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Fuzzy implicative filters of BE-algebras

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Abstract. The concept of fuzzy implicative filters is introduced in BE-algebras. A characterization of fuzzy implicative filters is derived in terms of fuzzy filters of a BE-algebra. Some properties of fuzzy relations and homomorphisms are studied with respect to a fuzzy implicative filter.

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1. Introduction

The notion of BE-algebras was introduced and extensively studied by H.S. Kim and Y.H. Kim in [6]. Some properties of filters of BE-algebras were studied by S. S. Ahn and K.S. So in [1] and then by H.S. Kim and Y.H. Kim in [6]. In 1996, Y. B. Jun and S. M. Hong [5] discussed the fuzzy deductive systems of Hilbert algebras. Later, W. A. Dudek and Y. B. Jun [4] considered the fuzzification of ideals in Hilbert algebras and discussed the relation between fuzzy ideals and fuzzy deductive systems. In [9], the author introduced the notion of fuzzy filters in BE-algebras and discussed some related properties. In [6], the authors characterized the fuzzy quasi-ideals and fuzzy bi-ideals of ternary semirings. Recently, F. A. Azam et.al.[2] studied the properties of anti-fuzzy ideals in rings. Y. Xu and K. Y. Qin [10] studied the properties of filters in lattice implication algebras.

In this paper, the fuzzification of implicative filters of BE-algebras is considered and discussed the related properties. We discuss a characterization of fuzzy implicative filters of BE-algebras in terms of fuzzy level filters. It is also observed that every fuzzy implicative filter of a BE-algebra is a fuzzy filter but not the converse. Some equivalent conditions are derived for a fuzzy filter of a BE-algebra to become a fuzzy implicative filter. An extension property of fuzzy implicative filters is also studied. The properties of homomorphic images of fuzzy implicative filters are studied. Finally, fuzzy relations including the cartesian products of fuzzy implicative filters are discussed in BE-algebras.

2. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1], [7], [8], [9] and [10] for the ready reference of the reader.

Definition 2.1 ([7]). An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:
(1) $x * x = 1$
(2) $x * 1 = 1$
(3) $1 * x = x$
(4) $x * (y * z) = y * (x * z)$
for all $x, y, z \in X$

Theorem 2.2 ([7]). *Let $(X, *, 1)$ be a BE-algebra. Then we have the following:*
(1) $x * (y * x) = 1$
(2) $x * ((x * y) * y)) = 1$

We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ implies $x * y = 1$. A BE-algebra X is called self-distributive if $x*(y*z) = (x*y)*(x*z)$ for all $x, y, z \in X$. A non-empty subset S of a BE-algebra X is called a subalgebra of X if $x, y \in S$, then $x * y \in S$.

Definition 2.3 ([8]). A BE-algebra $(X, *, 1)$ is said to be transitive if for all $x, y, z \in X$, it satisfies $y * z \leq (x * y) * (x * z)$.

Definition 2.4 ([1]). Let $(X, *, 1)$ be a BE-algebra. A non-empty subset F of X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:
(a) $1 \in F$
(b) $x \in F$ and $x * y \in F$ imply that $y \in F$

Definition 2.5 ([1]). Let $(X_1, *, 1)$ and $(X_2, \circ, 1')$ be two BE-algebras. Then a mapping $f : X_1 \longrightarrow X_2$ is called a homomorphism if $f(x * y) = f(x) \circ f(y)$ for all $x, y \in X_1$.

It it clear that if $f : X_1 \longrightarrow X_2$ is a homomorphism, then $f(1) = 1'$.

Definition 2.6 ([10]). Let X be a set. Then a fuzzy set in X is a function $\mu : X \longrightarrow [0, 1]$.

Definition 2.7 ([9]). A fuzzy set μ in X is called a fuzzy filter of X if it satisfies:
(F_1) $\mu(1) \geq \mu(x)$
(F_2) $\mu(y) \geq min\{\mu(x), \mu(x * y)\}$ for all $x, y \in X$

Lemma 2.8 ([9]). *Let μ be a fuzzy filter of a BE-algebra X. Then the following conditions hold for all $x, y \in X$.*
(1) $\mu(x * y) = \mu(1)$ *implies* $\mu(x) \leq \mu(y)$
(2) $x \leq y$ *implies* $\mu(x) \leq \mu(y)$

Definition 2.9 ([9]). Let μ be a fuzzy set in a BE-algebra X. For any $\alpha \in [0,1]$, the set $\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$ is called a level subset of μ.

Definition 2.10 ([9]). Let μ be a fuzzy filter of a BE-algebra X. Then the filters $\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}, \alpha \in [0,1]$, are called level filters of X.

Theorem 2.11 ([9]). *A fuzzy set μ of a BE-algebra X is a fuzzy filter in X if and only if it satisfies the following conditions:*
(1) $\mu(1) \geq \mu(x)$ *for all* $x \in X$
(2) $\mu(x * z) \geq min\{\mu(x * (y * z)), \mu(y)\}$ *for all* $x, y, z \in X$

3. Fuzzy implicative filters of BE-algebras

In this section, we introduce the concept of implicative filters in BE-algebras and then we discuss the fuzzification of implicative filters in BE-algebras.

Definition 3.1. A non-empty subset F of a BE-algebra X is called an implicative filter if, for all $x, y, z \in X$, it satisfies the following conditions.
(1) $1 \in F$
(2) $x * (y * z) \in F$ and $x * y \in F$ imply that $x * z \in F$

Definition 3.2. A fuzzy set μ of a BE-algebra X is called a fuzzy implicative filter if it satisfies, for all $x, y, z \in X$, the following conditions.
(1) $\mu(1) \geq \mu(x)$
(2) $\mu(x * z) \geq min\{\mu(x * (y * z)), \mu(x * y)\}$

If we replace x of the above Definitions 3.1 and 3.2 by the element 1, then it can be easily observed that every implicative filter is a filter as well as every fuzzy implicative filter is a fuzzy filter. However, every fuzzy filter is not a fuzzy implicative filter as shown in the following example.

Example 3.3. Let $X = \{1, a, b, c\}$ be a non-empty set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then it can be easily verified that $(X, *, 1)$ is a BE-algebra. Define a fuzzy set μ on X as follows:

$$\mu(x) = \begin{cases} 0.9 & \text{if } x = a, 1 \\ 0.2 & \text{otherwise} \end{cases}$$

for all $x \in X$. Then clearly μ is a fuzzy filter of X, but μ is not a fuzzy implicative filter of X since $\mu(b * c) \ngeq min\{\mu(b * (d * c)), \mu(b * d)\}$.

Theorem 3.4. *Let X be a distributive BE-algebra. Then every fuzzy filter is a fuzzy implicative filter.*

Proof. Let F be a fuzzy filter of a distributive BE-algebra X. Then clearly $\mu(1) \geq \mu(x)$ for all $x \in X$. Let $x, y, z \in X$. Since F is a fuzzy filter, by Theorem 2.11, we get $\mu(x * z) \geq min\{\mu((x * y) * (x * z)), \mu(x * y)\} = min\{\mu(x * (y * z)), \mu(x * y)\}$, because of X is distributive. Therefore μ is a fuzzy implicative filter in X. □

We now derive a sufficient condition for every fuzzy filter of a transitive BE-algebra to become a fuzzy implicative filter.

Theorem 3.5. *Let μ be a fuzzy filter of a transitive BE-algebra X such that*

$$\mu(y * z) \geq min\{\mu(x), \mu(x * (y * (y * z)))\}$$

for all $x, y, z \in X$.

Proof. Assume that μ is a fuzzy filter of X satisfying the given condition. Let $x, y, z \in X$. Since X is a transitive BE-algebra, we get

$$\begin{aligned} \mu(x * (y * z)) &= \mu(y * (x * z)) \\ &\leq \mu((x * y) * (x * (x * z))) \end{aligned}$$

Hence by assumed condition, we get

$$\begin{aligned} \mu(x * z) &\geq min\{\mu(x * y), \mu((x * y) * (x * (x * z)))\} \\ &\geq min\{\mu(x * y), \mu(x * (y * z))\} \end{aligned}$$

Therefore μ is a fuzzy implicative filter in X. □

In the following, we derive a set of equivalent conditions for every fuzzy filter of a transitive BE-algebra to become a fuzzy implicative filter.

Theorem 3.6. *Let μ be a fuzzy filter of a transitive BE-algebra X. Then the following conditions are equivalent.*

(1) μ *is a fuzzy implicative filter in* X
(2) $\mu(x * y) = \mu(x * (x * y))$ *for all* $x, y \in X$
(3) $\mu((x * y) * (x * z)) \geq \mu(x * (y * z))$ *for all* $x, y, z \in X$

Proof. (1) ⇒ (2) : Assume that μ is a fuzzy implicative filter in X. Let $x, y \in X$. Since $x * y \leq x * (x * y)$, we get $\mu(x * y) \leq \mu(x * (x * y))$. Since μ is a fuzzy filter in X, we get that

$$\begin{aligned} \mu(x * y) &\geq min\{\mu(x * (x * y)), \mu(x * x)\} \\ &= min\{\mu(x * (x * y)), \mu(1)\} \\ &= \mu(x * (x * y)) \end{aligned}$$

Hence $\mu(x * y) \geq \mu(x * (x * y))$. Therefore $\mu(x * y) = \mu(x * (x * y))$.

(2) ⇒ (3) : Assume the condition (2). Let $x, y, z \in X$. Since X is transitive, we get $y * z \leq (x * y) * (x * z)$ and hence we get that $x * (y * z) \leq x * ((x * y) * (x * z))$. Hence $\mu(x * (y * z)) \leq \mu(x * ((x * y) * (x * z)))$. Now, we get

$$\begin{aligned} \mu((x * y) * (x * z)) &= \mu(x * ((x * y) * z)) \\ &= \mu(x * (x * ((x * y) * z)) && \text{by (2)} \\ &= \mu(x * ((x * y) * (x * z)) \\ &\geq \mu(x * (y * z)) \end{aligned}$$

$(3) \Rightarrow (1)$: Assume that μ satisfies the condition (2). Let $x, y, z \in X$. Since μ is a fuzzy filter in X, we have the following

$$\begin{aligned} \mu(x*z) &\geq min\{\mu((x*y)*(x*z)), \mu(x*y)\} \\ &\geq min\{\mu(x*(y*z)), \mu(x*y)\} \end{aligned}$$

Therefore μ is a fuzzy implicative filter in X. □

Proposition 3.7. *Let μ be a fuzzy set in a BE-algebra X. Then μ is a fuzzy implicative filter in X if and only if for each $\alpha \in [0,1]$, the level subset μ_α is an implicative filter in X, when $\mu_\alpha \neq \varnothing$.*

Proof. Assume that μ is a fuzzy implicative filter of X. Then $\mu(1) \geq \mu(x)$ for all $x \in X$. In particular, $\mu(1) \geq \mu(x) \geq \alpha$ for all $x \in \mu_\alpha$. Hence $1 \in \mu_\alpha$. Let $x*(y*z), x*y \in \mu_\alpha$. Then $\mu(x*(y*z)) \geq \alpha$ and $\mu(x*y) \geq \alpha$. Since μ is a fuzzy implicative filter, we get $\mu(x*z) \geq min\{\mu(x*(y*z)), \mu(x*y)\} \geq \alpha$. Thus $x*z \in \mu_\alpha$. Therefore μ_α is an implicative filter in X.

Conversely, assume that μ_α is an implicative filter of X for each $\alpha \in [0,1]$ with $\mu_\alpha \neq \varnothing$. Suppose there exists $x_0 \in X$ such that $\mu(1) < \mu(x_0)$. Again, let $\alpha_0 = \frac{1}{2}(\mu(1) + \mu(x_0))$. Then $\mu(1) < \alpha_0$ and $0 \leq \alpha_0 < \mu(x_0) \leq 1$. Hence $x_0 \in \mu_{\alpha_0}$ and $\mu_{\alpha_0} \neq \varnothing$. Since μ_{α_0} is an implicative filter in X, we get $1 \in \mu_{\alpha_0}$ and hence $\mu(1) \geq \alpha_0$, which is a contradiction. Therefore $\mu(1) \geq \mu(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $\mu(x*(y*z)) = \alpha_1$ and $\mu(x*y) = \alpha_2$. Then $x*(y*z) \in \mu_{\alpha_1}$ and $x*y \in \mu_{\alpha_2}$. Without loss of generality, assume that $\alpha_1 \leq \alpha_2$. Then clearly $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$. Hence $x*y \in \mu_{\alpha_1}$. Since μ_{α_1} is an implicative filter in X, we get $x*z \in \mu_{\alpha_1}$. Thus $\mu(x*z) \geq \alpha_1 = min\{\alpha_1, \alpha_2\} = \min\{\mu(x*(y*z)), \mu(x*y)\}$. Therefore μ is a fuzzy implicative filter of X. □

Theorem 3.8. *Let F be an implicative filter of a BE-algebra. Then there exists a fuzzy implicative filter μ of X such that $\mu_\alpha = F$ for some $\alpha \in (0,1)$.*

Proof. Let μ be a fuzzy set in a BE-algebra X defined by

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in F \\ 0 & \text{otherwise} \end{cases}$$

where α is a fixed number $(0 < \alpha < 1)$. Since $1 \in F$, we get $\mu(1) = \alpha \geq \mu(x)$ for all $x \in X$. Let $x, y, z \in X$. Suppose $x*(y*z), x*y \in F$. Since F is an implicative filter, we get $x*z \in F$. Then $\mu(x*y) = \mu(x*(y*z)) = \mu(x*z) = \alpha$. Hence $\mu(x*z) \geq min\{\mu(x*(y*z)), \mu(x*y)\}$. Suppose $x*(y*z) \notin F$ and $x*y \notin F$. Then $\mu(x*y) = \mu(x*(y*z)) = 0$. Hence $\mu(*z) \geq min\{\mu(x*(y*z)), \mu(x*y)\}$. If exactly one of $x*(y*z)$ and $x*y$ is in F, then exactly one of $\mu(x*(y*z))$ and $\mu(x*y)$ is equal to 0. Hence $\mu(x*z) \geq min\{\mu(x*(y*z)), \mu(x*y)\}$. By summarizing the above results, we get $\mu(x*z) \geq min\{\mu(x*(y*z)), \mu(x*y)\}$ for all $x, y, z \in X$. Therefore μ is a fuzzy implicative filter of X. Clearly $\mu_\alpha = F$. □

Theorem 3.9. *Let μ be a fuzzy implicative filter of a BE-algebra X. Then two level implicative filters μ_{α_1} and μ_{α_2}(with $\alpha_1 < \alpha_2$) of μ are equal if and only if there is no $x \in X$ such that $\alpha_1 \leq \mu(x) < \alpha_2$.*

Proof. Assume that $\mu_{\alpha_1} = \mu_{\alpha_2}$ for $\alpha_1 < \alpha_2$. Suppose there exists some $x \in X$ such that $\alpha_1 \leq \mu(x) < \alpha_2$. Then μ_{α_2} is a proper subset of μ_{α_1}, which is impossible. Conversely, assume that there is no $x \in X$ such that $\alpha_1 \leq \mu(x) < \alpha_2$. Since $\alpha_1 < \alpha_2$, we get that $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$. If $x \in \mu_{\alpha_1}$, then $\mu(x) \geq \alpha_1$. Hence by assume condition, we get that $\mu(x) \geq \alpha_2$. Hence $x \in \mu_{\alpha_2}$ and so $\mu_{\alpha_1} \subseteq \mu_{\alpha_2}$. Therefore $\mu_{\alpha_1} = \mu_{\alpha_2}$. □

Theorem 3.10. *Let μ be a fuzzy implicative filter of X with $Im(\mu) = \{\alpha_i \mid i \in \Delta\}$ and $\mathcal{F} = \{\mu_{\alpha_i} \mid i \in \Delta\}$ where Δ is an arbitrary indexed set. If μ attains its infimum on all implicative filters of X, then $\mathcal{F}$ contains all level implicative filters of μ.*

Proof. Suppose μ attains its infimum on all implicative filters of X. Let μ_α be a level implicative filter of μ. If $\alpha = \alpha_i$ for some $i \in \Delta$, then clearly $\mu_\alpha \in \mathcal{F}$. Assume that $\alpha \neq \alpha_i$ for all $i \in \Delta$. Then there exists no $x \in X$ such that $\mu(x) = \alpha$. Let $F = \{x \in X \mid \mu(x) > \alpha\}$. Clearly $1 \in F$. Let $x, y, z \in X$ be such that $x * y \in F$ and $x * (y * z) \in F$. Then $\mu(x * y) > \alpha$ and $\mu(x * (y * z)) > \alpha$. Since μ is a fuzzy implicative filter in X, we get

$$\mu(x * z) \geq min\{\mu(x * (t * z)), \mu(x * y)\} > \alpha$$

Hence $\mu(x * z) > \alpha$, which implies that $x * z \in F$. Therefore F is an implicative filter of X. By the hypothesis, there exists $y \in F$ such that

$$\mu(y) = inf\{\mu(x) \mid x \in X\}$$

Hence $\mu(y) \in Im(\mu)$, which yields that $\mu(y) = \alpha_i$ for some $i \in \Delta$. It is clear that $\alpha_i \geq \alpha$. Hence, by assumption, we get $\alpha_i > \alpha$. Thus there exists no $x \in X$ such that $\alpha \leq \mu(x) < \alpha_i$. Hence by above Theorem 3.9, we get $\mu_\alpha = \mu_{\alpha_i}$. Therefore $\mu_\alpha \in \mathcal{F}$. This completes the proof. □

We now present an extension property of fuzzy implicative filters

Theorem 3.11. *Let μ and ν be two fuzzy filters of a transitive BE-algebra X with $\mu \leq \nu$ and $\mu(1) = \nu(1)$. If μ is a fuzzy implicative filter of X, then so is ν.*

Proof. Assume that μ and ν are two fuzzy filters of a transitive BE-algebra X such that $\mu \leq \nu$ and $\mu(1) = \nu(1)$. Suppose μ is a fuzzy implicative filter of X. Let $x, y, z \in X$. Since $\mu \subseteq \nu$ and μ is a fuzzy implicative filter, by Theorem 3.6(3), we can obtain the following consequence.

$$\begin{aligned} \nu((x * y) * (x * ((x * (y * z)) * z))) &\geq \mu((x * y) * (x * ((x * (y * z)) * z))) \\ &\geq \mu(x * (y * ((x * (y * z)) * z))) \\ &= \mu(x * ((x * (y * z)) * (y * z))) \\ &= \mu((x * (y * z)) * (x * (y * z))) \\ &= \mu(1) \\ &= \nu(1) \end{aligned}$$

whence

$$\begin{aligned} \nu((x * (y * z)) * ((x * y) * (x * z))) &= \nu((x * y) * ((x * (y * z)) * (x * z))) \\ &= \nu((x * y) * (x * ((x * (y * z)) * z))) \\ &\geq \nu(1) \end{aligned}$$

Hence $\nu((x*(y*z))*((x*y)*(x*z))) = \nu(1)$. Since ν is a fuzzy filter, by Lemma 2.8, we get that

$$\nu((x*y)*(x*z)) \geq \nu(x*(y*z))$$

By Theorem 3.6(3), it yields that ν is a fuzzy implicative filter in X. □

Definition 3.12. Let $f : X \longrightarrow Y$ be a homomorphism of BE-algebras and μ is a fuzzy set in Y. Then define a mapping $\mu^f : X \longrightarrow [0,1]$ such that $\mu^f(x) = \mu(f(x))$ for all $x \in X$.

Clearly the above mapping μ^f is well-defined and a fuzzy set in X.

Theorem 3.13. *Let $f : X \longrightarrow Y$ be an onto homomorphism of BE-algebras and μ is a fuzzy set in Y. Then μ is a fuzzy implicative filter in Y if and only if μ^f is a fuzzy implicative filter in X.*

Proof. Assume that μ is a fuzzy implicative filter of Y. For any $x \in X$, we have $\mu^f(1) = \mu(f(1)) = \mu(1') \geq \mu(f(x)) = \mu^f(x)$. Let $x, y, z \in X$. Then

$$\begin{aligned}
\mu^f(x*z) &= \mu(f(x*z)) \\
&= \mu(f(x)*f(z)) \\
&\geq min\{\mu(f(x)*(f(y)*f(z))), \mu(f(x)*f(y))\} \\
&= min\{\mu(f(x*(y*z))), \mu(f(x*y))\} \\
&= min\{\mu^f(x*(y*z)), \mu^f(x*y)\}
\end{aligned}$$

Hence μ^f is a fuzzy implicative filter of X. Conversely, assume that μ^f is a fuzzy implicative filter of X. Let $x \in Y$. Since f is onto, there exists $y \in X$ such that $f(y) = x$. Then $\mu(1') = \mu(f(1)) = \mu^f(1) \geq \mu^f(y) = \mu(f(y)) = \mu(x)$. Let $x, y, z \in Y$. Then there exist $a, b, c \in X$ such that $f(a) = x, f(b) = y$ and $f(c) = z$. Hence we get

$$\begin{aligned}
\mu(x*z) &= \mu(f(a)*f(c)) \\
&= \mu(f(a*c)) \\
&= \mu^f(a*c) \\
&\geq min\{\mu^f(a*(b*c)), \mu^f(a*b)\} \\
&= min\{\mu(f(a)*(f(b)*f(c)), \mu(f(a)*f(b))\} \\
&= min\{\mu(x*(y*z)), \mu(x*y)\}
\end{aligned}$$

Therefore μ is a fuzzy implicative filter in Y. □

4. Cartesian products of fuzzy implicative filters

In this section, we discuss some properties of the cartesian products of fuzzy implicative filters of BE-algebras. The notion of fuzzy relations [2] are extended to the case of fuzzy implicative filters of BE-algebras.

Definition 4.1 ([3])**.** A fuzzy relation on a set S is a fuzzy set $\mu : S \times S \longrightarrow [0,1]$.

Definition 4.2. Let μ be a fuzzy relation on a set S and ν a fuzzy set in S. Then μ is a fuzzy relation on ν if for all $x, y \in S$

$$\mu(x,y) \leq min\{\nu(x),\nu(y)\}$$

Definition 4.3. Let μ and ν be two fuzzy sets in a BE-algebra X. The cartesian product of μ and ν is defined by

$$(\mu \times \nu)(x,y) = min\{\mu(x),\nu(y)\}$$

for all $x, y \in X$

The following Lemma is a direct consequence of the above definitions.

Lemma 4.4. *Let μ, ν be two fuzzy sets in a BE-algebra X. Then the following hold.*

(1) $\mu \times \nu$ is a fuzzy relation on X
(2) $(\mu \times \nu)_\alpha = \mu_\alpha \times \nu_\alpha$ for all $\alpha \in [0,1]$

For any two BE-algebras X and Y, define an operation $*$ on $X \times Y$ as follows:

$$(x,y) * (x',y') = (x * x', y * y') \text{ for all } x, x' \in X \text{ and } y, y' \in Y$$

Then it can be easily observed that $(X \times Y, *, (1,1))$ is a BE-algebra.

Proposition 4.5. *Let F and G be two implicative filters of the BE-algebras X and Y respectively. Then $F \times G$ is an implicative filter in $X \times Y$.*

Proof. It can be routinely verified. □

Theorem 4.6. *Let μ and ν be two fuzzy implicative filters of a BE-algebra X. Then $\mu \times \nu$ is a fuzzy implicative filter in $X \times X$.*

Proof. Let $(x,y) \in X \times X$. Since μ, ν are fuzzy implicative filters in X, we get

$$\begin{aligned} (\mu \times \nu)(1,1) &= min\{\mu(1),\nu(1)\} \\ &\geq min\{\mu(x),\nu(y)\} \qquad \text{for all } x, y \in X \\ &= (\mu \times \nu)(x,y) \end{aligned}$$

Let $(x,x'), (y,y'), (z,z') \in X \times X$. Put $t = x * (y * z)$ and $t' = x' * (y' * z')$. Clearly $(t,t') = (x,x') * ((y,y') * (z,z'))$. Since μ and ν are fuzzy implicative filters in X, we can obtain the following consequence.

$$\begin{aligned} (\mu \times \nu)((x,x') * (z,z')) &= (\mu \times \nu)(x * z, x' * z') \\ &= min\{\mu(x * z), \nu(x' * z')\} \\ &\geq min\{min\{\mu(x * y), \mu(t)\}, min\{\nu(x' * y'), \nu(t')\}\} \\ &= min\{min\{\mu(x * y), \nu(x' * y')\}, min\{\mu(t), \nu(t')\}\} \\ &= min\{(\mu \times \nu)(x * y, x' * y), (\mu \times \nu)(t,t')\} \\ &= min\{(\mu \times \nu)((x,x') * (y,y')), (\mu \times \nu)(t,t')\} \end{aligned}$$

Therefore $\mu \times \nu$ is a fuzzy implicative filter in $X \times X$. □

Theorem 4.7. *Let μ and ν be two fuzzy sets in a BE-algebra X such that $\mu \times \nu$ is a fuzzy filter of $X \times X$. Then we have the following:*

(1) either $\mu(x) \leq \mu(1)$ or $\nu(x) \leq \nu(1)$ for all $x \in X$
(2) if $\mu(x) \leq \mu(1)$ for all $x \in X$, then either $\mu(x) \leq \nu(1)$ or $\nu(x) \leq \nu(1)$
(3) if $\nu(x) \leq \nu(1)$ for all $x \in X$, then either $\mu(x) \leq \mu(1)$ or $\mu(x) \leq \mu(1)$
(4) either μ or ν is a fuzzy implicative filter of X

Proof. (1). Suppose $\mu(x) > \mu(1)$ and $\nu(y) > \nu(1)$ for some $x, y \in X$. Then $(\mu \times \nu)(x, y) = min\{\mu(x), \nu(y)\} > min\{\mu(1), \nu(1)\} = (\mu \times \nu)(1, 1)$, which is a contradiction. Hence either $\mu(x) \leq \mu(1)$ or $\nu(x) \leq \nu(1)$ for all $x \in X$.

(2). Assume that $\mu(x) \leq \mu(1)$ for all $x \in X$. Suppose $\mu(x) > \mu(1)$ and $\nu(y) > \nu(1)$ for some $x, y \in X$. Then $(\mu \times \nu)(1, 1) = min\{\mu(1), \nu(1)\} = \nu(1)$. Hence

$$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} > \nu(1) = (\mu \times \nu)(1, 1)$$

which is a contradiction. Therefore (2) holds.

(3). It can be obtained in a similar fashion.

(4). Since, by (1), either $\mu(x) \leq \mu(1)$ or $\nu(x) \leq \nu(1)$ for all $x \in X$. Without loss of generality, we may assume that $\mu(x) \leq \mu(1)$ for all $x \in X$. From (2), we can get either $\mu(x) \leq \nu(1)$ or $\nu(x) \leq \nu(1)$ for all $x \in X$.

Case. I: Suppose $\mu(x) \leq \nu(1)$ for all $x \in X$. Then

$$(\mu \times \nu)(x, 1) = min\{\mu(x), \nu(1)\} = \mu(x) \text{ for all } x \in X.$$

Since $\mu \times \nu$ is a fuzzy implicative filter in $X \times X$, we get $\mu(x) = min\{\mu(x), \nu(1)\} = (\mu \times \nu)(x, 1) \leq (\mu \times \nu)(1, 1) = \mu(1)$. Also

$$\begin{aligned}
\mu(x * z) &= min\{\mu(x * z), \nu(1)\} \\
&= (\mu \times \nu)(x * z, 1) \\
&= (\mu \times \nu)(x * z, x * 1) \\
&= (\mu \times \nu)((x, x) * (z, 1)) \\
&\geq min\{(\mu \times \nu)((x, x) * (y, 1)), (\mu \times \nu)((x, x) * ((y, 1) * (z, 1)))\} \\
&= min\{(\mu \times \nu)(x * x, y * 1), (\mu \times \nu)(x * (y * z), x * (1 * 1))\} \\
&= min\{min\{\mu(x * y), \nu(y * 1)\}, min\{\mu(x * (y * z)), \nu(x * (1 * 1))\}\} \\
&= min\{min\{\mu(x * y), \nu(1)\}, min\{\mu(x * (y * z)), \nu(1)\}\} \\
&= min\{\mu(x * y), \mu(x * (y * z))\}
\end{aligned}$$

Therefore μ is a fuzzy implicative filter in X.

Case. II: Suppose $\nu(x) \leq \nu(1)$ for all $x \in X$. Suppose $\mu(x) \leq \nu(1)$ for all $x \in X$. Then it leads to case I, from which we can conclude that μ is a fuzzy implicative filter. Suppose $\mu(t) > \nu(1)$ for some $t \in X$. Then $\mu(1) \geq \mu(t) > \nu(1)$. Since $\nu(x) \leq \nu(1)$ for all $x \in X$, it yields that $\mu(1) > \nu(x)$ for all $x \in X$. Hence

$$\begin{aligned}
\nu(x * z) &= min\{\mu(1), \nu(x * z)\} \\
&= (\mu \times \nu)(1, x * z) \\
&= (\mu \times \nu)(x * 1, x * z) \\
&= (\mu \times \nu)((x, x) * (1, z)) \\
&\geq min\{(\mu \times \nu)((x, x) * (1, y)), (\mu \times \nu)((x, x) * ((1, y) * (1, z)))\} \\
&= min\{(\mu \times \nu)(x * 1, x * y), (\mu \times \nu)(x * (1 * 1), x * (y * z))\} \\
&= min\{(\mu \times \nu)(1, x * y), (\mu \times \nu)(1, x * (y * z))\} \\
&= min\{min\{\mu(1), \nu(x * y)\}, min\{\mu(1), \nu(x * (y * z))\}\} \\
&= min\{\nu(x * y), \nu(x * (y * z))\}
\end{aligned}$$

Therefore ν is a fuzzy implicative filter in X. □

In the following, we present an example to show that if $\mu\times\nu$ is a fuzzy implicative filter of $X \times X$, then μ and ν both need not be fuzzy implicative filters of X.

Example 4.8. Let X be a BE-algebra with $|X| > 2$ and let $\alpha, \beta \in [0, 1]$ be such that $0 \leq \alpha \leq \beta < 1$. Define fuzzy sets μ and $\nu : X \to [0, 1]$ by $\mu(x) = \alpha$ and

$$\nu(x) = \begin{cases} \beta & \text{if } x = 1 \\ 1 & \text{if } x \neq 1 \end{cases}$$

for all $x \in X$, respectively. Then we get $(\mu \times \nu)(x, y) = min\{\mu(x), \nu(y)\} = \alpha$ for all $(x, y) \in X \times X$. Hence $\mu \times \nu : X \times X \to [0, 1]$ is a constant function. Thus $\mu \times \nu$ is a fuzzy implicative filter of $X \times X$. Now μ is a fuzzy implicative filter of X but ν is not a fuzzy implicative filter of X because ν does satisfy F_1.

Definition 4.9. Let ν be a fuzzy set in a BE-algebra X. Then the strongest fuzzy relation μ_ν is a fuzzy relation X defined by

$$\mu_\nu(x, y) = min\{\nu(x), \nu(y)\}$$

for all $x, y \in X$.

Theorem 4.10. *Let ν be a fuzzy set in a BE-algebra X and μ_ν be the strongest fuzzy relation on X. Then ν is a fuzzy implicative filter in X if and only if μ_ν is a fuzzy implicative filter of $X \times X$.*

Proof. Assume that ν is a fuzzy implicative filter of X. Then for any $(x, y) \in X\times X$, we have

$$\mu_\nu(x, y) = min\{\nu(x), \nu(y)\} \leq min\{\nu(1), \nu(1)\} = \mu_\nu(1, 1)$$

Let $(x, x'), (y, y')$ and $(z, z') \in X \times X$. Then we have the following:

$$\begin{aligned}
\mu_\nu((x, x') * (z, z')) &= \mu_\nu(x * z, x' * z') \\
&= min\{\nu(x * z), \nu(x' * z')\} \\
&\geq min\{min\{\nu(x * y), \nu(t)\}, min\{\nu(x' * y'), \nu(t'))\}\} \\
&\qquad \text{where } t = x * (y * z) \text{and } t' = x' * (y' * z') \\
&= min\{min\{\nu(x * y), \nu(x' * y')\}, min\{\nu(t), \nu(t'))\}\} \\
&= min\{\mu_\nu(x * y, x' * y'), \mu_\nu(x * (y * z), x' * (y' * z'))\} \\
&= min\{\mu_\nu((x, x') * (y, y')), \mu_\nu((x, x') * ((y, y') * (z, z')))\}
\end{aligned}$$

Therefore μ_ν is a fuzzy implicative filter in $X \times X$.

Conversely, assume that μ_ν is a fuzzy implicative filter in $X \times X$. Then

$$\nu(1) = min\{\nu(1), \nu(1)\} = \mu_\nu(1, 1) \geq \mu_\nu(x, y) = min\{\nu(x), \nu(y)\}$$

for all $x, y \in X$. Hence it yields that $\nu(x) \leq \nu(1)$ for all $x \in X$. Let $(x, x'), (y, y')$ and $(z, z') \in X \times X$. Then we have the following consequence.

$$\begin{aligned} \nu(x * z) &= min\{\nu(x * z), \nu(1)\} \\ &= \mu_\nu(x * z, 1) \\ &= \mu_\nu(x * z, x * 1) \\ &= \mu_\nu((x, x) * (z, 1)) \\ &\geq min\{\mu_\nu((x, z) * (y, 1)), \mu_\nu((x, z) * ((y, 1) * (z, 1)))\} \\ &= min\{\mu_\nu(x * y, z * 1), \mu_\nu(x * (y * z), z * (1 * 1))\} \\ &= min\{\mu_\nu(x * y, 1), \mu_\nu(x * (y * z), 1)\} \\ &= min\{min\{\nu(x * y), \nu(1)\}, min\{\nu(x * (y * z), \nu(1)\}\} \\ &= min\{\nu(x * y), \nu(x * (y * z))\} \end{aligned}$$

Therefore ν is a fuzzy implicative filter in X. □

References

[1] S. S. Ahn and K. S. So, On ideals and upper sets in BE-algebras, Sci. Math. Jpn., Online e-2008 351-357.
[2] F. A. Azam, A. A. Mamun and F. Nasrin, Anti fuzzy ideals of a ring, Annal. Fuzzy Math. Inform. 5(2) (2013) 349–360.
[3] P. Bhattacharya and N. P. Mukherjee, Fuzzy relations and fuzzy groups, Inform. Sci. 36 (1985) 267–282.
[4] W. A. Dudek and Y. B. Jun, On fuzzy ideals in Hilbert algebras, Novi Sad J. Math. 29(2) (1999) 193–207.
[5] Y. B. Jun and S. M. Hong, Fuzzy deductive systems of Hilbert algebras, Indian J. Pure Appl. Math. 27(2) (1996) 141-151.
[6] S. Kar and P. Sarkar, Fuzzy quasi-ideals and fuzzy bi-ideals of ternary semigroups, Annal. Fuzzy Math. Inform. 4(2) (2012) 407–423.
[7] H. S. Kim and Y. H. Kim, On BE-algebras, Sci. Math. Jpn. 66(1) (2007) 113–116.
[8] A. Rezaei and A. B. Saeid, On fuzzy subalgebras of BE-algebras, Afr. Mat. 22(2) (2011) 115–127.
[9] M. Sambasiva Rao, Fuzzy filters of BE-algebras, to appear in International Journal of Mathematical Archive.
[10] Y. Xu and K. Y. Qin, On filters of lattice implication algebras, J. Fuzzy Math. 1(2) (1993) 251–260.

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