alpha]\}$$
and
$$q_1(x,y,z,t,\alpha) = \min\{Q(x,y,z,t,k)|k \in \bar{K}[\alpha]\},$$
$$q_2(x,y,z,t,\alpha) = \max\{Q(x,y,z,t,k)|k \in \bar{K}[\alpha]\}$$

for all x, y, z, t and α.

Assume that the $f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)$ and the $z_i(x,y,z,t,\alpha)$ have a continuous partials so that $(z_i)_t + f_1(x,y,z)(z_i)_{xx} + f_2(x,y,z) + (z_i)_{yy} + f_3(x,y,z)(z_i)_{zz}$ is continuous for $i = 1,2$ and all $(x,y,z,t) \in \prod_{j=1}^4 I_j$ and $\alpha \in [0,1]$. Define

$$\begin{aligned} \Gamma(x,y,z,\alpha) \quad = \quad & [(z_1)_t + f_1(x,y,z)(z_1)_{xx} + f_2(x,y,z) + (z_1)_{yy} + f_3(x,y,z)(z_1)_{zz}, \\ & (z_2)_t + f_1(x,y,z)(z_2)_{xx} + f_2(x,y,z) + (z_2)_{yy} + f_3(x,y,z)(z_2)_{zz}] \end{aligned}$$

for all $(x,y,z,t) \in \prod_{j=1}^4 I_j$ and all α. If, for each fixed $(x,y,z,t) \in \prod_{j=1}^4 I_j$, $\Gamma(x,y,z,t,\alpha)$ defines α-cut of fuzzy number, then will say that $\bar{Z}(x,y,z,t)$ is differentiable and is written

$$\begin{aligned} \Gamma(x,y,z,t,\alpha) \quad &= \quad \bar{Z}_t[\alpha] + f_1(x,y,z)\bar{Z}_{xx}[\alpha] \\ &\qquad + f_2(x,y,z)\bar{Z}_{yy}[\alpha] + f_3(x,y,z)\bar{Z}_{zz}[\alpha] \end{aligned} \tag{2.7}$$

for all $(x,y,z,t) \in \prod_{j=1}^4 I_j$ and all α. $\Gamma(x,y,z,t,\alpha)$ defines an α-cut of a fuzzy number if the following conditions hold:

(i): $(z_1(x,y,z,t,\alpha))_t + f_1(x,y,z)(z_1(x,y,z,t,\alpha))_{xx}$
$+f_2(x,y,z)(z_1(x,y,z,t,\alpha))_{yy} + f_3(x,y,z)(z_1(x,y,z,t,\alpha))_{zz}$
is an increasing function of α for each $(x,y,z,t) \in \prod_{j=1}^{4} I_j$:

(ii): $(z_2(x,y,z,t,\alpha))_t + f_1(x,y,z)(z_2(x,y,z,t,\alpha))_{xx}$
$+f_2(x,y,z)(z_2(x,y,z,t,\alpha))_{yy} + f_3(x,y,z)(z_2(x,y,z,t,\alpha))_{zz}$
is an decreasing function of α for each $(x,y,z,t) \in \prod_{j=1}^{4} I_j$; and

(iii): $(z_1(x,y,z,t,\alpha))_t + f_1(x,y,z)(z_1(x,y,z,t,\alpha))_{xx}$
$+f_2(x,y,z)(z_1(x,y,z,t,\alpha))_{yy} + f_3(x,y,z)(z_1(x,y,z,t,\alpha))_{zz}$
$\leq (z_2(x,y,z,t,\alpha))_t + f_1(x,y,z)(z_2(x,y,z,t,\alpha))_{xx}$
$+f_2(x,y,z)(z_2(x,y,z,t,\alpha))_{yy} + f_3(x,y,z)(z_2(x,y,z,t,\alpha))_{zz}$
for all $(x,y,z,t) \in \prod_{j=1}^{4} I_j$.

We had already assumed that $z_i(x,y,z,t,\alpha)$ had continuous partials so

$$(z_i)_t + f_1(x,y,z)(z_i)_{xx} + f_2(x,y,z) + (z_i)_{yy} + f_3(x,y,z)(z_i)_{zz}$$

is continuous on $\prod_{j=1}^{4} I_j \times [0,1]$ for $i = 1,2$. Hence, if the conditions (i)-(iii) hold, $\bar{Z}(x,y,z,t)$ is differentiable. $\bar{Z}(x,y,z,t)$ will be a BF-solution of the fuzzy heat-like equation if: (i) $\bar{Z}(x,y,z,t)$ be differential; (ii) Eq.(2.1) holds for $\bar{U}(x,y,z,t) = \bar{Z}(x,y,z,t)$; and (c) $\bar{Z}(x,y,z,t)$ satisfies the initial and boundary conditions. Since there is not specified particular initial and boundary conditions then only is checked if Eq.(2.1) holds. We will say that $\bar{Z}(x,y,z,t)$ is a BF-solution (without the initial and boundary conditions) if $\bar{Z}(x,y,z,t)$ is differentiable and

$$\bar{Z}_t + f_1(x,y,z)\bar{Z}_{xx} + f_2(x,y,z)\bar{Z}_{yy} + f_3(x,y,z)\bar{Z}_{zz} = \bar{Q}(x,y,z,t,k) \tag{2.8}$$

or the following equations must hold:

$$\begin{aligned}(z_1)_t + f_1(x,y,z)(z_1)_{xx} + f_2(x,y,z) + (z_1)_{yy} + f_3(x,y,z)(z_1)_{zz}\\ = q_1(x,y,z,t,\alpha),\end{aligned} \tag{2.9}$$

$$\begin{aligned}(z_2)_t + f_1(x,y,z)(z_2)_{xx} + f_2(x,y,z) + (z_2)_{yy} + f_3(x,y,z)(z_2)_{zz}\\ = q_2(x,y,z,t,\alpha),\end{aligned} \tag{2.10}$$

for all $(x,y,z,t) \in \prod_{j=1}^{4} I_j$ and α.

If $\bar{Z}(x,y,z,t)$ is BF-solution and it satisfies the boundary conditions we will say that $\bar{Z}(x,y,z,t)$ is a BF-solution satisfying the boundary conditions.

If $\bar{Z}(x,y,z,t)$ is not a BF-solution, then we will consider the Seikkala solution (S-solution) [17].

Let us define the S-solution. Let $\bar{U}(x,y,z,t)[\alpha] = [u_1(x,y,z,t,\alpha), u_2(x,y,z,t,\alpha)]$. Consider the system of heat-like equations

$$(u_1)_t + f_1(x,y,z)(u_1)_{xx} + f_2(x,y,z)(u_1)_{yy} + f_3(x,y,z)(u_1)_{zz} = q_1(x,y,z,t,\alpha),$$
$$(u_2)_t + f_1(x,y,z)(u_2)_{xx} + f_2(x,y,z)(u_2)_{yy} + f_3(x,y,z)(u_2)_{zz} = q_2(x,y,z,t,\alpha),$$

for all $(x,y,z,t) \in \prod_{j=1}^{4} I_j$ and $\alpha \in [0,1]$. We append to Eqs.(2.9) and (2.10) any boundary conditions. For example, if they were $\bar{U}(x,y,z,0) = \bar{C}_1$ and $\bar{U}(x,y,z,M_1)$

$= \bar{C}_2$ then we add

$$\left.\begin{array}{l} u_1(x,y,z,0,\alpha) = c_{11}(\alpha) \\ u_2(x,y,z,0,\alpha) = c_{12}(\alpha) \\ u_1(x,y,z,M_1,\alpha) = c_{21}(\alpha) \\ u_2(x,y,z,M_1,\alpha) = c_{22}(\alpha) \end{array}\right\} \tag{2.11}$$

where $\bar{C}_i[\alpha] = [c_{i1}(\alpha), c_{i2}], i = 1,2$. Let $u_i(x,y,z,t,\alpha)$ solve Eqs. (2.9) and (2.10), plus the boundary conditions. If

$$[u_1(x,y,z,t,\alpha), u_2(x,y,z,t,\alpha)] \tag{2.12}$$

defines the α-cut of fuzzy number, for all $(x,y,z,t) \in \prod_{j=1}^{4} I_j$ and α, then $\bar{U}(x,y,z,t,\alpha)$ is the S-Solution. Clearly if the BF-solution satisfying the boundary conditions is $\bar{Z}(x,y,z,t)$, then $\bar{Z}(x,y,z,t)$ is also the S-solution. As we shall see, the S-solution can exist when the BF-solution fails to exist.

Now we will present a sufficient condition for the BF-solution to exist such as Buckley and Feuring. Since there are such a variety of possible initial and boundary conditions, so we will omit them from the following theorem. One must separately check out the initial and boundary conditions. So, we will omit the constants c_r, $1 \leq r \leq m_2$, from the problem. Hence, Eq.(2.2) becomes $U(x,y,z,t) = G(x,y,z,t,k)$, so $\bar{Z}(x,y,z,t) = \bar{G}(x,y,z,t,\bar{K})$.

Theorem 2.1. *Assume $\bar{Z}(x,y,z,t)$ is differentiable.*

(a) If for all $i \in \{1,\ldots,n\}, G(x,y,z,t,k)$ and $Q(x,y,z,t,k)$ are both increasing (or both decreasing) in k_i, for $(x,y,z,t) \in \prod_{j=1}^{4}$ and $k \in J$, then $\bar{Z}(x,y,z,t)$ is a BF-solution.

(b) If there is an $i \in \{1,\ldots,n\}$ so that for variable $k_i, G(x,y,z,t,k)$ is strictly increasing and $Q(x,y,z,t,k)$ is strictly decreasing (or $G(x,y,z,t,k)$ is strictly decreasing $Q(x,y,z,t,k)$ is strictly increasing), for $(x,y,z,t) \in \prod_{j=1}^{4}$ and $k \in J$, then $\bar{Z}(x,y,z,t)$ is not a BF-solution.

Proof. Proof is similar to proof of Theorem 1 in [1] □

Corollary 2.2. *Assume $\bar{Z}(x,y,z,t)$ is differentiable.*

(a) $\bar{Z}(x,y,z,t)$ is a BF-solution if

$$f_1(x,y,z) > 0, f_2(x,y,z) > 0, f_3(x,y,z) > 0, \quad (x,y,z) \in \prod_{j=1}^{3} I_j and \tag{2.13}$$

$$\frac{\partial G}{\partial k_l}\frac{\partial Q}{\partial k_l} > 0, for\ all \quad l = 1,\ldots,n, \quad (x,y,z,t) \in \prod_{j=1}^{4} I_j and \quad k \in J. \tag{2.14}$$

(b) If the (2.13) do not hold or the relation (2.14) does not hold for some l, then $\bar{Z}(x,y,z,t)$ is not BF-solution.

We will say that derivative conditions holds for three dimensional fuzzy heat-like equation with variable coefficients when Eqs. (2.13) and (2.14) are true.

Theorem 2.3. (1) *If $\bar{Z}(x,y,z,t)$ is a BF-solution then $\bar{Z}(x,y,z,t)$ is a S-solution.*

(2) *If $\bar{U}(x,y,z,t)$ and the derivative conditions holds, then $\bar{U}(x,y,z,t)$ is a BF-solution.*

Proof. Proof is similar to proof of Theorem 4.2 and Theorem 4.3 in [1]. □

3. Some examples

Example 3.1. We consider a fuzzy heat-like equation with variable coefficients described by a three-dimensional of the form

$$u_t + \frac{1}{36}\left(x^2U_{xx} + y^2U_{yy} + z^2U_{zz}\right) = kx^2y^2z^2 \tag{3.1}$$

and the initial condition $U(x,y,z,0) = c_1x^2 + c_2y^2 - c_3z^2$, where $x \in [0,1],\ t \in [0,1]$ and the value of parameters $k \in [0, J_1]$, and $c_1, c_2, c_3 \in [0, L]$.

By using the variational iteration method for three dimensional in equation (3.1), the following solution can be obtained

$$\begin{aligned} U(x,y,z,t) = G(x,y,z,t,c_1,c_2,c_3,k) &= \frac{1}{36}(c_1x^2 + c_2y^2 - c_3z^2)(e^{-t} - 35) \\ &\quad -\frac{1}{6}kx^2y^2z^2(e^{-6t} - 1). \end{aligned}$$

Now, we fuzzify the function $Q(x,y,z,k)$ and $G(x,y,z,c_1,c_2,c_3,k)$ producing their α-cuts

$$\begin{aligned} z_1(x,y,z,t,\alpha) &= \frac{1}{36}(c_{11}(\alpha)x^2 + c_{21}(\alpha)y^2 - c_{32}(\alpha)z^2)(e^{-t} - 35) \\ &\quad -\frac{1}{6}k_2(\alpha)x^2y^2z^2(e^{-6t} - 1), \end{aligned} \tag{3.2}$$

$$\begin{aligned} z_2(x,y,z,t,\alpha) &= \frac{1}{36}(c_{12}(\alpha)x^2 + c_{22}(\alpha)y^2 - c_{31}(\alpha)z^2)(e^{-t} - 35) \\ &\quad -\frac{1}{6}k_1(\alpha)x^2y^2z^2(e^{-6t} - 1), \end{aligned} \tag{3.3}$$

and also obtain for $Q(x,y,z,k)$

$$Q_1(x,y,z,t,\alpha) = k_1(\alpha)x^2y^2z^2,$$
$$Q_2(x,y,z,t,\alpha) = k_2(\alpha)x^2y^2z^2,$$

where $\bar{K}(\alpha) = [k_1(\alpha), k_2(\alpha)]$, and $\bar{C}_j(\alpha) = [c_{j1}(\alpha), c_{j2}(\alpha)]$ for $j = 1,2,3$.
we first need to check to see if $\bar{Z}(x,y,z,t)$ is differentiable. We compute

$$(z_i)_t + \frac{1}{36}x^2(z_i)_{xx} + \frac{1}{36}y^2(z_i)_{yy} + \frac{1}{36}z^2(z_i)_{zz} = k_ix^2y^2z^2,$$

for $i = 1,2$ which are α-cuts of $\bar{K}x^2y^2z^2$ that is α-cut of a fuzzy number. Hence, $\bar{Z}(x,y,z,t)$ is differentiable. Due to the corollary $\bar{Z}(x,y,z,t)$ is BF-solution because of

$$f_1(x,y,z) > 0, \quad f_2(x,y,z) > 0, \quad f_3(x,y,z) > 0 \ \text{ and} \frac{\partial Q}{\partial k} > 0, \quad \frac{\partial G}{\partial k} > 0.$$

The initial conditions are

$$z_i(x,y,0,\alpha) = c_{1i}(\alpha)x^2 + c_{2i}(\alpha)y^2 - c_{3i}(\alpha)z^2$$

for all $i = 1, 2$. Also satisfies the initial condition. Therefore, BF-solution holds the initial condition may written as

$$\bar{Z}(x,y,t,\alpha) \;=\; \frac{1}{36}(\bar{C}_1x^2 + \bar{C}_2y^2 - \bar{C}_3z^2)(e^{-t} - 35) - \frac{1}{6}\bar{K}x^2y^2z^2(e^{-6t} - 1).$$

α-Cuts of above equation produces Eqs.(3.2) and (3.3) for all $x, y, z \in [0,1]$ and $t \in [0,1]$.

Example 3.2. Consider the three dimensional fuzzy heat like equation as

$$U_t - \frac{1}{2}\left(U_{xx} + U_{yy} + U_{zz}\right) = k_1x^2 + k_2y^2 + k_3z^2, \tag{3.4}$$

subject to the initial condition $U(x,y,z,0) = 0$, where $x, y, z \in [0,1],\ t > 0$ and the value of parameters k_1, k_2 and k_3 are in intervals $[0, J]$.

Now, we apply the variational iteration scheme of Eq. (3.4) has the form

$$\begin{aligned} U_{n+1}(x,y,z,t) \;&=\; U_n(x,y,z,t) + \int_0^t \lambda(s)\left\{(U_n(X))_s - \frac{1}{2}\left[(\tilde{U}_n(X))_{xx}\right.\right. \\ &\left.\left.+(\tilde{U}_n(X))_{yy} + (\tilde{U}_n(X))_{zz}\right] - k_1x^2 - k_2y^2 - k_3z^2\right\} \mathrm{d}s \end{aligned} \tag{3.5}$$

where $X = (x,y,z,s)$ $n \geq 0$ and $U(x,y,z,0) = 0$. This implies that, the stationary conditions are

$$1 + \lambda|_{s=t} = 0, \quad \lambda'(s) = 0$$

Hence, the Lagrange multiplier is $\lambda = -1$. Substituting this value of the Lagrange multiplier into the functional (3.5) yields the iteration formula

$$\begin{aligned} U_{n+1}(x,y,z,t) = U_n(x,y,z,t) - \int_0^t &\left\{(U_n(X))_s - \frac{1}{2}\left[(\tilde{U}_n(X))_{xx}\right.\right. \\ &\left.\left.+(\tilde{U}_n(X))_{yy} + (\tilde{U}_n(X))_{zz}\right] - k_1x^2 - k_2y^2 - k_3z^2\right\} \mathrm{d}s \end{aligned} \tag{3.6}$$

We begin with an initial approximation: $U_0(x,y,z,t) = U(x,y,z,0) = 0$, and using the iteration formula (3.6), we obtain the closed form of the exact solution after the fourth iteration

$$\begin{aligned} U(x,y,z,t) &= G(x,y,z,t,k_1,k_2,k_3) \\ &= k_1\left(x^2t + x\frac{t^2}{2} + \frac{t^3}{12}\right) + k_2\left(y^2t + y\frac{t^2}{2} + \frac{t^3}{12}\right) + k_3\left(z^2t + z\frac{t^2}{2} + \frac{t^3}{12}\right) \end{aligned}$$

There is no BF-solution because $f_1(x,y,z) = -1 < 0,\ f_2(x,y,z) = -1 < 0$ and $f_3(x,y,z) = -1 < 0$ by the corollary (2.3.1). We proceed to look for a S-solution. We have to solve the following equations.

$$\begin{aligned} &(u_1(x,y,z,t,\alpha))_t - \tfrac{1}{2}(u_2(x,y,z,t,\alpha))_{xx} - \tfrac{1}{2}(u_2(x,y,z,t,\alpha))_{yy} - \tfrac{1}{2}(u_2(x,y,z,t,\alpha))_{zz} \\ &= k_{11}(\alpha)x^2 + k_{21}(\alpha)y^2 + k_{31}(\alpha)z^2, (u_1(x,y,z,t,\alpha))_t - \tfrac{1}{2}(u_1(x,y,z,t,\alpha))_{xx} \\ &\qquad -\tfrac{1}{2}(u_1(x,y,z,t,\alpha))_{yy} - \tfrac{1}{2}(u_1(x,y,z,t,\alpha))_{zz} \\ &= k_{12}(\alpha)x^2 + k_{22}(\alpha)y^2 + k_{32}(\alpha)z^2, \end{aligned}$$

subject to initial condition

$$u_j(x,y,z,0,\alpha) = 0$$

for $j = 1, 2$, where $\bar{K}_i = [k_{i1}(\alpha), k_{i2}(\alpha)]$ for $i = 1, 2, 3$. By VIM, The solution is

$$\begin{aligned} u_1(x,y,z,t,\alpha) &= k_{11}(\alpha)\left(x^2t+\frac{t^3}{12}\right)+k_{21}(\alpha)\left(y^2t+\frac{t^3}{12}\right)+k_{31}(\alpha)\left(z^2t+\frac{t^3}{12}\right) \\ &\quad + k_{12}(\alpha)x+k_{22}(\alpha)y+k_{32}(\alpha)z, u_2(x,y,z,t,\alpha) \\ &= k_{12}(\alpha)\left(x^2t+\frac{t^3}{12}\right)+k_{22}(\alpha)\left(y^2t+\frac{t^3}{12}\right)+k_{32}(\alpha)\left(z^2t+\frac{t^3}{12}\right) \\ &\quad + k_{11}(\alpha)x+k_{21}(\alpha)y+k_{31}(\alpha)z. \end{aligned}$$

Now we need to check if $[u_1(x,t,\alpha),\ u_2(x,t,\alpha)]$ defines α-cuts of a fuzzy number for $x \in [0,\pi/2],\ t \in [0,\pi/2]$. Since the $u_i(x,y,z,t,\alpha)$ are continuous and $u_1(x,y,z,t,1) = u_2(x,y,z,t,1)$, Now we only to check is $\frac{\partial u_1}{\partial\alpha} > 0$ and $\frac{\partial u_2}{\partial\alpha} < 0$. There is a region R contained in $[0,1]\times[0,1]$ for which the S-solution exists and in $[0,1]\times[0,1]-R$ There may be no S-solution. Since $\bar{K}_i,\ \forall\ i = 1,2,3$ are triangular fuzzy numbers we know that $k'_{i1}(\alpha),\ \forall\ i = 1,2,3$ are all positive numbers while $k'_{i2}(\alpha),\ \forall\, i = 1,2,3$ therefore, we pick simple fuzzy parameter so that $k'_{i1}(\alpha) = b > 0$ and $k'_{i2}(\alpha) = -b$. The "prime" denotes differentiation with respect to α. Hence, for a S-solution we require

$$\begin{aligned} &\frac{\partial u_1}{\partial\alpha} = b\left((x^2+y^2+z^2)t-(x+y+z)\frac{t^2}{2}+\frac{t^3}{4}\right) > 0, \\ (3.7)\qquad & \\ &\frac{\partial u_2}{\partial\alpha} = -b\left((x^2+y^2+z^2)t-(x+y+z)\frac{t^2}{2}+\frac{t^3}{4}\right) < 0. \end{aligned}$$

The inequality (3.7) holds if each

$$\frac{1}{2t}\left(\frac{t^2}{2}+\sqrt{\frac{t^4}{4}-4(y^2+z^2)t-(y+z)\frac{t^2}{2}+\frac{t^3}{4}}\right) < x \le 1,\ \ y,z\in[0,1],$$

and $t > 0$, therefore $\bar{Z}(x,y,z,t)$ is S-solution and

$$\begin{aligned} \bar{Z}(x,y,z,t) \quad = \quad & \bar{K}_1(\alpha)\left(x^2t+\frac{t^3}{12}\right)+\bar{K}_2(\alpha)\left(y^2t+\frac{t^3}{12}\right)+\bar{K}_3(\alpha)\left(z^2t+\frac{t^3}{12}\right) \\ & +\bar{K}_1(\alpha)x+\bar{K}_2(\alpha)y+\bar{K}_3(\alpha)z, \end{aligned}$$

for all $k_i \in [0,J],\ \forall\ \ i = 1,2,3$ and

$$\frac{1}{2t}\left(\frac{t^2}{2}+\sqrt{\frac{t^4}{4}-4(y^2+z^2)t-(y+z)\frac{t^2}{2}+\frac{t^3}{4}}\right) < x \le 1,\ \ y,z\in[0,1] \text{ and } t>0.$$

4. Conclusion

The VIM has been successfully applied to find exact solution of time dependent fuzzy heat-like equations three dimensions with variable co-efficients. This method solve the problem without any need for discretization of variables. The results shows that the VIM is a simple and reliable method for finding exact solution to fuzzy heat-like equations. Our strategy based on Buckley-Feuring[1] consist of two type of solutions (1) BF-solution, (2) S-solution. We presented examples showing the situation where BF-solution exists and does not exist. If the BF-solution fails to exist we check if the S-solution exists and when the S-solution fails to exist, we offer no solution to the fuzzy heat-like equations.

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