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Fuzzy hypergroups based on fuzzy spaces

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Abstract. Our interest in this paper is to define and study the concept of a fuzzy hypergroup, which depends on the concept of a fuzzy space. Indeed, the paper is a continuation of ideas presented by Davvaz [Fuzzy Sets and Systems, 101 (1999) 191-195]. A relation between a fuzzy hypergroup based on a fuzzy space and a fuzzy hypergroup in the sense of Davvaz is obtained

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1. Introduction

The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld [26]. Since the first paper by Rosenfeld, many papers are published in this area, for example see [1, 19, 24, 27]. The main problem in fuzzy mathematics is how to carry out the ordinary concepts to the fuzzy case. The difficulty lies in how to pick out the rational generalization from the large number of available approaches. Dib in [14] remarked the absence of the fuzzy universal set and discussed some problems in Rosenfeld's [26] approach. Its absence has strong effect on the introduced structure of fuzzy theory. A new approach how to define and study fuzzy groups and fuzzy subgroups is given in [14, 15]. This access depends on the concept of fuzzy space which serves as the concept of the universal set in the ordinary algebra, also see [16]. This approach can be considered as a generalization and a new formulation of Rosenfeld's approach.

Hyperstructures represent a natural extension of classical algebraic structures and they were introduced by the French mathematician F. Marty. Since then, hundreds of papers and several books have been written on this topic, see [3, 4, 29]. A recent book on hyperstructures [4] points out on their applications in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. The concept of H_v-structures [30] constitute a generalization of the well-known algebraic hyperstructures (hypergroups, hyperrings, hypermodules and so on). Actually some axioms concerning the above hyperstructures such as the associative law, the distributive law and so on are replaced by their corresponding weak axioms. The study of fuzzy hyperstructures is an interesting research topic of fuzzy sets. There is a considerable amount of work on the connections between fuzzy sets and hyperstructures, for example see [4, 5, 7, 8, 10, 11, 12, 13, 22, 23, 25]. In [5], Davvaz applied the concept of fuzzy sets to the theory of algebraic hyperstructures and defined fuzzy sub-hypergroup (respectively H_v-subgroup) of a hypergroup (respectively H_v-group) which is a generalization of the concept of Rosenfeld's fuzzy subgroup of a group. Further investigations are contained in [6, 9, 17, 18, 21, 28].

In this paper, we use the notion of fuzzy space to define fuzzy hyperstructure and fuzzy hyperoperation as a generalization of fuzzy groupoid and fuzzy group in the sense of Dib. Also, we introduce and discuss the concepts of fuzzy hypergroup (fuzzy H_v-group).

This paper is constructed as follows: After the introduction, in Section 2 we recall some basic notions and results on hypergroups and H_v-groups, in Section 3 we review basic facts about fuzzy spaces and fuzzy groups, in Section 4 using the concept of a fuzzy space, we introduce the concept of a fuzzy hypergroup in a natural way, in Section 5 a relation between a fuzzy hypergroup based on a fuzzy space and a fuzzy hypergroup in the sense of Davvaz is obtained. At last, a conclusion is presented.

2. Preliminaries

In this section, we summarize the preliminary definitions and results required in the sequel.

Let H be a non-empty set and let $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ and the couple $(H, \circ)$ is called a *hypergroupoid*.

If A and B are non-empty subsets of H, then we denote

$$A \circ B = \bigcup_{a \in A,\, b \in B} a \circ b, \quad x \circ A = \{x\} \circ A \;\text{ and }\; A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

We say that a semihypergroup $(H, \circ)$ is a *hypergroup* if for all $x \in H$, we have $x \circ H = H \circ x = H$. Now, we look at some examples of hypergroups.

(1) Let $(S, \cdot)$ be a semigroup and let P be a non-empty subset of S. For all x, y of S, we define $x \circ y = xPy$. Then $(S, \circ)$ is a semihypergroup. If $(S, \cdot)$ is a group, then $(S, \circ)$ is a hypergroup.

(2) If G is a group and for all x, y of G, $< x, y >$ denotes the subgroup generated by x and y, then we define $x \circ y =< x, y >$. We obtain that $(G, \circ)$ is a hypergroup.
(3) Let $(G, \cdot)$ be a group and let H be a non-normal subgroup of it. If we denote $G/H = \{xH \ : \ x \in G\}$, then $(G/H, \circ)$ is a hypergroup, where for all xH, yH of G/H, we have $xH \circ yH = \{zH \ : \ z \in xHy\}$.

A *subhypergroup* $(K, \circ)$ of $(H, \circ)$ is a non-empty set K, such that for all $k \in K$, we have $k \circ K = K \circ k = K$.

The hypergroupoid $(H, \circ)$ is called an *H_v-group*, if for all $x, y, z \in H$, the following conditions hold:

(1) $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \varnothing$,
(2) $x \circ H = H \circ x = H$.

Some examples of H_v-groups can be found in [29, 2, 20].

(1) Let $(G, \cdot)$ be a group and R be an equivalence relation on G. In G/R consider the hyperoperation $\odot$ defined by $\overline{x} \odot \overline{y} = \{\overline{z} | \ z \in \overline{x} \cdot \overline{y}\}$, where $\overline{x}$ denotes the equivalence class of the element x. Then $(G, \odot)$ is an H_v-group which is not always a hypergroup.
(2) On the set $\mathbb{Z}_{mn}$ consider the hyperoperation $\oplus$ defined by setting $0 \oplus m = \{0, m\}$ and $x \oplus y = x + y$ for all $(x, y) \in \mathbb{Z}^2_{mn} - \{(0, m)\}$. Then $(\mathbb{Z}_{mn}, \oplus)$ is an H_v-group. $\oplus$ is weak associative but not associative.

An H_v-subgroup $(K, \circ)$ of $(H, \circ)$ is a non-empty set K, such that for all $k \in K$, we have $k \circ K = K \circ k = K$. Davvaz applied in [5] fuzzy sets to the theory of algebraic hyperstructures and studied their fundamental properties.

Definition 2.1 ([5])**.** Let $(H, \circ)$ be a hypergroup (respectively, H_v-group) and let A be a fuzzy subset of H. Then A is said to be a *fuzzy sub-hypergroup* (respectively, *fuzzy H_v-subgroup*) of H if the following axioms hold:

1) $\min\{A(x), A(y)\} \leq \inf\{A(z) \ : \ z \in x \circ y\}$ for all $x, y \in H$,
2) for all $x, a \in H$ there exists $y \in H$ such that $x \in a \circ y$ and
$$\min\{A(a), A(x)\} \leq A(y),$$
3) for all $x, a \in H$ there exists $z \in H$ such that $x \in z \circ a$ and
$$\min\{A(a), A(x)\} \leq A(z).$$

3. Basic facts about fuzzy spaces and fuzzy groups

Throughout the paper, we shall adopt the notations:
X: for a non-empty set,
I: for the closed interval $[0, 1]$ of real numbers.

The concept of *fuzzy space* (X, I) was introduced and discussed by Dib [14], where (X, I) is the set of all ordered pairs $(x, I); x \in X$; i.e., $(X, I) = \{(x, I) \ : \ x \in X\}$, where $(x, I) = \{(x, r) \ : \ r \in I\}$. The ordered pair (x, I) is called a *fuzzy element* in the fuzzy space (X, I).

A *fuzzy subspace* U of the fuzzy space (X, I) is the collection of all ordered pairs (x, u_x), where $x \in U_\circ$ for some $U_\circ \in X$ and u_x is a subset of I, which contains at least one element beside the zero element. If it happens that $x \notin U_\circ$, then $u_x = 0$. An empty fuzzy subspace is defined as $\{(x, \phi_x) \ : \ x \in \phi\}$.

Let $U = \{(x, u_x) \ : \ x \in U_\circ\}$ and $V = \{(x, v_x) \ : \ x \in V_\circ\}$ be fuzzy subspaces of (X, I). The *union* and *intersection* of U and V are defined respectively as follows:

$$U \cup V = \{(x, u_x \cup v_x) \ : \ x \in U_\circ \cup V_\circ\},$$
$$U \cap V = \{(x, u_x \cap v_x) \ : \ x \in U_\circ \cap V_\circ\}.$$

Clearly both of $U \cup V$ and $U \cap V$ are fuzzy subspaces of the fuzzy space (X, I).

Let (X, I) be a fuzzy space and let A be a fuzzy subset of X with $A_\circ$ denoted the support of the fuzzy subset A, i.e., $A_\circ = \{x : A(x) \neq 0\}$. The fuzzy subset A induces the following fuzzy subspaces of the fuzzy space (X, I):

- The lower fuzzy subspace $\underline{H}(A) = \{(x, [0, A(x)]) : x \in A_\circ\}$.
- The upper fuzzy subspace $\overline{H}(A) = \{(x, \{0\} \cup [A(x), 1]) : x \in A_\circ\}$.
- The finite fuzzy subspace $H_\circ(A) = \{(x, \{0, A(x)\}) : x \in A_\circ\}$.

Given two fuzzy spaces namely, (X, I) and (Y, I). A *fuzzy function* $\underline{F}$ from (X, I) into (Y, I) is defined as an ordered pair $\underline{F} = (F, \{f_x\}_{x \in X})$, where F is a function from X into Y, and $\{f_x\}_{x \in X}$ is a family of onto functions (called *co-membership functions*) $f_x : I \to I$, satisfying the conditions:

(1) f_x is non decreasing on I,
(2) $f_x(0) = 0$, $f_x(1) = 1$.

A *fuzzy binary operation* $\underline{F} = (F, f_x)$ on the fuzzy space (X, I) is a fuzzy function from $(X, I)\Box(X, I) \to (X, I)$, where $F : X \times X \to X$ with onto co-membership functions $f_{xy} : I\Box I \to I$ which satisfies $f_{xy}(r, s) \neq 0$ if $r \neq 0$ and $s \neq 0$, where $I\Box I$ is the vector lattice with partial order defined for all $r_1, r_2, s_1, s_2 \in I$ by

(i) $(r_1, r_2) \leq (s_1, s_2)$ if and only if $r_1 \leq s_1$ and $r_2 \leq s_2$ whenever $s_1 \neq 0$ and $s_2 \neq 0$.
(ii) $(0, 0) \leq (s_1, s_2)$ whenever $s_1 = 0$ or $s_2 = 0$.

The fuzzy binary operation $\underline{F} = (F, f_x)$ on (X, I) is said to be *uniform* if the associated co-membership functions f_{xy} are identical for all $x, y \in X$, i.e., $f_{xy} = f$ for all $x, y \in X$.

A fuzzy space (X, I) together with a fuzzy binary operation $\underline{F} = (F, f_x)$ is said to be a *fuzzy groupoid* and is denoted by $((X, I); \underline{F})$. A *fuzzy semigroup* is a fuzzy groupoid which is associative. A *fuzzy monoid* is a fuzzy semigroup admits an identity (e, I) such that for every $(x, I) \in (X, I)$ we have $(x, I)\underline{F}(e, I) = (e, I)\underline{F}(x, I) = (x, I)$.

A *fuzzy group* is a fuzzy monoid in which each fuzzy element (x, I) has an inverse $(x, I)^{-1} = (x^{-1}, I)$ such that $(x, I)\underline{F}(x, I)^{-1} = (x, I)^{-1}\underline{F}(x, I) = (e, I)$. A fuzzy group is said to be *abelian (commutative)* if and only if for any $(x, I), (y, I) \in (X, I)$ we have $(x, I)\underline{F}(y, I) = (y, I)\underline{F}(x, I)$.

4. Fuzzy hypergroup

In this section we define fuzzy hypergroups, we give a correspondence relation between these fuzzy notions and classical ordinary notions.

Definition 4.1. Let (H, I) be a non-empty fuzzy space. A fuzzy hyperstructure (hypergroupoid), denoted by $\langle (H, I), \Diamond \rangle$ is a fuzzy space together with a fuzzy function having onto co-membership functions (referred as a fuzzy hyperoperation) $\Diamond : (H,I)\times(H,I) \to \mathcal{P}^*((H,I))$, where $\mathcal{P}^*((H,I))$ denotes the set of all nonempty fuzzy subspaces of (H,I) and

$$\Diamond = (\triangle, \bigtriangledown_{xy}) \text{ with } \triangle : H\times H \to \mathcal{P}^*(H) \text{ and } \bigtriangledown_{xy} : I\times I \to I.$$

A fuzzy hyperoperation $\Diamond = (\triangle, \bigtriangledown_{xy})$ on (H,I) is said to be *uniform* if the associated co-membership functions $\bigtriangledown_{xy}$ are identical, i.e., $\bigtriangledown_{xy} = \bigtriangledown$ for all $x,y\in H$. A uniform fuzzy hyperstructure $\langle (H,I),\Diamond\rangle$ is a fuzzy hyperstructure $\langle (H,I),\Diamond\rangle$ with uniform fuzzy hyperoperation.

Recall that the action of the fuzzy function $\Diamond = (\triangle, \bigtriangledown_{xy})$ on fuzzy elements of the fuzzy space (H,I) can be symbolized as follows: $(x,I)\Diamond(y,I) = (x\triangle y, \bigtriangledown_{xy}(I\square I)) = (\triangle(x,y), I)$.

Remark 4.2. For the fuzzy hyperstructure $\langle (H,I),\Diamond\rangle$ when $\Diamond : (H,I)\times(H,I) \to (H,I) \subseteq \mathcal{P}^*((H,I))$, then the fuzzy hyperstructure $\langle (H,I),\Diamond\rangle$ becames a fuzzy groupoid in the sense of Dib.

The next theorem gives a correspondence relation between fuzzy hyperstructures and ordinary hyperstructures.

Theorem 4.3. *To each fuzzy hyperstructure $\langle (H,I),\Diamond\rangle$ there is an associated ordinary hyperstructure $\langle H,\triangle\rangle$ which is isomorphic to the fuzzy hyperstructure $\langle (H,I),\Diamond\rangle$ by the correspondence $(x,I)\leftrightarrow x$.*

Proof. It is straightforward. $\square$

Example 4.4. (1) Let $Q=\{a\}$. Define the fuzzy hyperoperation $\Diamond = (\triangle, \bigtriangledown_{xy})$ over the fuzzy space (Q,I) such that $\triangle : Q\times Q\to\mathcal{P}^*(Q)$ with $\triangle(a,a) = a\triangle a = \{a\}$, and $\bigtriangledown_{aa} : I\times I\to I$ with $\bigtriangledown_{aa}(r,s) = r\wedge s$. That is,

$$(a,I)\Diamond(a,I) = \{(a, \bigtriangledown_{aa}(I\square I))\},$$

for $(a,I)\in(Q,I)$. Clearly $\langle (Q,I),\Diamond\rangle$ defines a uniform fuzzy hyperstructure (the trivial fuzzy hyperstructure).

(2) Let $R=\{a,b\}$. Define the fuzzy hyperoperation $\Diamond = (\triangle, \bigtriangledown_{xy})$ over the fuzzy space (R,I) such that $\triangle : R\times R\to\mathcal{P}^*(R)$ with

$$\begin{aligned}&\triangle(a,a) = a\triangle a = \{a\},\\ &\triangle(a,b) = \triangle(b,a) = \triangle(b,b) = \{a,b\}\end{aligned}$$

and $\bigtriangledown_{xy} : I\times I\to I$ such that

$$\bigtriangledown_{aa}(r,s) = r\wedge s,\ \bigtriangledown_{ab}(r,s) = \bigtriangledown_{ba}(r,s) = \bigtriangledown_{bb}(r,s) = r\vee s.$$

That is,

$$(x,I)\Diamond(y,I) = \{(x, \bigtriangledown_{xy}(I\square I)), ((y, \bigtriangledown_{yx}(I\square I))\},$$

for all $(x,I),(y,I)\in(R,I)$. Thus $\langle (R,I),\Diamond\rangle$ defines a (nonuniform) fuzzy hyperstructure.

(3) Let $Z=\{1,2,3\}$. Define the fuzzy hyperoperation $\Diamond = (\triangle, \bigtriangledown_{xy})$ over the fuzzy space (Z,I) such that $\triangle : Z\times Z\to\mathcal{P}^*(Z)$ with

$$\begin{aligned}&\triangle(x,x) = \{x\},\\ &\triangle(x,y) = \triangle(y,x) = \{x,y\},\end{aligned}$$

and $\bigtriangledown_{xy} : I \times I \rightarrow I$ such that $\bigtriangledown_{xy}(r,s) = r \cdot s$, for all $(x,I),(y,I) \in (Z,I)$. That is,

$$(x,I)\Diamond(y,I) = \{(x, \bigtriangledown_{xy}(I\Box I)), ((y, \bigtriangledown_{yx}(I\Box I))\},$$

for all $(x,I),(y,I) \in (Z,I)$. Thus $\langle (Z,I), \Diamond \rangle$ defines a uniform fuzzy hyperstructure.

(4) Let $H = \{-1, 1, -i, i\}$ where i is the imaginary number $\sqrt{-1}$. Define the fuzzy hyperoperation $\Diamond = (\bigtriangleup, \bigtriangledown_{xy})$ over the fuzzy space (H,I) such that

$$\begin{array}{l} \bigtriangleup : H \times H \rightarrow \mathcal{P}^*(H) \ \text{ with } \ \bigtriangleup(x,y) = x \bigtriangleup y = \{x,y\}, \text{ and} \\ \bigtriangledown_{xy} : I \times I \rightarrow I \ \text{ with } \bigtriangledown_{xy}(r,s) = r \wedge s, \end{array}$$

that is,

$$(x,I)\Diamond(y,I) = \{(x, \bigtriangledown_{xy}(I\Box I)), ((y, \bigtriangledown_{yx}(I\Box I))\},$$

for all $(x,I),(y,I) \in (H,I)$. Therefore $\langle (H,I), \Diamond \rangle$ is a uniform fuzzy hyperstructure, and from Theorem 4.3, we have $\langle H, \bigtriangleup \rangle$ is the associated ordinary hyperstructure to the fuzzy hyperstructure $\langle (H,I), \Diamond \rangle$.

After introducing fuzzy hyperstructure, we are able now to define the notion of a fuzzy hypergroup.

Definition 4.5. A *fuzzy hypergroup* is a fuzzy hyperstrucure $\langle (H,I), \Diamond \rangle$ satisfying the following axioms:

(i) $((x,I)\Diamond(y,I))\Diamond(z,I) = (x,I)\Diamond((y,I)\Diamond(z,I))$, for all $(x,I),(y,I),(z,I)$ in (H,I),

(ii) $(x,I)\Diamond(H,I) = (H,I)\Diamond(x,I) = (H,I)$, for all (x,I) in (H,I).

Definition 4.6. A *fuzzy H_v-group* is a fuzzy hyperstructure $\langle (H,I), \Diamond \rangle$ satisfying the following conditions:

(i) $((x,I)\Diamond(y,I))\Diamond(z,I) \cap (x,I)\Diamond((y,I)\Diamond(z,I)) \neq \phi$, for all $(x,I),(y,I),(z,I) \in (H,I)$,

(ii) $(x,I)\Diamond(H,I) = (H,I)\Diamond(x,I) = (H,I)$, for all $(x,I) \in (H,I)$.

For the sake of simplicity we will denote $(x,I)\Diamond(H,I)$ by $(x,I)\Diamond H$.

If $\langle (H,I), \Diamond \rangle$ satisfies only the first condition of Definition 4.4 (Definition 4.5) then it is called a *fuzzy semihypergroup* (*fuzzy H_v-semigroup*). A *uniform fuzzy hypergroup* (*fuzzy H_v-group*) is a fuzzy hypergroup (fuzzy H_v-group) having uniform co-membership functions, i.e., $\Diamond = (\bigtriangleup, \bigtriangledown_x = \bigtriangledown)$ for all $x \in H$. A fuzzy hypergroup (fuzzy H_v-group) $\langle (H,I), \Diamond \rangle$ is a *commutative* fuzzy hypergroup if $((x,I)\Diamond(y,I)) = ((y,I)\Diamond(x,I))$ for all $(x,I),(y,I)$ in (H,I).

Similar to Theorem 4.3, the next theorem gives a relation between fuzzy hypergroups and ordinary hypergroups.

Theorem 4.7. *For each fuzzy hypergroup (fuzzy H_v-group) there is an associated ordinary hypergroup (H_v-group) $\langle H, \bigtriangleup \rangle$ which is isomorphic to the fuzzy hypergroup (fuzzy H_v-group) $\langle (H,I), \Diamond \rangle$ by the correspondence $(x,I) \leftrightarrow x$.*

Proof. It is straightforward. □

Definition 4.8. Let $\langle (H,I), \Diamond \rangle$ be a fuzzy hypergroup (fuzzy H_v-group) and let

$$U = \{(x, u_x) : \ x \in U_\circ\}$$

be a fuzzy subspace of (H, I). Then $\langle U; \Diamond\rangle$ is called a *fuzzy sub-hypergroup* (*fuzzy H_v-subgroup*) of the fuzzy hypergroup $\langle (H, I), \Diamond\rangle$ if $\Diamond$ is closed on the fuzzy subspace U and $\langle U; \Diamond\rangle$ satisfies the conditions of an ordinary hypergroup (H_v-group).

Example 4.9. (1) Let $\langle (Q, I), \Diamond\rangle$ be as in Example 4.4 (1). Then $\langle (Q, I), \Diamond\rangle$ defines a fuzzy hypergroup (the trivial fuzzy hypergroup). Let Q' be a fuzzy subspace of the fuzzy space (Q, I) such that $Q' = \{(a, \alpha)\}$ for some fixed number $\alpha < 1$. Then $(Q', \Diamond)$ defines a fuzzy sub-hypergroup of the fuzzy hypergroup $\langle (Q, I), \Diamond\rangle$. If we redefine the co-membership functions $\bigtriangledown_{aa}$ to be $\bigtriangledown_{aa}(r, s) = r \cdot s$, then $(Q', \Diamond)$ is not a fuzzy sub-hypergroup of $\langle (Q, I), \Diamond\rangle$ since $(a, \alpha)\Diamond Q' = \{(a, \bigtriangledown_{aa}(r, s))\} = \{(a, \alpha^2)\} \neq Q'$. Note that, different from the classical case, the trivial fuzzy hypergroup $\langle (Q, I), \Diamond\rangle$ can have more than one fuzzy sub-hypergroup, for instance $(Q'', \Diamond)$ where $Q'' = \{(a, \beta)\}$ for some fixed number $\beta < \alpha < 1$ is a fuzzy sub-hypergroup of the fuzzy group $\langle (Q, I), \Diamond\rangle$ and $(Q', \Diamond) \neq (Q'', \Diamond)$.

(2) Let $\langle (R, I), \Diamond\rangle$ be as in Example 4.4 (2). Then $\langle (R, I), \Diamond\rangle$ defines a (non-uniform) fuzzy hypergroup. Also, the fuzzy subspaces $R_1 = \{(a, \alpha)\}$ and $R_2 = \{(b, \beta)\}$ defines a fuzzy sub-hypergroup for any $\alpha, \beta \in I$. On the other hand, the fuzzy subspace $R_3 = \{(a, \alpha), (b, \beta)\}$ defines a fuzzy sub-hypergroup of the fuzzy hypergroup $\langle (R, I), \Diamond\rangle$ if $\alpha = \beta$. That is $R_3 = \{(a, \alpha), (b, \alpha)\}$.

(3) Let $\langle (Z, I), \Diamond\rangle$ be as in Example 4.4 (3). Then $\langle (Z, I), \Diamond\rangle$ defines a fuzzy hypergroup. Now, we consider the fuzzy subspace $Z' = \{(1, \alpha), (3, \beta)\}$ with $\beta < \alpha < 1$ are fixed. Then $(Z', \Diamond)$ is not a fuzzy sub-hypergroup of $\langle (Z, I), \Diamond\rangle$. If we redefine the co-membership functions $\bigtriangledown_{xy} : I \times I \to I$ to be:

$$\bigtriangledown_{13}(r, s) = \begin{cases} \frac{rs}{\beta} & \text{if } rs \leq \alpha\beta \\ \frac{\alpha - rs}{1-\beta} & \text{if } rs \geq \alpha\beta \end{cases} \qquad \bigtriangledown_{31}(r, s) = \begin{cases} \frac{rs}{\alpha} & \text{if } rs \leq \alpha\beta \\ \frac{\beta + rs}{1+\alpha} & \text{if } rs \geq \alpha\beta \end{cases}$$

and $\bigtriangledown_{xy}(r, s) = r \vee s$ otherwise, such that

$$(x, v_x)\Diamond(y, v_y) = \{(x, \bigtriangledown_{xy}(r, s)), (y, \bigtriangledown_{yx}(r, s))\}$$

for all $x, y \in Z$. Then $(Z', \Diamond)$ is a fuzzy sub-hypergroup of $\langle (Z, I), \Diamond\rangle$.

(4) Let H be as in Example 4.4 (4). Define the fuzzy hyperoperation $\Diamond = (\bigtriangleup, \bigtriangledown_{xy})$ over the fuzzy space (H, I) such that

$$\bigtriangleup : H \times H \to \mathcal{P}^*(H) \ \text{ with } \bigtriangleup(x, y) = x \bigtriangleup y = \{x, y\},$$

and $\bigtriangledown_{xy} : I \times I \to I$ such that

$$\bigtriangledown_{11}(r, s) = \begin{cases} \frac{rs}{a} & \text{if } rs \leq a^2 \\ \frac{a+rs}{1+a} & \text{if } rs \geq a^2 \end{cases},$$

$$\bigtriangledown_{-1-1}(r, s) = \begin{cases} \frac{rs}{b} & \text{if } rs \leq b^2 \\ \frac{b+rs}{1+b} & \text{if } rs \geq b^2 \end{cases},$$

$$\bigtriangledown_{-11}(r, s) = \begin{cases} \frac{(rs)}{a} & \text{if } rs \leq ab \\ \frac{b+rs}{1+a} & \text{if } rs \geq ab \end{cases},$$

$$\bigtriangledown_{1-1}(r, s) = \begin{cases} \frac{(rs)}{b} & \text{if } rs \leq ab \\ \frac{a+rs}{1+b} & \text{if } rs \geq ab \end{cases},$$

and the other co-membership functions are given by $r \vee s$, where a, b are given fixed numbers satisfying $0 < b < a < 1$. Clearly, $\langle (H, I), \Diamond \rangle$ is a fuzzy hypergroup and the fuzzy subspace

$$U = \{(-1, [0, a]), (1, [0, a])\}$$

defines a fuzzy sub-hypergroup $\langle U, \Diamond \rangle$ of $\langle (H, I), \Diamond \rangle$.

On the other hand, the fuzzy subspace $V = \{(-1, [0, b]), (1, [0, a])\}$ will define a fuzzy sub-hypergroup $\langle V, \Diamond \rangle$ of $\langle (H, I), \Diamond \rangle$ if

$$(x, v_x)\Diamond(y, v_y) = \{(x, \bigtriangledown_{xy}(r, s)), (y, \bigtriangledown_{yx}(r, s))\}$$

such that $x, y \in \{-1, 1\}$. If we assume that

$$(x, v_x)\Diamond(y, v_y) = \{(x, \bigtriangledown_{xy}(r, s)), (y, \bigtriangledown_{xy}(r, s))\},$$

then

$$\begin{array}{rcl} (-1, [0, b])\Diamond V & = & (-1, [0, b])\Diamond(1, [0, a]) \cup (-1, [0, b])\Diamond(-1, [0, b]) \\ & = & \{(-1, \bigtriangledown_{-11}(r, s)), (1, \bigtriangledown_{-11}(r, s))\} \cup \{(-1, \bigtriangledown_{-11}(r, s))\} \\ & = & \{(-1, [0, \bigtriangledown_{-11}(r, s)]), (1, [0, \bigtriangledown_{-11}(r, s)])\} \cup \{(-1, [0, b])\} \\ & = & \{(-1, [0, b]), (1, [0, b])\} \cup \{(-1, [0, b])\} \neq V. \end{array}$$

While by assuming $(x, v_x)\Diamond(y, v_y) = \{(x, \bigtriangledown_{xy}(r, s)), (y, \bigtriangledown_{yx}(r, s))\}$ we have

$$\begin{array}{rcl} (-1, [0, b])\Diamond V & = & (-1, [0, b])\Diamond(1, [0, a]) \cup (-1, [0, b])\Diamond(-1, [0, b]) \\ & = & \{(-1, \bigtriangledown_{-11}(r, s)), (1, \bigtriangledown_{1-1}(r, s))\} \cup \{(-1, \bigtriangledown_{-1-1}(r, s))\} \\ & = & \{(-1, [0, \bigtriangledown_{-11}(r, s)]), (1, [0, \bigtriangledown_{1-1}(r, s)])\} \cup \{(-1, [0, b])\} \\ & = & \{(-1, [0, b]), (1, [0, a])\} \cup \{(-1, [0, b])\} = V = V\Diamond(-1, [0, b]), \end{array}$$

$$\begin{array}{rcl} (1, [0, a])\Diamond V & = & (1, [0, a])\Diamond(-1, [0, b]) \cup (1, [0, a])\Diamond(1, [0, a]) \\ & = & \{(1, \bigtriangledown_{1-1}(r, s)), (1, \bigtriangledown_{-11}(r, s))\} \cup \{(1, \bigtriangledown_{11}(r, s))\} \\ & = & \{(1, [0, \bigtriangledown_{1-1}(r, s)]), (-1, [0, \bigtriangledown_{-11}(r, s)])\} \cup \{(1, [0, a])\} \\ & = & \{(1, [0, a]), (-1, [0, b])\} \cup \{(1, [0, a])\} = V = V\Diamond(1, [0, a]). \end{array}$$

That is $\langle V, \Diamond \rangle$ is a fuzzy sub-hypergroup of $\langle (H, I), \Diamond \rangle$.

Theorem 4.10. *$\langle U; \Diamond \rangle$ is a fuzzy sub-hypergroup (fuzzy H_v-subgroup) of the fuzzy hypergroup (fuzzy H_v-group) $\langle (H, I), \Diamond \rangle$ if and only if*

(i) *$\langle U; \triangle \rangle$ is an ordinary sub-hypergroup (ordinary H_v-subgroup) of the ordinary hypergroup (ordinary H_v-group) $\langle H, \triangle \rangle$.*

(ii) $\bigtriangledown_{xy}(u_x, u_y) = u_{x\triangle y}$.

Proof. Assume that conditions (i) and (ii) are satisfied, we want to show that $\langle U; \Diamond \rangle$ is a fuzzy sub-hypergroup of the fuzzy hypergroup $\langle (H, I), \Diamond \rangle$.

(1) U is closed under $\Diamond$: Let $(x, u_x), (y, u_y) \in U$. Then

$$\begin{array}{rcl} (x, u_x)\Diamond(y, u_y) & = & (x \triangle y, \bigtriangledown_{xy}(u_x, u_y)) \\ & = & (x \triangle y, u_{x\triangle y}) \in U. \end{array}$$

(2) $\langle U;\Diamond\rangle$ satisfies the conditions of an ordinary hypergroup:
Let $(x,u_x),(y,u_y)$ and (z,u_z) be in U. Then

$$\begin{aligned}((x,u_x)\Diamond(y,u_y))\,\Diamond(z,u_z) &= (x\triangle y, u_{x\triangle y})\,\Diamond(z,u_z)\\ &= \big((x\triangle y)\triangle z, u_{(x\triangle y)\triangle z}\big)\\ &= \big(x\triangle(y\triangle z), u_{x\triangle(y\triangle z)}\big)\\ ((x,u_x)\Diamond(y,u_y))\,\Diamond(z,u_z) &= (x,u_x)\Diamond\left((y,u_y)\Diamond(z,u_z)\right).\end{aligned}$$

Also, for any $(x,u_x)\in U$ we have

$$\begin{aligned}(x,u_x)\Diamond U &= \bigcup_{y\in U}(x,u_x)\Diamond(y,u_y)\\ &= \bigcup_{y\in U}((x\triangle y),\bigtriangledown_{xy}(u_x,u_x))\\ &= U;(\text{by assumptions}).\end{aligned}$$

Similarly $U\Diamond(x,u_x)=U$. Therefore by (1) and (2) we have $\langle U;\Diamond\rangle$ is a fuzzy sub-hypergroup.

Conversely, assume that $\langle U;\Diamond\rangle$ is a fuzzy sub-hypergroup then (i) follow directly from Theorem 4.7. For (ii) let $(x,u_x),(y,u_y)\in U$ then

$$(x,u_x)\Diamond(y,u_y) \quad=\quad ((x\triangle y),\bigtriangledown_{xy}(u_x,u_y))\in U.$$

Thus $\bigtriangledown_{xy}(u_x,u_y)$ is the corresponding possible membership values for $x\triangle y$. That is, $\bigtriangledown_{xy}(u_x,u_y)=u_{x\triangle y}$. In order to prove that $\langle(H,I),\Diamond\rangle$ is a fuzzy H_v-subgroup the same method can be applied. □

5. Fuzzy sub-hypergroups induced by fuzzy subsets

In this section, first we introduce fuzzy sub-hypergroups and fuzzy H_v-subgroups induced by fuzzy subsets and then we obtain a relationship between the induced fuzzy sub-hypergroup, fuzzy H_v-subgroup and fuzzy sub-hypergroup and fuzzy H_v-subgroup in the sense of Davvaz [5].

Let A be a non-empty fuzzy subset of H and let $H_\circ(A),\underline{H}(A)$ and $\overline{H}(A)$ be fuzzy subspaces of (H,I) induced by the fuzzy set A. Then we have the following theorem:

Theorem 5.1. *$\langle H_\circ(A),\Diamond\rangle,\langle\underline{H}(A),\Diamond\rangle$ and $\langle\overline{H}(A),\Diamond\rangle$ are fuzzy sub-hypergroups (fuzzy H_v-subgroups) of the fuzzy hypergroup (fuzzy H_v-group) $\langle(H,I),\Diamond\rangle$ if and only if*

1. *$(A_\circ,\triangle)$ is an ordinary sub-hypergroup (H_v-subgroup) of the ordinary hypergroup (H_v-subgroup) $(H,\triangle)$.*
2. *$\bigtriangledown_{xy}(A(x),A(y))=\inf\{A(z)\ :\ z\in x\triangle y\}$, for all $x,y\in A_\circ$.*

Proof. The proof follow immediately from Theorem 4.10. □

Remark 5.2. We will refer to the fuzzy sub-hypergroup and the fuzzy H_v-subgroup defined by Davvaz [5] by the $\mathcal{D}$-fuzzy sub-hypergroup and the $\mathcal{D}$-fuzzy H_v-subgroup, respectively. Also, by

$$\langle H_i(A),\Diamond\rangle;\quad H_i(A)=\{(x,I_{H_i}):I_{H_i}\subseteq I,x\in A_\circ\}$$

we mean the fuzzy sub-hypergroups (fuzzy H_v-subgroups) of $\langle(H,I),\Diamond\rangle$ induced by the fuzzy subset A.

Theorem 5.3. *Let $\langle (H, I), \Diamond \rangle$ be a uniform fuzzy hypergroup and let A be a non-empty fuzzy subset of H which induces fuzzy sub-hypergroups $\langle H_i(A), \Diamond \rangle$ of $\langle (H, I), \Diamond \rangle$. Then the co-membership function $\bigtriangledown$ satisfies the following conditions:*

(1) $\bigtriangledown (\bigtriangledown (A(x), A(y)), A(z)) = \bigtriangledown (A(x), \bigtriangledown (A(y), A(z)))$, *for all* $x, y, z \in A_\circ$.

(2) $\bigcup_{y \in H_i} \bigtriangledown (A(x), A(y)) = I_{H_i}$, *for all* $x \in A_\circ$.

Proof. For condition (1): If the induced fuzzy sub-hypergroup is

$$\langle H_\circ(A), \Diamond \rangle = \{(x, \{0, A(x)\}) \mid x \in A_\circ\},$$

then using associativity we have

$$\begin{aligned} &((x, \{0, A(x)\}) \Diamond (y, \{0, A(y)\})) \Diamond (z, \{0, A(z)\}) \\ &= (x, \{0, A(x)\}) \Diamond ((y, \{0, A(y)\}) \Diamond (z, \{0, A(z)\})), \end{aligned}$$

that is,

$$((x \bigtriangleup y) \bigtriangleup z, \bigtriangledown(\bigtriangledown(A(x), A(y)), A(z))) = (x \bigtriangleup (y \bigtriangleup z), \bigtriangledown(A(x), \bigtriangledown(A(y), A(z)))).$$

Therefore, $\bigtriangledown$ satisfies

$$\bigtriangledown (\bigtriangledown (A(x), A(y)), A(z)) = \bigtriangledown (A(x), \bigtriangledown (A(y), A(z))),$$

for all $x, y, z \in A_\circ$. If the induced fuzzy sub-hypergroups are $\langle \underline{H}(A), \Diamond \rangle, \langle \overline{H}(A), \Diamond \rangle$, then using the same argument one can prove that $\bigtriangledown$ satisfies condition (1).

For condition (2): If the induced fuzzy sub-hypergroup is

$$\langle H_\circ(A), \Diamond \rangle = \{(x, \{0, A(x)\}) : x \in A_\circ\},$$

then $I_H = \{0, A(y)\}$ for all $y \in A_\circ$. Clearly, $I_H = \bigcup_{y \in H_\circ(A)} \bigtriangledown (A(x), A(y))$, for all $x \in A_\circ$. If the induced fuzzy sub-hypergroups are $\langle \underline{H}(A), \Diamond \rangle, \langle \overline{H}(A), \Diamond \rangle$, then using the same method one can prove that $\bigtriangledown$ satisfies condition (2) of Theorem 5.3. □

Theorem 5.4. *Let $\langle (H, I), \Diamond \rangle$ be a uniform fuzzy H_v-group and let A be a non-empty fuzzy subset of H which induces fuzzy H_v-subgroups $\langle H_i(A), \Diamond \rangle$ of $\langle (H, I), \Diamond \rangle$. Then the co-membership function $\bigtriangledown$ satisfies the following conditions:*

(1) $\bigtriangledown (\bigtriangledown (A(x), A(y)), A(z)) \cap \bigtriangledown (A(x), \bigtriangledown (A(y), A(z))) \neq \{0\}$, *for all* $x, y, z \in A_\circ$.

(2) $\bigcup_{y \in H_i} \bigtriangledown (A(x), A(y)) = I_{H_i}$, *for all* $x \in A_\circ$.

Proof. The proof is similar to the proof of Theorem 5.3. □

Theorem 5.5. *Let $\langle (H, I), \Diamond \rangle$ be a uniform fuzzy hypergroup (or fuzzy H_v-group) and let the co-membership function $\bigtriangledown$ have the t-norm property. Then every fuzzy subset A of H which induces fuzzy sub-hypergroups (or fuzzy H_v-subgroup) is a $\mathcal{D}$-fuzzy sub-hypergroup (or $\mathcal{D}$-fuzzy H_v-subgroup) of $(H, \bigtriangleup)$.*

Proof. If the fuzzy subsets A induces fuzzy sub-hypergroups of the fuzzy hypergroup $\langle (H, I), \Diamond \rangle$, then by Theorems 5.1 and 5.4 the co-membership function $\bigtriangledown$ satisfies

$$\bigtriangledown (A(x), A(y)) = \inf \{A(z) \ : \ z \in x \bigtriangleup y\},$$

for all $A(x) \neq 0$ and$A(y) \neq 0$. Therefore, if the fuzzy subset A induces fuzzy sub-hypergroup (or fuzzy H_v-subgroup), then A (without loss of generality) satisfies

$$\bigtriangledown (A(x), A(y)) \leq \inf \{A(z) \ : \ z \in x \bigtriangleup y\},$$

for all $x, y \in H$. That is,

$$\min\{A(x), A(y)\} \leq \inf\{A(z) \ : \ z \in x \bigtriangleup y\},$$

for all $x, y \in H$. Hence the first condition of the $\mathcal{D}$-fuzzy sub-hypergroup (or $\mathcal{D}$-fuzzy H_v-subgroup) definition satisfied. Now, let $(x, u_x), (x', u_{x'})$ be in the induced fuzzy sub-hypergroups (or fuzzy H_v-subgroup). Then the inequality

$$\bigtriangledown(A(x), A(x')) \leq \inf\{A(z) \ : \ z \in x \bigtriangleup x'\}$$

satisfied, also $x \in x \bigtriangleup x'$, but $\min\{A(x), A(x')\} \leq A(x)$. Hence if we choose $x = y$, then the second condition of the $\mathcal{D}$-fuzzy sub-hypergroup (or $\mathcal{D}$-fuzzy H_v-subgroup) definition will hold for all $x, x' \in G$. The proof of third condition is similar. □

Remark 5.6. The above theorem can be obtained directly from Proposition 2 [5] if we assume that to each induced fuzzy sub-hypergroup $\langle H_i(A), \Diamond\rangle$ by the fuzzy subset A the corresponding ordinary sub-hypergroup $\langle H_i(A), \bigtriangleup\rangle$ defines an ordinary group.

Theorem 5.7. *Let $\langle \mathbb{A}, \bigtriangleup\rangle$ be an ordinary sub-hypergroup of the ordinary hypergroup $\langle H, \bigtriangleup\rangle$. Then every fuzzy subset A of H for which $A_\circ = \{x \in H : A(X) \neq 0\} = \mathbb{A}$, induces a fuzzy sub-hypergroup of a fuzzy hypergroup $\langle (H, I), \overline{\Diamond}\rangle$ where $\overline{\Diamond} = \{\overline{\bigtriangleup}, \overline{\bigtriangledown}_{xy}\}$ with $\overline{\bigtriangleup} = \bigtriangleup$ and $\overline{\bigtriangledown}_{xy}$ are suitable co-membership functions.*

Proof. Assume that $\langle \mathbb{A}, \bigtriangleup\rangle$ is an ordinary sub-hypergroup of the ordinary hypergroup $\langle H, \bigtriangleup\rangle$. Let A be a fuzzy subset of H such that $A_\circ = \mathbb{A}$ and $\theta(r, s)$ be any given t-norm. Define the fuzzy hypergroup $\langle (H, I), \overline{\Diamond}\rangle$ as follows:

$$\overline{\Diamond} = \{\overline{\bigtriangleup}, \overline{\bigtriangledown}_{xy}\},$$

where $\overline{\bigtriangleup} = \bigtriangleup$ and $\overline{\bigtriangledown}_{xy}(r, s) = \psi_{xy}(\theta(r, s))$ such that if $\theta(A(x), A(y)) \neq 0$ then

$$\psi_{xy}(t) = \begin{cases} \frac{A(x \bigtriangleup y)}{\theta(A(x), A(y))}(t) & \text{if} \ \ t \leq \theta(A(x), A(y)) \\ \\ 1 + \frac{1 - A(x \bigtriangleup y)}{1 - \theta(A(x), A(y))}(t - 1) & \text{if} \ \ t \geq \theta(A(x), A(y)) \end{cases}$$

If $\theta(A(x), A(y)) = 0$, then $\psi_{xy}(t) = t$ for all $t \in I$. Clearly $\overline{\bigtriangledown}_{xy}(r, s)$ are continuous co-membership functions for $x, y \in H$, and $\overline{\bigtriangledown}_{xy}(r, s) = 0$ if and only if either $r = 0$ or $s = 0$. That is $\overline{\Diamond}$ is a fuzzy hyperoperation.

Now, based on the property of the given t-norm $\theta(r, s)$ and the construction of $\overline{\bigtriangledown}_{xy}(r, s)$ we notice that $\theta(A(x), A(y)) \neq 0$ whenever both $A(x), A(y)$ have nonzero values. That is

$$\overline{\bigtriangledown}_{xy}(A(x), A(y)) = \psi_{xy}(\theta(A(x), A(y))) = \inf\{A(z) \ : \ z \in x \bigtriangleup y\}.$$

Therefore, by Theorem 5.1 and the assumption that $\langle A_\circ, \bigtriangleup\rangle$ is an ordinary sub-hypergroup A induces a fuzzy sub-hypergroups of the fuzzy hypergroup $\langle (H, I), \overline{\Diamond}\rangle$. □

The next corollary can be obtained directly from the above theorem.

Corollary 5.8. *Every $\mathcal{D}$-fuzzy sub-hypergroup A of $(G, \bigtriangleup)$ induces a fuzzy sub-hypergroup relative to some fuzzy hypergroup $\langle (H, I), \overline{\Diamond}\rangle$.*

6. Conclusion

In this paper, we continue the study initiated in [14] about fuzzy groups to the context of fuzzy hypergroups. We know that a hypergroup is a universal set with a hyperoperation, but a fuzzy sub-hypergroup is not so. Davvaz in [5] found an adequate outlet, although partial, to overcome the absence of the fuzzy universal set and fuzzy hyperoperation. He assumed a hypergroup structure on a non-empty set and then introduced the notion of a fuzzy sub-hypergroup. In the absence of the fuzzy universal set, formulation of the intrinsic definition for a sub-hypergroup is not evident. In this paper we define the notion of a fuzzy hypergroup and its sub-hypergroups using the notion of a fuzzy space. The use of fuzzy space as a universal set corrects the deviation in the definition of fuzzy sub-hypergroups. This concept can be considered as a new formulation of the classical theory of fuzzy hypergroups.

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