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Using fuzzy goal programming in solving quadratic bi-level fractional multi-objective programming problems

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Abstract. This paper presents a fuzzy goal programming (FGP) technique for solving Quadratic Bi-Level Fractional Multi-Objective Programming (QBL-FMOP) Problems. The present approach is an extension work of B. B Pal and B. N. Moitra in [10] and I. A. Baky in [2]. In a bi-level programming problem (BLPP), two decision makers (DMs) are located at two different hierarchical levels, each independently controlling one set of decision variables with different and perhaps conflicting objectives. In the present article both the lower level decision maker (LLDM) and upper level decision maker (ULDM) solve the problem for the decision variables and if the obtained result is not satisfactory from the DMs point of view, then both the DMs make a balance of decision powers i.e. the leader and follower would have to give possible relaxations of their decisions which depends on the decision-making context. At the first phase of the solution process, we transform the fractional quadratic programming model into an equivalent nonlinear quadratic problem. In this stage the above obtained quadratic problem is again transformed into an equivalent linear membership function by using first order Taylor series expansion. An illustrative numerical example is given in the end to demonstrate the procedure.

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1. Introduction

A bi-level programming problem is formulated for a problem in which two decision makers make decisions successively. For example, in a decentralized firm, top management, an executive or headquarters makes a decision such as a budget of the firm, and then each division determines a production plan in the full knowledge of the budget. We can find many instances of decision problem, which are formulated as bi-level programming problem, and concerning the above mentioned hierarchical decision problem in decentralized firm, it is natural that the decision makers behave cooperatively rather than noncooperatively.

Edmund and Bard [6] dealt with nonlinear bi-level mathematical problems in 1991. Savard and Gauvin [13] proposed steepest decent direction for the nonlinear bi-level programming. Vicente et. al. [15] discussed descent approach for quadratic bi-level programming problem (QBLPP) in 1994. Pal and Moitra [10] proposed FGP procedure to QBLPP.

During the mid-1960s and early 1970s of the last century, fractional programming (FP) was studied extensively [4, 3] in the literature. A usual linear fractional programming problem is a special case of a nonlinear programming problem, but it can be transformed into a linear programming problem by using the variable transformation method by Charnes and Cooper (1962). It can also transform the quadratic fractional programming problem into a quadratic programming problem by using the proper transformation.

In a BLPP, if the objective functions are linear fractional forms, then the problems are termed as linear fractional bi-level programming problem (LFBLPPs) and if they are of nonlinear fractional forms, they are termed as nonlinear fractional bi-level programming problems (NLFBLPPs). Quadratic fractional bi-level programming problem (QFBLPP) is one type of NLFBLPP.

The use of the fuzzy set theory [16] for decision problems with several conficting objectives was first introduced by Zimmermann [19]. Thereafter, various versions of fuzzy programming (FP) have been investigated and widely circulated in literature[5, 18].

Abo-Sinha [1] discussed multi-objective optimization for solving non-linear multi-objective bi-level programming problem in fuzzy environment. Baky [2] studied FGP algorithm for solving decentralized bi-level multi-objective programming problems. Zhang et al. [17] presented an algorithm to fuzzy linear multi objective bi-level programming problems by using λ-cut method. Gao et al [7] studied fuzzy linear BLMOPP based on λ-cut and goal programming.

In the present study, the FGP method is used to solve QBL-FMOP. In the model formulation process, the membership functions defined for the fuzzy goal of the problem are transformed into flexible forms by assigning the highest degree (unity) of the membership functions as their aspiration level. A linearization technique is adopted to linearize the quadratic fractional goals and to arrive at the most satisfactory solution in the decision making context.

The model formulation of the problem is presented in the next section.

2. Problem formulation

Complete residuated lattices, first introduced in the 1930sAssume that there are two levels in a hierarchy structure with upper-level decision maker (ULDM) and lower-level decision maker (LLDM). Let the vector of decision variables $X = (X_1, X_2) \in R^n$ be partitioned between the two planners. The upper-level decision maker has control over the vector $X_1 \in R^{n_1}$ and the lower-level decision maker has control over the vector $X_2 \in R^{n_2}$, where $n = n_1 + n_2$. We also assume that

$$F_i(X_1, X_2) : R^{n_1} \times R^{n_2} \longrightarrow R^{m_i} \qquad i = 1, 2,$$

are the upper-level and lower-level vector objective functions, respectively. So the QBL-FMOP problem of minimization type may be formulated as follows [2]:

(Upper Level)

$$\min_{X_1} F_1(X_1, X_2) = \min_{X_1} \big(f_{11}(X_1, X_2), f_{12}(X_1, X_2), \ldots, f_{1m_1}(X_1, X_2)\big),$$

where X_2 solves

(Lower Level)

$$\min_{X_2} F_2(X_1, X_2) = \min_{X_2} \big(f_{21}(X_1, X_2), f_{22}(X_1, X_2), \ldots, f_{2m_2}(X_1, X_2)\big),$$

subject to

$$X \in S = \{X = (X_1, X_2) \in R^n | A_1X_1 + A_2X_2 \begin{pmatrix} \leqslant \\ = \\ \geqslant \end{pmatrix} b, X \geq 0, b \in R^m\}. \tag{2.1}$$

Where

$$f_{ij}(X_1, X_2) = \frac{P_{ij}(X_1, X_2)}{Q_{ij}(X_1, X_2)}, \tag{2.2}$$

here

$$P_{ij}(X_1, X_2) = C_{ij}X + \frac{1}{2}X^T D_{ij}X \quad \text{and} \quad Q_{ij}(X_1, X_2) = \bar{C}_{ij}X + \frac{1}{2}X^T \bar{D}_{ij}X$$

$$j = 1, 2, \ldots, m_1, \quad i = 1 \quad \text{for ULDM objective functions,}$$

$$j = 1, 2, \ldots, m_2, \quad i = 2 \quad \text{for LLDM objective functions,}$$

and where

1. $X_1 = (x_1^1, x_1^2, \ldots, x_1^{n_1}), X_2 = (x_2^1, x_2^2, \ldots, x_2^{n_2})$.
2. Feasible region $S(\neq \varnothing)$ is convex.
3. $m_1$is the number of upper-level objective functions.
4. $m_2$is the number of lower-level objective functions.
5. m is the number of the constraints.
6. $A_i : m \times n_i$ matrix $i = 1, 2$.
7. $C_{ij}, \bar{C}_{ij}$ and b are constant vectors.
8. D_{ij} and $\bar{D}_{ij}$ are constant symmetric matrices.
9. $Q_{ij}(X_1, X_2) > 0$ for all $X \in S$.

3. Fuzzy programming formulation of QBL-FMOP

In QBL-FMOP problems, if an imprecise aspiration level is assigned to each of the objectives $(f_{ij}(X), i = 1, 2, \quad j = 1, 2, \dots, m_i)$, then these fuzzy objectives are termed as fuzzy goals. They are to be characterized by their associated membership functions by defining the tolerance limits for achievement of their aspired levels.

Let $(X_1^{1j}, X_2^{1j}; f_{1j}^{min}, j = 1, 2, \dots, m_1)$ and $(X_1^{2j}, X_2^{2j}; f_{2j}^{min}, j = 1, 2, \dots, m_2)$ be the optimal solutions of ULDM and LLDM objective functions, respectively, when calculated in isolation over the feasible solution space S. Let $l_{ij} \geq f_{ij}^{min}$ be the aspiration level assigned to the ijth objective $f_{ij}(X_1, X_2)$ (the subscript ij means that $j = 1, 2, \dots, m_1$ when $i = 1$ for ULDM problem and $j = 1, 2, \dots, m_2$ when $i = 2$ for LLDM problem). Then, the fuzzy goals appear as

$$f_{ij}(X_1, X_2) \lesssim l_{ij},$$

where " $\lesssim$ " indicate the fuzziness of the aspiration levels and is to be understood as "essentially less than" [19, 12].

Now, using the individual optimal solutions we find the values of all the objective functions at each individual optimal solution and formulate a payoff matrix as follows:

$$\begin{bmatrix} f_{11}(X^{11}) & \cdots & f_{1m_1}(X^{11}) & f_{21}(X^{11}) & \cdots & f_{2m_2}(X^{11}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{11}(X^{1m_1}) & \cdots & f_{1m_1}(X^{1m_1}) & f_{21}(X^{1m_1}) & \cdots & f_{2m_2}(X^{1m_1}) \\ f_{11}(X^{21}) & \cdots & f_{1m_1}(X^{21}) & f_{21}(X^{21}) & \cdots & f_{2m_2}(X^{21}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{11}(X^{2m_2}) & \cdots & f_{1m_1}(X^{2m_2}) & f_{21}(X^{2m_2}) & \cdots & f_{2m_2}(X^{2m_2}) \end{bmatrix} \tag{3.1}$$

where

$$X^{ij} = (X_1^{ij}, X_2^{ij}) \qquad i = 1, 2, \quad j = 1, 2, \dots, m_i.$$

The maximum value of each column can be considered as the upper tolerance limit u_{ij}, $i = 1, 2, \quad j = 1, 2, \dots, m_i$ of the fuzzy goal to the objective functions $(f_{ij}(X_1, X_2), i = 1, 2, \quad j = 1, 2, \dots, m_i)$.
The solution usually are different because the objectives of ULDM and the objectives of LLDM are conflicting in natural, therefore, it can easily be assumed that all values larger than or equal to $u_{ij}(i = 1, 2, \quad j = 1, 2, \dots, m_i,)$ are absolutely unacceptable to ULDM and LLDM. So the membership functions $\mu_{f_{ij}}(f_{ij}(X_1, X_2))$ for the ijth fuzzy goal can be expressed algebraically as:

$$\mu_{f_{ij}}(f_{ij}(X_1, X_2) = \begin{cases} 1 & f_{ij} \leqslant l_{ij} \\ \dfrac{u_{ij} - f_{ij}(X_1, X_2)}{u_{ij} - l_{ij}} & l_{ij} \leqslant f_{ij} \leqslant u_{ij} \\ 0 & f_{ij} > u_{ij} \end{cases} \tag{3.2}$$

Let

$$X_1^L = \min\{X_1^{2j} | j = 1, 2, \dots, m_2\}, \qquad X_2^L = \min\{X_2^{1j} | j = 1, 2, \dots, m_1\},$$
$$X_1^U \leq \max\{X_1^{1j} | j = 1, 2, \dots, m_1\}, \qquad X_2^U \leq \max\{X_2^{2j} | j = 1, 2, \dots, m_2\}.$$

Now, it is mentioned that the decisions lower than X_1^L and X_2^L are absolutely acceptable to the respective DMs. But, to make a balance of decision powers, the leader and follower would have to give possible relaxations of their decisions X_1^L and X_2^L, respectively, and that depends on the decision-making context.

Let X_1^U and X_2^U be the upper tolerance limits of the respective decisions. Then,

$$X_1^L \leq X_1 \leq X_1^U$$

$$X_2^L \leq X_2 \leq X_2^U \tag{3.3}$$

appear as constraints in making decisions.

4. FGP model of QBL-FMOP

In decision making situation, the aim of each DM is to achieve highest membership value (unity) of the associated fuzzy goal in order to obtain the absolute satisfactory solution. However, in real practice, achievement of all membership values to the highest degree (unity) is not possible due to conficting objectives. Therefore, decision policy for minimizing the regrets of the DMs for all the levels should be taken into consideration. Therefore, each DM should try to maximize his or her membership function by making them as close as possible to unity by minimizing its negative-deviational variables. In FP approaches, the highest degree of membership function is one. So, as in Mohamed [9], for the defined membership functions in (3.2), the flexible membership goals for both the levels can be presented as

$$\mu_{f_{ij}}(f_{ij}(X_1, X_2)) + d_{ij}^- - d_{ij}^+ = 1, \quad i = 1, 2, \quad j = 1, 2, \ldots, m_i,$$

or equivalently as

$$\frac{u_{ij} - f_{ij}(X_1, X_2)}{u_{ij} - l_{ij}} + d_{ij}^- - d_{ij}^+ = 1, \quad i = 1, 2, \quad j = 1, 2, \ldots, m_i, \tag{4.1}$$

where $d_{ij}^-, d_{ij}^+ \geq 0$ with $d_{ij}^- \times d_{ij}^+ = 0$ represent the under-and over-deviational, respectively, from the aspired levels. In this paper GP approach to fuzzy multi-objective decision making problems introduced by Mohamed [9] is extended to solve QBL-FMOP problems. Therefore, considering the goal achievement problem of the goals at the same priority level, the equivalent fuzzy quadratic bi-level fractional multi-objective goal programming model of the problem can be presented as

$$MinZ = \sum_{j=1}^{m_1} w_{1j}^- d_{1j}^- + \sum_{j=1}^{m_2} w_{2j}^- d_{2j}^-$$

subject to

$$\frac{u_{1j} - f_{1j}(X_1, X_2)}{u_{1j} - l_{1j}} + d_{1j}^- - d_{1j}^+ = 1 \qquad j = 1, 2, \ldots, m_1,$$

$$\frac{u_{2j} - f_{2j}(X_1, X_2)}{u_{2j} - l_{2j}} + d_{2j}^- - d_{2j}^+ = 1 \qquad j = 1, 2, \ldots, m_2,$$
$$X_1^L \leq X_1 \leq X_1^U$$
$$X_2^L \leq X_2 \leq X_2^U$$

$$(X_1, X_2) \in S \tag{4.2}$$
$$d_{ij}^-, d_{ij}^+ \geq 0 \text{ with } d_{ij}^- \times d_{ij}^+ = 0 \quad i = 1, 2, \quad j = 1, 2, \ldots, m_i,$$

where the numerical weights w_{ij}^- represent the relative importance of achieving the aspired levels of the respective fuzzy goals subject to the constraints set in the decision situation. In the present formulation, these values are determined as [9]

$$w_{ij}^- = \frac{1}{u_{ij} - l_{ij}} \quad i = 1, 2, \quad j = 1, 2, \ldots, m_i. \tag{4.3}$$

It can be easily realized that the membership goals in (4.2) are in the form of quadratic fractional. So, the existing FGP procedure cannot be used directly to solve the problem. To avoid such problems, a linearization procedure is presented in the following section.

5. Linearization of membership goals

In this section a linearization process for the quadratic fractional objectives on using the method of changing variable the under- and over-deviational variables of the membership goals associated with the fuzzy goals of the model is introduced to solve the problem. Considering the ijth membership goal in (4.2) can be presented as

$$h_{ij}u_{ij} - h_{ij}f_{ij}(X_1, X_2) + d_{ij}^- - d_{ij}^+ = 1 \text{ where } h_{ij} = \frac{1}{u_{ij} - l_{ij}} \;\; i = 1, 2, \;\; j = 1, 2, \ldots, m_i.$$

Introducing the expression of $f_{ij}(X_1, X_2)$ from (2.2). The above goal can be presented as

$$h_{ij}u_{ij} - h_{ij}\frac{P_{ij}(X_1, X_2)}{Q_{ij}(X_1, X_2)} + d_{ij}^- - d_{ij}^+ = 1,$$

or equivalently as

$$(h_{ij}u_{ij} - 1)Q_{ij}(X_1, X_2) - h_{ij}P_{ij}(X_1, X_2) + d_{ij}^- Q_{ij}(X_1, X_2) - d_{ij}^+ Q_{ij}(X_1, X_2) = 0.$$

Hence we have

$$G_{ij} + d_{ij}^- Q_{ij}(X_1, X_2) - d_{ij}^+ Q_{ij}(X_1, X_2) = 1, \tag{5.1}$$

where

$$G_{ij} = (h_{ij}u_{ij} - 1)Q_{ij}(X_1, X_2) - h_{ij}P_{ij}(X_1, X_2) + 1. \tag{5.2}$$

Now, using the method of variable change as presented by Kornbluth and Steuer [8] Pal et al. [11], and Steuer [14], the goal expression in (5.1) can be linearized as follows.

Let $D_{ij}^{-} = d_{ij}^{-}Q_{ij}(X_1, X_2)$ and $D_{ij}^{+} = d_{ij}^{+}Q_{ij}(X_1, X_2)$; the quadratic form of the expression in (9) is obtained as

$$G_{ij} + D_{ij}^{-} - D_{ij}^{+} = 1, \tag{5.3}$$

with $D_{ij}^{-}, D_{ij}^{+} \geq 0$ and $D_{ij}^{-}D_{ij}^{+} = 0$ since $d_{ij}^{-}, d_{ij}^{+} \geq 0$ and $Q_{ij}(x_1, x_2) > 0$.

Here, clearly the equation (5.3) contains only quadratic forms without any fractional part. Next, we transform the quadratic membership functions in (5.3), into equivalent linear membership functions, by first order Taylor series as follows.

Let, $X^{\circ} = (X_1^{\circ}, X_2^{\circ})$ determined by maximize the each of the objectives in upper level and lower level membership functions $\mu_{f_{ij}}(X)$ associated to upper level and lower level $f_{ij}(X_1, X_2)(i = 1, 2, \quad j = 1, 2, \ldots, m_i)$. Then linear approximation to the ijth membership goal in (5.3), by using first-order Taylor polynomial series can be obtained as

$$\begin{aligned} 1 - D_{ij}^{-} + D_{ij}^{+} = G_{ij}(X) &\cong G_{ij}(X^{\circ}) \\ &+ \Big((x_1 - x_1^{\circ})\frac{\partial}{\partial x_1} + (x_2 - x_2^{\circ})\frac{\partial}{\partial x_2} + \cdots + (x_n - x_n^{\circ})\frac{\partial}{\partial x_n}\Big)G_{ij}(X^{\circ}), \end{aligned}$$

or equivalently as

$$1 - D_{ij}^{-} + D_{ij}^{+} = G_{ij}(X) \cong G_{ij}(X^{\circ}) + \sum_{k=1}^{n}(x_k - x_k^{\circ})\frac{\partial G_{ij}(X^{\circ})}{\partial x_k}, \tag{5.4}$$

where n is the number of decision variables, x_k° is the kth component of $X^{\circ} = (X_1^{\circ}, X_2^{\circ})$ and $x_k$is the kth component of the new solution $X = (X_1, X_2)$.

In vector notation, the linear approximation of the ijth membership goal can be rewritten as

$$G_{ij}(X^{\circ}) + [\nabla G_{ij}(X^{\circ})]^T(X - X^{\circ}) + D_{ij}^{-} - D_{ij}^{+} = 1, \tag{5.5}$$

where $\nabla G_{ij}(X^{\circ})$ is the gradient of $G_{ij}(X^{\circ})$ and the superscript T denotes transpose of $\nabla G_{ij}(X^{\circ})$.

By following the linearization process, the extenuated proposed FGP model of the QBL-FMOP problem can be presented as

$$MinZ = \sum_{j=1}^{m_1} w_{1j}^{-}d_{1j}^{-} + \sum_{j=1}^{m_2} w_{2j}^{-}d_{2j}^{-}$$

subject to

$$\begin{aligned} &G_{1j}(X^{\circ}) + [\nabla G_{1j}(X^{\circ})]^T(X - X^{\circ}) + D_{1j}^{-} - D_{1j}^{+} = 1 \quad j = 1, 2, \ldots, m_1 \\ &G_{2j}(X^{\circ}) + [\nabla G_{2j}(X^{\circ})]^T(X - X^{\circ}) + D_{2j}^{-} - D_{2j}^{+} = 1 \quad j = 1, 2, \ldots, m_2 \\ &\quad X_1^L \leq X_1 \leq X_1^U \\ &\quad X_2^L \leq X_2 \leq X_2^U \end{aligned}$$

$$(5.6) \qquad (X_1, X_2) \in S$$
$$D^-_{ij}, D^+_{ij} \geq 0 \text{ with } D^-_{ij} \times D^+_{ij} = 0 \quad i = 1,2, \quad j = 1,2,\ldots,m_i,$$

where Z represents the fuzzy achievement function consisting of the weighted under-deviational variables, where the numerical weights $w^-_{ij} \quad i = 1,2, \quad j = 1,2,\ldots,m_i$, that defined in(5.6), represent the relative importance of achieving the aspired levels of the respective fuzzy goals subject to the constraints set in the decision situation.

The FGP model (5.6) provides the most satisfactory decision for both the ULDM and the LLDM by achieving the aspired levels of the membership goals to the extent possible in the decision environment. The solution procedure is straightforward and illustrated via the following example.

6. The FGP Algorithm for QBL-FMOP Problems

Following the above discussion, we can now construct the proposed FGP algorithm for solving the QBL-FMOP problems.

Step1. Calculate the individual minimum of each objective function in the two levels under the given constraints.

Step2. Formulate the payoff matrix as given by (3.1). Then set the goals and the upper tolerance limits for all the objective functions in the two levels.

Step3. Elicit the quadratic fractional membership functions for each of the objective functions in the two levels.

Step4. Determine $X^\circ = (X^\circ_1, X^\circ_2)$ which is the value(s) that is used to maximize the ijth membership function $\mu_{f_{ij}}(f_{ij}(X_1, X_2)$ associated with ijth ($i = 1,2, j = 1,2,\ldots,m_i$) objective.

Step5. Linearize the reduced quadratic programming from quadratic fractional membership functions by using (5.3) and (5.5).

Step6. Determine the preference bounds on the decision variables provide by the DMs in (3.3)

Step7. Formulate the Model (5.6) for the QBL-FMOP problem.

Step8. Solve Model (5.6) to get a candidate solution to the QBL-FMOP problem.

Step9. If the DM is satisfied with the candidate solution in Step 8, go to Step 10, else go to Step 11.

Step10. Stop with a satisfactory solution to the QBL-FMOP problem.

Step11. Modify the upper tolerance limits all the decision variables to reach a compromise optimal solutions i.e. go to Step 6.

7. Numerical examples

To demonstrate proposed FGP procedure, consider the following quadratic bi-level fractional multi-objective programming problem:
(Upper Level)

$$\min_{X_1} \big(f_{11} = \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2 + 2}, f_{12} = \frac{(x_1-2)^2 - x_2}{(x_2-1)^2 + 5}\big)$$

where X_2 solves
(Lower Level)

$$\min_{X_2}\left(f_{21}=\frac{(x_1-1)^2+(x_2+3)^2}{x_1^2+x_2+10}, f_{22}=\frac{8x_1^2-9x_2^2-4}{x_1^2+x_2^2+8}, f_{23}=8x_1^2+x_1-(x_2-2)^2\right)$$

subject to

$$\begin{aligned} x_1+x_2 &\leqslant 10\\ -5x_1+3x_2 &\leqslant 15\\ x_1,x_2 &\geqslant 0. \end{aligned}$$

The individual optimal solution of the leader and follower are $(x_1^{11},x_2^{11})=(0.32,5.53)$ with $f_{11}^{min}=-0.93, (x_1^{12},x_2^{12})=(2,2.44)$ with $f_{12}^{min}=-0.34, (x_1^{21},x_2^{21})=(3.16,0)$ with $f_{21}^{min}=0.683, (x_1^{22},x_2^{22})=(1.19,6.19)$ with $f_{22}^{min}=-7.41$ and $(x_1^{23},x_2^{23})=(0.86,6.43)$ with $f_{23}^{min}=-12.84$, respectively, then the fuzzy objectives goals appear as

$$f_{11}\lesssim -0.93,\quad f_{12}\lesssim -0.34,\quad f_{21}\lesssim 0.68,\quad f_{22}\lesssim -7.41,\quad f_{23}\lesssim -12.84.$$

Now, using the individual optimal solution, we formulate a payoff matrix as follows:

X^{ij}	f_{11}	f_{12}	f_{21}	f_{22}	f_{23}
(0.32,5.53)	-0.93	-10	4.68	-7.19	-11.32
(2,2.44)	-0.16	-0.34	1.86	-1.42	33.8
(3.16,0)	0.83	0.22	0.68	4.21	79.04
(1.19,6.19)	-0.88	-0.17	4.79	-7.41	-5.03
(0.86,6.43)	-0.92	-0.148	5.18	-7.39	-12.84

The following table summarizes the aspiration levels and upper tolerance limits, of all objective functions for the two levels of the QBL-FMOP problem.

	f_{11}	f_{12}	f_{21}	f_{22}	f_{23}
u_{ij}	0.83	0.22	5.18	4.21	79.04
l_{ij}	-0.93	-0.34	0.68	-7.41	-12.84

Now, by using the above tolerance ranges the quadratic fractional membership functions of ULDM are:

$$\mu_{f_{11}}(f_{11}(x_1,x_2))=\frac{0.83-\dfrac{x_1^2-x_2^2}{x_1^2+x_2^2+2}}{0.83+0.93}+d_{11}^- - d_{11}^+=1,$$

$$\mu_{f_{12}}(f_{12}(x_1,x_2))=\frac{0.22-\dfrac{(x_1-2)^2-x_2}{(x_2-1)^2+5}}{0.22+0.34}+d_{12}^- - d_{12}^+=1,$$

$$d_{1j}^-,d_{1j}^+\geq 0 \text{ with } d_{1j}^-\times d_{1j}^+=0\quad j=1,2.$$

The quadratic fractional membership functions of LLDM are:

$$\mu_{f_{21}}(f_{21}(x_1,x_2)) = \frac{5.18 - \dfrac{(x_1-1)^2+(x_2+3)^2}{x_1^2+x_2+10}}{5.18-0.68} + d_{21}^- - d_{21}^+ = 1,$$

$$\mu_{f_{22}}(f_{22}(x_1,x_2)) = \frac{4.21 - \dfrac{8x_1^2-9x_2^2-4}{x_1^2+x_2^2+8}}{4.21+7.41} + d_{22}^- - d_{22}^+ = 1,$$

$$\mu_{f_{23}}(f_{23}(x_1,x_2)) = \frac{79.04 - (8x_1^2+x_1-(x_2-2)^2)}{79.04+12.84} + d_{23}^- - d_{23}^+ = 1,$$

$$d_{2j}^-, d_{2j}^+ \geq 0 \text{ with } d_{2j}^- \times d_{2j}^+ = 0 \quad j=1,2,3.$$

After transforming the quadratic fractional membership functions of ULDM into quadratic forms, in (5.3) the model takes the form as

$$-1.09x_1^2 + 0.03x_2^2 - 0.06 + D_{11}^- - D_{11}^+ = 1,$$
$$-1.78(x_1-2)^2 - 0.6(x_2-1)^2 + 1.78x_2 - 2.04 + D_{12}^- - D_{12}^+ = 1,$$

where

$$D_{11}^- = d_{11}^-(x_1^2+x_2^2+2), D_{11}^+ = d_{11}^+(x_1^2+x_2^2+2) \text{ and } D_{12}^- = d_{12}^-((x_2-1)^2+5),$$
$$D_{12}^+ = d_{12}^+((x_2-1)^2+5), D_{1j}^-, D_{1j}^+ \geq 0 \text{ with } D_{1j}^- \times D_{1j}^+ = 0 \quad j=1,2.$$

The transformed quadratic fractional membership functions of LLDM are:

$$-0.22(x_1-1)^2 + 0.13x_1^2 - 0.22(x_2+3)^2 + 0.13x_2 + 2.3 + D_{21}^- - D_{21}^+ = 1,$$
$$-1.3x_1^2 + 0.06x_2^2 - 3.96 + D_{22}^- - D_{22}^+ = 1,$$
$$-0.08x_1^2 - 0.01x_1 + 0.01(x_2-2)^2 + 0.86 + D_{23}^- - D_{23}^+ = 1,$$

where

$$D_{21}^- = d_{21}^-(x_1^2+x_2+10), D_{21}^+ = d_{21}^+(x_1^2+x_2+10), D_{22}^- = d_{22}^-(x_1^2+x_2^2+8),$$
$$D_{22}^+ = d_{22}^+(x_1^2+x_2^2+8), D_{2j}^-, D_{2j}^+ \geq 0 \text{ with } D_{2j}^- \times D_{2j}^+ = 0 \quad j=1,2,3.$$

The membership functions for ULDM and LLDM are maximal at the points $\mu^*_{11}(0.32, 5.53)$, $\mu^*_{12}(2, 2.44)$, $\mu^*_{21}(3.16, 0)$, $\mu^*_{22}(1.19, 6.99)$ and $\mu^*_{23}(0.86, 6.43)$, respectively. Then membership functions are transformed using first-order Taylor polynomial series in (5.5) of ULDM and LLDM as

$$\begin{aligned}
1 - D^-_{11} + D^+_{11} &\cong G_{11}(0.32, 5.53) + \big((x_1 - 0.32)\frac{\partial}{\partial x_1} + (x_2 - 5.53)\frac{\partial}{\partial x_2}\big)G_{11}(0.32, 5.53)\\
&= -1.86x_1 + 0.32x_2 - 0.26.\\
1 - D^-_{12} + D^+_{12} &\cong G_{12}(2, 2.44) + \big((x_1 - 2)\frac{\partial}{\partial x_1} + (x_2 - 2.44)\frac{\partial}{\partial x_2}\big)G_{12}(2, 2.44)\\
&= 0.052x_1 + 0.93.\\
1 - D^-_{21} + D^+_{21} &\cong G_{21}(3.16, 0) + \big((x_1 - 3.16)\frac{\partial}{\partial x_1} + (x_2 - 0)\frac{\partial}{\partial x_2}\big)G_{21}(3.16, 0)\\
&= 1.77x_1 + 1.45x_2 - 2.96.\\
1 - D^-_{22} + D^+_{22} &\cong G_{22}(1.19, 6.99) + \big((x_1 - 1.19)\frac{\partial}{\partial x_1} + (x_2 - 6.99)\frac{\partial}{\partial x_2}\big)G_{22}(1.19, 6.99)\\
&= -3.09x_1 + 0.83x_2 - 5.02.\\
1 - D^-_{23} + D^+_{23} &\cong G_{23}(0.86, 6.43) + \big((x_1 - 0.86)\frac{\partial}{\partial x_1} + (x_2 - 6.43)\frac{\partial}{\partial x_2}\big)G_{23}(0.86, 6.43)\\
&= 0.1476x_1 - 0.88x_2 + 0.66.
\end{aligned}$$

Introducing the upper tolerance limits for x_1 and x_2, their tolerance ranges are obtained as

$$0.86 \leq x_1 \leq 2$$
$$2.44 \leq x_2 \leq 6.43.$$

Then, the proposed FGP model for solving QBL-FMOP is formulated as follows:
$MinZ = 0.56D^-_{11} + 1.78D^-_{12} + 0.22D^-_{21} + 0.08D^-_{22} + 0.01D^-_{23}$
subject to

$$\begin{aligned}
-1.86x_1 + 0.33x_2 + D^-_{11} - D^+_{11} &= 1.26\\
0.052x_2 + D^-_{12} - D^+_{12} &= 0.07\\
1.77x - 1 + 1.45x_2 + D^-_{21} - D^+_{21} &= 3.96\\
-3.09x_1 + 0.83x_2 + D^-_{22} - D^+_{22} &= 6.02\\
-0.147x_1 + 0.88x_2 + D^-_{23} - D^+_{23} &= 0.33\\
x_1 + x_2 &\leqslant 10\\
-5x_1 + 3x_2 &\leqslant 15\\
0.86 \leq x_1 &\leq 2\\
2.44 \leq x_2 &\leq 4\\
x_1, x_2 &\geqslant 0
\end{aligned}$$

$$D^-_{ij}, D^+_{ij} \geq 0,\ \ D^-_{ij} \times D^+_{ij} = 0 \quad i = 1, 2,\ \ j = 1, 2,\ \text{and}\ i = 1, 2,\ \ j = 1, 2, 3.$$

The *software* LINGO (ver. 11.0) is used to solve the problem. Optimal compromise solution of the problem is given by $x_1^* = 0.86$ and $x_2^* = 4$ with objective functions values $f_{11} = -0.81$, $f_{12} = -0.19$, $f_{21} = 3.32$, $f_{22} = -5.74$ and $f_{23} = 2.77$, with membership functions values $\mu_{11} = 0.93$, $\mu_{12} = 0.73$, $\mu_{21} = 0.41$, $\mu_{22} = 0.85$ and $\mu_{23} = 0.83$, respectively.

8. Conclusion

In this paper, a FGP algorithm is proposed to solve QBL-FMOP problems. This technique can be easily extended for other QBL-FMOP problems where the decision variables are integers. In the same fashion the present problem can be also considered and extended for the case of decentralized and QML-FMOP problems.

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