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Fuzzy proper functions and net-convergence in smooth fuzzy topological spaces

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Abstract. The objective of the paper is to find interrelations among smooth fuzzy continuity, weakly smooth fuzzy continuity, q_n-weakly smooth fuzzy continuity, net 1 continuity, net 2 continuity and the well known inequality connecting closure operator and continuity, for a fuzzy proper function between smooth fuzzy topological spaces.

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1. Introduction

The concept of fuzzy topological space was first introduced by Chang in [9]. Later, fuzzy topological space is generalized in different ways, one of which is developed by Šostak [45]. Höhle and Šostak [22], Kubiak and Šostak [25] introduced the concept of an L-fuzzy topological space even further to situations, where L is more general than $[0, 1]$, in 1995 and 1997, respectively. The respective categories of L-fuzzy topological spaces and L-fuzzy continuous maps are studied by Roadabaugh in [21, 36, 37]. Ramadan [33] renamed Šostak's I fuzzy topological space "*smooth fuzzy topological space*". Then plenty of works on Šostak's fuzzy topological spaces have been done in order to extend various concepts in classical topology. To mention a few, we refer to [1, 2, 4, 10, 16, 20, 24, 26, 34, 35, 41, 42, 43, 44, 46, 47].

The net convergence of fuzzy points in Chang fuzzy topology was first introduced by Pu and Liu [27, 28]. In [40], Ranta and Ajmal introduced the notion of convergence of a fuzzy net of fuzzy sets, which generalizes the notion of fuzzy net of fuzzy points defined in [27, 28]. Geogiou and Papadopoulos [18] defined fuzzy upper limit, fuzzy lower limit and fuzzy convergence of a fuzzy net of fuzzy sets and discussed their properties. In [19], the authors introduced and studied the notion of fuzzy

θ-convergence and weakly fuzzy θ-convergence in a fuzzy topological space. Sheikh [15] introduced another notion of fuzzy convergence in terms of fuzzy upper (lower) δ-limit, which generalizes the convergence discussed in [18]. Further, there are many more works on convergence in L-fuzzy topological spaces in various contexts. See [3, 5, 6, 7, 13, 23, 29, 30, 31, 32].

Fuzzy proper function from a fuzzy set into a fuzzy set was first introduced by Chakraborty and Ahsanulla [8]. Chaudhuri and Das [11] proved some characterizations of continuity of fuzzy proper function between Chang fuzzy topological spaces. In [17], the authors introduced the fuzzy graph, strong fuzzy graph of a fuzzy proper function on Chang fuzzy topological space. They proved the closed graph theorem under some sufficient conditions and also established various results relating separation axioms. The notions of smooth fuzzy continuity, weakly smooth fuzzy continuity and q_n-weakly smooth fuzzy continuity of a fuzzy proper function on smooth fuzzy topological spaces and their properties are discussed in [33, 38]. In [39], the results combining various types of smooth fuzzy continuities including the one, "*If $F : (\mu, \tau) \to (\nu, \sigma)$ is weakly smooth fuzzy continuous, then $F^{\leftarrow}(V^{\circ}) \leq (F^{\leftarrow}(V))^{\circ}$ for every $V \leq \nu$.*" are established and smooth connectedness of smooth fuzzy topological space is also discussed.

In [11, Remark 3.4], it was pointed out that the conditions,

(1) $F^{-1}(V) \in c(T), \forall V \in c(T')$, where $c(T)$ is the family of all fuzzy closed sets in a Chang fuzzy topological space (A, T)
(2) $F(\overline{H}) \leq \overline{F(H)}$, $\forall$ maximal fuzzy subsets H of μ,

are neither necessary nor sufficient for F is fuzzy continuous. The same drawback holds for continuous fuzzy proper function on smooth fuzzy topological spaces. However, by assuming that a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ between smooth fuzzy topological spaces satisfies $\nu = F(\mu)$ with some more conditions, we are able to get these results successfully.

In Section 4, we establish that net 1 continuity on a fuzzy set implies q_n-weak smooth fuzzy continuity on a fuzzy set. We also obtain the converse of this result, under some additional conditions. Further, we show that net 1 continuity implies weak continuity if F is one-to-one and the domain of F is a positive minimum smooth fuzzy topological space and we point out that net 1 continuity does not imply smooth fuzzy continuity. The same results are obtained for net 2 continuity in Section 5, by a similar set of arguments employed in the previous section.

2. Preliminaries

Throughout this paper X, S denote fixed non-empty sets, μ, ν denote fuzzy subsets of X, S, respectively, I denotes the unit interval $[0, 1]$, and I^X denotes the set of all fuzzy subsets of X. A fuzzy point [27] in X is defined by $P_x^{\lambda}(t) = \begin{cases} \lambda & \text{if } t = x \\ 0 & \text{if } t \neq x \end{cases}$, $\forall t \in X$, where $0 < \lambda \leq 1$. By $P_x^{\lambda} \in \mu$, we mean that $\lambda \leq \mu(x)$.

Definition 2.1 ([33])**.** A smooth fuzzy topology on a fuzzy set $\mu \in I^X$ is a map $\tau : \mathfrak{I}_{\mu} \to I$, where $\mathfrak{I}_{\mu} = \{U \in I^X : U \leq \mu\}$, satisfying the following axioms:

(1) $\tau(0_X) = \tau(\mu) = 1$,

(2) $\tau(A_1 \wedge A_2) \geq \tau(A_1) \wedge \tau(A_2), \forall A_1, A_2 \in \mathfrak{I}_\mu$,
(3) $\tau(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \tau(A_i)$ for every family $(A_i)_{i\in\Gamma} \subseteq \mathfrak{I}_\mu$.

The pair (μ, τ) is called a smooth fuzzy topological space.

Definition 2.2 ([33]). A smooth fuzzy cotopology on a fuzzy set $\mu \in I^X$ is a map $\mathscr{F} : \mathfrak{I}_\mu \to I$ satisfying, the following axioms:

(1) $\mathscr{F}(0_X) = \mathscr{F}(\mu) = 1$,
(2) $\mathscr{F}(A_1 \vee A_2) \geq \mathscr{F}(A_1) \vee \mathscr{F}(A_2), \forall A_1, A_2 \in \mathfrak{I}_\mu$,
(3) $\mathscr{F}(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathscr{F}(A_i)$ for every family $(A_i)_{i\in\Gamma} \subseteq \mathfrak{I}_\mu$.

The pair $(\mu, \mathscr{F})$ is called a smooth fuzzy cotopological space.

One may see that for a given smooth fuzzy topology $\tau : \mathfrak{I}_\mu \to I$, if $\mathscr{F}_\tau : \mathfrak{I}_\mu \to I$ is defined by $\mathscr{F}_\tau(U) = \tau(\mu - U)$, then $\mathscr{F}_\tau$ is a smooth fuzzy cotopology on μ.

Definition 2.3 ([8]). $U, V \in \mathfrak{I}_\mu$ are said to be quasi-coincident referred to μ (written as $UqV[\mu]$) if there exists $x \in X$ such that $U(x) + V(x) > \mu(x)$. If U is not quasi-coincident with V, then we write $U\not{q}V[\mu]$.

A fuzzy set $U \in \mathfrak{I}_\mu$ is called a q-neighborhood of a fuzzy point P_x^λ in μ if $P_x^\lambda qU[\mu]$ and $\tau(U) > 0$.

Definition 2.4 ([12]). Let (μ, τ) be a smooth fuzzy topological space and $U \in \mathfrak{I}_\mu$. Then the fuzzy closure of U is defined as follows,

$$\overline{U} = \bigwedge \{K \in \mathfrak{I}_\mu : \tau(\mu - K) > 0, K \geq U\}.$$

Definition 2.5 ([8]). Let $\mu \in I^X$ and $\nu \in I^S$. A non-zero fuzzy subset F of $X \times S$ is said to be a fuzzy proper function from μ to ν if

(1) $F(x, s) \leq \min\{\mu(x), \nu(s)\}, \forall (x, s) \in X \times S$,
(2) for each $x \in X$, there exists a unique $s_0 \in S$ such that $F(x, s_0) = \mu(x)$ and $F(x, s) = 0$ if $s \neq s_0$.

Definition 2.6 ([8]). Let F be a fuzzy proper function from μ to ν. If $U \in \mathfrak{I}_\mu$ and $V \in \mathfrak{I}_\nu$, then $F(U) : S \to I$ and $F^{-1}(V) : X \to I$ are defined by

$$\begin{aligned} (F(U))(s) &= \sup\{F(x, s) \wedge U(x) : x \in X\}, \forall s \in S, \\ (F^{-1}(V))(x) &= \sup\{F(x, s) \wedge V(s) : s \in S\}, \forall x \in X. \end{aligned}$$

The inverse image of a fuzzy subset V under a fuzzy proper function F can be easily obtained as $(F^{-1}(V))(x) = \mu(x) \wedge V(s)$, where $s \in S$ is unique such that $F(x, s) = \mu(x)$.

Definition 2.7 ([17]). A fuzzy proper function $F : \mu \to \nu$ is said to be injective (or one-to-one) if $F(x_1, s) > 0$ and $F(x_2, s) > 0$ for some $x_1, x_2 \in X$ and $s \in S$, then $x_1 = x_2$.

Definition 2.8 ([33]). Let $F : (\mu, \tau) \to (\nu, \sigma)$ be a fuzzy proper function and let (μ, τ), (ν, σ) be smooth fuzzy topological spaces. Then F is said to be a smooth fuzzy continuous function on μ if $\tau(F^{-1}(V)) \geq \sigma(V), \forall V \in \mathfrak{I}_\nu$.

Definition 2.9 ([33])**.** Let $F : (\mu, \tau) \to (\nu, \sigma)$ be a fuzzy proper function and let (μ, τ), (ν, σ) be smooth fuzzy topological spaces. Then F is said to be a weakly smooth fuzzy continuous function on μ, if $\tau(F^{-1}(V)) > 0$ whenever $\sigma(V) > 0$, $\forall V \in \Im_\nu$.

Definition 2.10 ([38])**.** Let $F : (\mu, \tau) \to (\nu, \sigma)$ be a fuzzy proper function and let (μ, τ), (ν, σ) be smooth fuzzy topological spaces. Then F is said to be q_n-weakly smooth fuzzy continuous at a fuzzy point $P_x^\lambda \in \mu$ if for every q-neighborhood $V \in \Im_\nu$ of $F(P_x^\lambda)$, there exists a q-neighborhood $U \in \Im_\mu$ of P_x^λ such that $F(U) \leq V$.

A fuzzy proper function F is said to be q_n-weakly smooth fuzzy continuous on μ if F is q_n-weakly smooth fuzzy continuous at every fuzzy point in μ.

Definition 2.11 ([14])**.** A fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ is called a weak open map if $\sigma(F(A)) > 0$, whenever $A \in \Im_\mu$ with $\tau(A) > 0$.

Definition 2.12 ([38])**.** Let (μ, τ) be a smooth fuzzy topological space. Then τ is said to be a positive minimum smooth fuzzy topology if $\bigwedge_{i \in \Gamma} \tau(U_i) > 0$, whenever $U_i \in \Im_\mu$ and $\tau(U_i) > 0$, $\forall i \in \Gamma$. The pair (μ, τ) is called a positive minimum smooth fuzzy topological space.

Definition 2.13. Let D be a directed set and μ be a fuzzy subset of set X. Let $\mathscr{I}_\mu$ be the collection of fuzzy points in μ. Any function $\zeta : D \to \mathscr{I}_\mu$ is called a fuzzy net in μ. For every $\lambda \in D$, $\zeta(\lambda)$ is often denoted by ζ_λ and hence the net ζ is denoted by $\{\zeta_\lambda : \lambda \in D\}$.

3. Some properties of certain fuzzy continuous functions

In [11], it has been found that $F^{-1}(\nu - V) \neq \mu - F^{-1}(V)$, for some $V \in \Im_\nu$ and the authors of [11] introduced the concept of maximal fuzzy set and pointed out that the above equality holds only for maximal fuzzy subsets of ν. But the following proposition shows that the same identity is true for all fuzzy subsets of ν if $F : \mu \to \nu$ is a one-to-one fuzzy proper function such that $\nu = F(\mu)$.

Proposition 3.1. *If $F : \mu \to \nu$ is a one-to-one fuzzy proper function such that $\nu = F(\mu)$, then $F^{-1}(\nu - V) = \mu - F^{-1}(V)$, $\forall V \in \Im_\nu$.*

Proof. Let $V \in \Im_\nu$ and let $x \in X$. Then there exists unique $s \in Y$ such that $F(x, s) = \mu(x)$.

$$\begin{aligned}
F^{-1}(\nu - V)(x) &= \mu(x) \wedge (\nu - V)(s) \\
&= \mu(x) \wedge (F(\mu) - V)(s) \\
&= \mu(x) \wedge \left[\bigvee_{k \in X} \{F(k, s) \wedge \mu(k)\} - V(s) \right] \\
&= \mu(x) \wedge [\mu(x) - V(s)] \qquad \text{(since } F \text{ is one-to-one)} \\
&= \mu(x) - V(s) \\
&= \mu(x) - [\mu(x) \wedge V(s)] \qquad \text{(since } V(s) \leq \mu(x)) \\
&= \mu(x) - F^{-1}(V)(x) = (\mu - F^{-1}(V))(x)
\end{aligned}$$

Hence, $F^{-1}(\nu - V) = \mu - F^{-1}(V)$, $\forall V \in \Im_\nu$. □

Theorem 3.2. *Let $F : (\mu, \tau) \to (\nu, \sigma)$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$. Then F is smooth fuzzy continuous if and only if $\mathscr{F}_\tau(F^{-1}(V)) \geq \mathscr{F}_\sigma(V)$, $\forall V \in \Im_\nu$.*

Proof. Let $V \in \Im_\nu$ be arbitrary. If $\mathscr{F}_\sigma(V) = 0$, then the required inequality is obvious. Suppose that $\mathscr{F}_\sigma(V) > 0$. Using the fact that F is smooth fuzzy continuous and by Proposition 3.1, we obtain $\tau(\mu - F^{-1}(V)) = \tau(F^{-1}(\nu - V)) \geq \sigma(\nu - V)$. Therefore, we get $\mathscr{F}_\tau(F^{-1}(V)) \geq \mathscr{F}_\sigma(V)$, $\forall V \in \Im_\nu$. To show the sufficiency, let us assume that $\mathscr{F}_\tau(F^{-1}(V)) \geq \mathscr{F}_\sigma(V)$, $\forall V \in \Im_\nu$. Now we choose $V \in \Im_\nu$ arbitrarily. If $\sigma(V) = 0$, then it is obvious that $\tau(F^{-1}(V)) \geq \sigma(V)$. If $\sigma(V) > 0$, using the hypothesis and Proposition 3.1, we get $\tau(F^{-1}(V)) = \mathscr{F}_\tau(\mu - F^{-1}(V)) = \mathscr{F}_\tau(F^{-1}(\nu - V)) \geq \mathscr{F}_\sigma(\nu - V) = \sigma(V)$. □

Theorem 3.3. *Let $F : (\mu, \tau) \to (\nu, \sigma)$ be a one-to-one fuzzy proper function and $\nu = F(\mu)$. Then F is weakly smooth fuzzy continuous if and only if $\mathscr{F}_\tau(F^{-1}(V)) > 0$ whenever $\mathscr{F}_\sigma(V) > 0$, $\forall V \in \Im_\nu$.*

Proof. The proof is similar to that of Theorem 3.2. □

Remark 3.4. The statements of Theorems 3.2, 3.3 are not true if F is not one-to-one fuzzy proper function as shown by the following counterexamples.

Counterexample 3.5. Let $X = \{x, y\}$, $S = \{s, t\}$ and $\mu^{[0.5,0.4]}_{[x,y]}$, $\nu^{[0.5,0]}_{[s,t]}$ be fuzzy subsets of X and S, respectively. Define two fuzzy subsets $U_1 \leq \mu$, $V_1 \leq \nu$ by ${U_1}^{[0.3,0.3]}_{[x,y]}$, ${V_1}^{[0.3,0]}_{[s,t]}$.
If $\tau : \Im_\mu \to I$ is defined by

$$\tau(U) \quad = \quad \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.6, & U = U_1, \\ 0, & \text{otherwise} \end{cases}$$

and $\sigma : \Im_\nu \to I$ is defined by

$$\sigma(V) \quad = \quad \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise,} \end{cases}$$

then obviously, (μ, τ), (ν, σ) are smooth fuzzy topological spaces. Let a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x, s) = 0.5, F(x, t) = 0, F(y, s) = 0.4, F(y, t) = 0.$$

Then, F is not one-to-one and $F(\mu)^{[0.5,0]}_{[s,t]} = \nu$. Since $F^{-1}(V_1)^{[0.3,0.3]}_{[x,y]} = U_1$ and $\tau(U_1) = 0.6 > 0.5 = \sigma(V_1)$, F is smooth fuzzy continuous and F is also weakly smooth fuzzy continuous on μ. But $\mathscr{F}_\tau(F^{-1}(\nu - V_1)) = 0 < 0.5 = \mathscr{F}_\sigma(\nu - V_1)$.

Counterexample 3.6. Let $X = \{x, y\}$, $S = \{s, t\}$, $\mu^{[0.5,0.4]}_{[x,y]}$, $\nu^{[0.5,0]}_{[s,t]}$, ${U_1}^{[0.2,0.1]}_{[x,y]}$ and ${V_1}^{[0.2,0]}_{[s,t]}$.

Let $\tau : \Im_\mu \to I$ be defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.6, & U = U_1, \\ 0, & \text{otherwise} \end{cases}$$

and let $\sigma : \Im_\nu \to I$ be defined by

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise.} \end{cases}$$

Let a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x, s) = 0.5, F(x, t) = 0, F(y, s) = 0.4, F(y, t) = 0.$$

Then F is not one-to-one and $F(\mu)^{[0.5,0]}_{[s,t]} = \nu$. Since $\mathscr{F}_\tau(F^{-1}(\nu - V_1)^{[0.3,0.3]}_{[x,y]}) = \mathscr{F}_\tau(\mu - U_1)$, $\mathscr{F}_\tau(F^{-1}(V)) \geq \mathscr{F}_\sigma(V)$, $\forall V \in \Im_\nu$. But $\tau(F^{-1}(V_1)^{[0.2,0.2]}_{[x,y]}) = 0 < 0.5 = \sigma(V_1)$.

Lemma 3.7. *Let (μ, τ) be a smooth fuzzy topological space. If $A \in \Im_\mu$ is such that $\mathscr{F}_\tau(A) > 0$, then $\overline{A} = A$.*

Proof of the lemma follows by the definition of fuzzy closure. The converse of Lemma 3.7 is not true.

Counterexample 3.8. Let $X = \{x, y\}$, $\mu^{[0.6,0.7]}_{[x,y]} \in I^X$ and $U^{[0.5-\frac{1}{n+1}, 0.6-\frac{1}{n+1}]}_{n[x,y]}$, $\forall n = 1, 2 \ldots.$
If $\tau : \Im_\mu \to I$ is defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ \frac{1}{n}, & U = U_n, n = 1, 2, \ldots \\ 0, & \text{otherwise }, \end{cases}$$

then (μ, τ) is a smooth fuzzy topological space. Let $A^{[0.1,0.1]}_{[x,y]} \in \Im_\mu$.

$$\begin{aligned} \overline{A} &= \bigwedge \{K \in \Im_\mu : \mathscr{F}_\tau(K) > 0, K \geq S\} \\ &= \mu \wedge \left(\bigwedge_{n \in N} (\mu - U_n) \right) = A. \end{aligned}$$

But $\mathscr{F}_\tau(\overline{A}) = \mathscr{F}_\tau(A) = 0$.

Remark 3.9. The converse of the Lemma 3.7 is true if (μ, τ) is a positive minimum smooth fuzzy topological space.

Theorem 3.10. *Let $F : (\mu, \tau) \to (\nu, \sigma)$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$. If F is q_n-weakly smooth fuzzy continuous, then $F(\overline{A}) \leq \overline{F(A)}, \forall A \in \Im_\mu$.*

Proof. Let us assume that F is q_n-weakly smooth fuzzy continuous on μ. If $P_x^\lambda \notin F^{-1}(\overline{F(A)})$, then $F(P_x^\lambda) \notin \overline{F(A)}$ and hence there is a q-neighborhood $V \in \Im_\nu$ of $F(P_x^\lambda)$ such that $V \not q F(A)[\nu]$. Therefore, $V(s) + F(A)(s) \leq \nu(s), \forall s \in S$. On the other hand, since F is q_n-weakly smooth fuzzy continuous, there exists a q-neighborhood U of P_x^λ such that $F(U) \leq V$. Since F is one-to-one, $U(x) + A(x) = F(U)(s) + F(A)(s) \leq V(s) + F(A)(s) \leq \nu(s) = \mu(x)$, for every $x \in X$. Hence, $F(\overline{A}) \leq \overline{F(A)}, \forall A \in \Im_\mu$. □

Remark 3.11. In [33, 38], it is proved that smooth fuzzy continuity implies weakly smooth fuzzy continuity and weakly smooth fuzzy continuity implies q_n-weakly smooth fuzzy continuity.

By Theorem 3.10 and Remark 3.11, we obtain the following theorems.

Theorem 3.12. *Let $F : (\mu, \tau) \to (\nu, \sigma)$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$. If F is weakly smooth fuzzy continuous, then $F(\overline{A}) \leq \overline{F(A)}, \forall A \in \Im_\mu$.*

Theorem 3.13. *Let $F : (\mu, \tau) \to (\nu, \sigma)$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$. If F is smooth fuzzy continuous, then $F(\overline{A}) \leq \overline{F(A)}$, $\forall A \in \Im_\mu$.*

Remark 3.14. The statements of Theorems 3.10, 3.12, 3.13 are not true if F is not one-to-one fuzzy proper function as shown by the following counterexample.

Counterexample 3.15. Let $X = \{x, y\}$, $S = \{s, t\}$ and let $\mu^{[0.6,0.5]}_{[x,y]}$, $\nu^{[0.6,0]}_{[s,t]}$ be fuzzy subsets of X and S, respectively. Define two fuzzy subsets $U_1 \leq \mu$, $V_1 \leq \nu$ by ${U_1}^{[0.4,0.4]}_{[x,y]}$, ${V_1}^{[0.4,0]}_{[s,t]}$.
If $\tau : \Im_\mu \to I$ is defined by

$$\tau(U) \quad = \quad \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.7, & U = U_1, \\ 0, & \text{otherwise} \end{cases}$$

and $\sigma : \Im_\nu \to I$ is defined by

$$\sigma(V) \quad = \quad \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.6, & V = V_1, \\ 0, & \text{otherwise,} \end{cases}$$

then obviously, (μ, τ), (ν, σ) are smooth fuzzy topological spaces. Let a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x, s) = 0.6, F(x, t) = 0, F(y, s) = 0.5, F(y, t) = 0.$$

Then F is not one-to-one, $F(\mu)^{[0.6,0]}_{[s,t]} = \nu$. Since $F^{-1}(0_S) = 0_X$, $F^{-1}(\nu) = \mu$, $F^{-1}(V_1)^{[0.4,0.4]}_{[x,y]} = U_1$ and $\tau(U_1) = 0.7 > 0.6 = \sigma(V_1)$, we conclude that F is smooth fuzzy continuous. It follows that F is weakly smooth fuzzy continuous and q_n-weakly smooth fuzzy continuous on μ. For $C = (F^{-1}V_1))^{[0.2,0.2]}_{[x,y]}$, we get $\overline{C} = \overline{F^{-1}(\nu - V_1)} = \mu$ and hence $F(\overline{C}) = \nu$. But $F(C) = F(F^{-1}(\nu - V_1)) \leq \nu - V_1$ implies that $\overline{F(C)} \leq \overline{\nu - V_1} = (\nu - V_1)^{[0.2,0]}_{[s,t]}$. Hence, $F(\overline{C}) \nleq \overline{F(C)}$.

The converse of the Theorem 3.10 holds without assuming that F is one-to-one.

Theorem 3.16. *Let $F : \mu \to \nu$ be a fuzzy proper function such that $\nu = F(\mu)$. If $F(\overline{A}) \leq \overline{F(A)}, \forall A \in \Im_\mu$, then F is q_n-weakly smooth fuzzy continuous on μ.*

Proof. Let us assume that $F(\overline{A}) \leq \overline{F(A)}, \forall A \in \Im_\mu$. Suppose that F is not q_n-weakly smooth fuzzy continuous. Then there exists a q-neighborhood V_1 of $F(P_x^\lambda)$ such that $F(U_\alpha) \nleq V_1$, for a given q-neighborhood U_α of P_x^λ. Therefore, for a given U_α, there exists $t_\alpha \in S$ such that $F(U_\alpha)(t_\alpha) > V_1(t_\alpha)$. Hence, for each U_α and for the corresponding t_α, there exists $y_\alpha \in X$ such that $F(y_\alpha, t_\alpha) = \mu(y_\alpha)$. Thus, $\mu(y_\alpha) - U_\alpha(y_\alpha) < \mu(y_\alpha) - V_1(t_\alpha) \leq \nu(t_\alpha) - V_1(t_\alpha)$. If we define a fuzzy set $A \in \Im_\mu$ by $A(x) = \begin{cases} \mu(y_\alpha) - V_1(t_\alpha), & x = y_\alpha, \\ 0, & x \notin \{y_\alpha\}, \end{cases}$, then $A(y_\alpha) + U_\alpha(y_\alpha) > \mu(y_\alpha)$, for every $U_\alpha \Rightarrow P_x^\lambda \in \overline{A} \Rightarrow F(P_x^\lambda) \in F(\overline{A})$. Since, $F(A)(t) \leq \begin{cases} (\nu - V_1)(t_\alpha), & t = t_\alpha, \\ 0, & t \notin \{t_\alpha\}, \end{cases}$ we have $V_1 \not q F(A)[\nu]$ and hence $F(P_x^\lambda) \notin \overline{F(A)}$. Thus, $F(\overline{A}) \nleq \overline{F(A)}$, which is a contradiction. This completes the proof of this theorem. $\square$

The converse of the Theorems 3.12 holds if (μ, τ) is a positive minimum smooth fuzzy topological space.

Theorem 3.17. *Let $F : \mu \to \nu$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$ and let (μ, τ) be a positive minimum smooth fuzzy topological space. If $F(\overline{A}) \leq \overline{F(A)}, \forall A \in \Im_\mu$, then F is weakly smooth fuzzy continuous on μ.*

Proof. Assume that $F(\overline{A}) \leq \overline{F(A)}, \forall A \in \Im_\mu$. Let $\mathscr{F}_\sigma(V) > 0$ and $P_x^\lambda \in \overline{F^{-1}(V)}$. Then, $F(P_x^\lambda) \in F(\overline{F^{-1}(V)}) \leq \overline{F(F^{-1}(V))} \leq \overline{V} = V \Rightarrow P_x^\lambda \in F^{-1}(V)$. Thus $\overline{F^{-1}(V)} = F^{-1}(V)$. Then by Remark 3.9, we have $\mathscr{F}_\tau(F^{-1}(V)) > 0$ and hence F is weakly smooth fuzzy continuous on μ. $\square$

The following examples show that the statement of Theorem 3.17 is not true when F is not one-to-one or the domain of F is not a positive minimum smooth fuzzy topological space.

Counterexample 3.18. Let $X = \{x, y\}$, $S = \{s, t\}$, $\mu_{[x,y]}^{[0.8,0.7]}$, $\nu_{[s,t]}^{[0.8,0]}$. Define two fuzzy subsets $U_1 \leq \mu$ and $V_1 \leq \nu$ by ${U_1}_{[x,y]}^{[0.3,0.2]}$, ${V_1}_{[s,t]}^{[0.3,0]}$.
If $\tau : \Im_\mu \to I$ is defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.6, & U = U_1, \\ 0, & \text{otherwise} \end{cases}$$

and $\sigma : \Im_\nu \to I$ is defined by

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise }, \end{cases}$$

then obviously (μ, τ), (ν, σ) are smooth fuzzy topological spaces. Let a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x,s) = 0.8, F(x,t) = 0, F(y,s) = 0.7, F(y,t) = 0.$$

Here, F is not one-to-one, $F(\mu)^{[0.8,0]}_{[s,t]} = \nu$. Since $\tau(F^{-1}(V_1)^{[0.3,0.3]}_{[x,y]}) = 0$ and $\sigma(V_1) > 0$, we get F is not weakly smooth fuzzy continuous on μ. Let $C \in \mathfrak{I}_\mu$ be arbitrary.
Case 1. Let $C \leq (\mu - U_1)^{[0.5,0.5]}_{[x,y]}$. Then we have, $F(C) \leq (\nu - V_1)^{[0.5,0]}_{[s,t]}$.
Subcase 1. If $C = 0_X$, then $F(C) = 0_S$. Therefore, $F(\overline{C}) = 0_S = \overline{F(C)}$.
Subcase 2. If $0_X \neq C \leq (\mu - U_1)$, then $0_S \neq F(C) \leq (\nu - V_1)$. In this case, $\overline{C} = (\mu - U_1)$, $F(\overline{C}) = (\nu - V_1)$ and $\overline{F(C)} = \nu - V_1$. Therefore, $F(\overline{C}) = \overline{F(C)}$.
Case 2. $C \nleq (\mu - U_1)$. If $C(x) > 0.5$ or $C(y) > 0.5$, then $F(C)(s) > 0.5$. Therefore, $F(C) \nleq (\nu - V_1)$ and hence $F(\overline{C}) \leq \nu = \overline{F(C)}$.

Counterexample 3.19. Let $X = \{x, y\}$, $S = \{s, t\}$ and let $\mu^{[0.6,0.7]}_{[x,y]} \in I^X$, $\nu^{[0.6,0.7]}_{[s,t]} \in I^S$, ${U_n}^{[0.5-\frac{1}{n+1}, 0.6-\frac{1}{n+1}]}_{[x,y]}$, $\forall n = 1, 2, \ldots$, ${V_1}^{[0.5,0.6]}_{[s,t]}$.
Let $\tau : \mathfrak{I}_\mu \to I$ be defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ \frac{1}{n}, & U = U_n, \forall n = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

and let $\sigma : \mathfrak{I}_\nu \to I$ be defined by

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.6, & V = V_1, \\ 0, & \text{otherwise .} \end{cases}$$

Here (μ, τ) is a positive minimum smooth fuzzy topological space. Let a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x,s) = 0.6, F(x,t) = 0, F(y,s) = 0, F(y,t) = 0.7.$$

Clearly F is one-to-one and $F(\mu)^{[0.6,0.7]}_{[s,t]} = \nu$. Since $\tau(F^{-1}(V_1)^{[0.5,0.6]}_{[x,y]}) = 0$ and $\sigma(V_1) > 0$, F is not weakly smooth fuzzy continuous on μ. We note that if $C^{[p,q]}_{[x,y]} \in \mathfrak{I}_\mu$, then $F(C)^{[p,q]}_{[s,t]}$.
Case 1. $p \leq 0.1$ and $q \leq 0.1$. Since $F(C) \leq (\nu - V_1)^{[0.1,0.1]}_{[x,y]}$, $\overline{F(C)} = \nu - V_1$ and $\overline{C} = \wedge(\mu - U_n)^{[0.1,0.1]}_{[x,y]}$. Therefore, $F(\overline{C})^{[0.1,0.1]}_{[s,t]} = \overline{F(C)}$.
Case 2. $p > 0.1$ or $q > 0.1$. In this case, $F(\overline{C}) \leq \nu = \overline{F(C)}$.

The converse of the Theorem 3.13 is not true.

Counterexample 3.20. Let $X = \{x, y\}$, $S = \{s, t\}$ and let $\mu^{[0.8,0.7]}_{[x,y]} \in I^X$, $\nu^{[0.8,0.7]}_{[s,t]} \in I^S$, ${U_1}^{[0.2,0.1]}_{[x,y]}$ and ${V_1}^{[0.2,0.1]}_{[s,t]}$. We define $\tau : \mathfrak{I}_\mu \to I$ by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.4, & U = U_1, \\ 0, & \text{otherwise} \end{cases}$$

and $\sigma : \Im_\nu \to I$ by

$$\sigma(V) \quad = \quad \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise .} \end{cases}$$

Let a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x, s) = 0.8, F(x, t) = 0, F(y, s) = 0, F(y, t) = 0.7.$$

Clearly, F is one-to-one and $F(\mu)^{[0.8,0.7]}_{[s,t]} = \nu$. Since, $\tau(F^{-1}(V_1)^{[0.2,0.1]}_{[x,y]}) = \tau(U_1) = 0.4 < 0.5 = \sigma(V_1)$, F is not smooth fuzzy continuous on μ. Let $C \in \Im_\mu$.
Case 1. Let $C \leq (\mu - U_1)^{[0.6,0.6]}_{[x,y]}$. Clearly, we have, $F(C) \leq (\nu - V_1)^{[0.6,0.6]}_{[s,t]}$.
Subcase 1. If $C = 0_X$, then $F(C) = 0_S$. Therefore, $F(\overline{C}) = 0 = \overline{F(C)}$.
Subcase 2. If $0_X \neq C \leq (\mu - U_1)$, then $0_S \neq F(C) \leq (\nu - V_1)$. In this case, $\overline{C} = (\mu - U_1)$, $F(\overline{C}) = \nu - V_1 = \overline{F(C)}$.
Case 2. $C \nleq (\mu - U_1)$. If $C(x) > 0.6$ or $C(y) > 0.6$, then $F(C)(s) > 0.6$. Therefore, $F(C) \nleq (\nu - V_1)$ and hence $F(\overline{C}) \leq \nu = \overline{F(C)}$.

Remark 3.21. Let (μ, τ) be a positive minimum smooth fuzzy topological space. From Theorems 3.10, 3.17, one can notice that a one-to-one fuzzy proper function F is weakly smooth fuzzy continuous if and only if F is q_n-weakly smooth fuzzy continuous.

4. Convergence of net of fuzzy sets

Definition 4.1. (Cf. [18]) Let $\{A_n : n \in D\}$ be a net of fuzzy sets in a fuzzy topological space μ. The fuzzy upper limit of the net $\{A_n : n \in D\}$ in $\Im_\mu$ is denoted by $\limsup_D(A_n)$ and is defined by the union of all fuzzy points P^λ_x in μ such that for a given $n_0 \in D$ and for a given q-neighborhood U of P^λ_x in μ, there exists $n \in D$ such that $n \geq n_0$ and $A_n q U[\mu]$. If there is no such P^λ_x, then let $\limsup_D(A_n) = 0_X$.

Definition 4.2. (Cf. [18])Let $\{A_n : n \in D\}$ be a net of fuzzy sets in a fuzzy topological space μ. The fuzzy lower limit of the net $\{A_n : n \in D\}$ in $\Im_\mu$ is denoted by $\liminf_D(A_n)$ and is defined by the union of all fuzzy points P^λ_x in μ such that for a given q-neighborhood U of P^λ_x in μ, there exists $n_0 \in D$ such that $A_n q U[\mu]$, for every $n \in D$ with $n \geq n_0$. If there is no such P^λ_x, then let $\liminf_D(A_n) = 0_X$.

Definition 4.3. (Cf. [18]) A net $\{A_n : n \in D\}$ of fuzzy sets in a smooth fuzzy topological space μ is said to be fuzzy convergent to the fuzzy set A if $\liminf_D(A_n) = \limsup_D(A_n) = A$.

In this case, we write $A = \lim_D(A_n)$.

Definition 4.4. A fuzzy proper function $F : \mu \to \nu$ is said to be net 1 continuous on μ if the net $\{F(A_n) : n \in D\}$ of fuzzy sets in ν converges to $F(A)$ whenever a net $\{A_n : n \in D\}$ of fuzzy sets converges to A in μ.

Theorem 4.5. *(Cf. [18]) Let $\{A_n : n \in D\}$ be a net of fuzzy sets in μ. If $A_n = A$ for every $n \in D$, then $\limsup_D(A_n) = \liminf_D(A_n) = \overline{A}$.*

Theorem 4.6. *Let $F : \mu \to \nu$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$. If F is q_n-weakly smooth fuzzy continuous and weak open, then*

$$F(\limsup_D(A_n)) = \limsup_D F(A_n),$$

for every net $\{A_n : n \in D\}$ in μ.

Proof. Let us assume that F is q_n-weakly smooth fuzzy continuous and F is weak open. Now we choose $P_s^\lambda \in F(\limsup_D(A_n))$ and a q-neighborhood $V \in \Im_\nu$ of P_s^λ arbitrarily. Then, there exists $P_x^\lambda \in \limsup_D(A_n)$ such that $F(P_x^\lambda) = P_s^\lambda$. Since F is q_n-weakly smooth fuzzy continuous, there exists a q-neighborhood U of P_x^λ such that $F(U) \leq V$. On the other hand, since $P_x^\lambda \in \limsup_D(A_n)$, $\forall n_0 \in D$, there exists $n \in D$ such that $n \geq n_0$ and $UqA_n[\mu]$. Therefore, $U(y) + A_n(y) > \mu(y)$, for some $y \in X$. For this y, there exists unique $t \in S$ such that $F(y,t) = \mu(y)$. By using the fact that F is one-to-one, we have $V(t) + F(A_n)(t) \geq F(U)(t) + F(A_n)(t) = U(y) + A_n(y) > \mu(y) = F(\mu)(t) = \nu(t)$. Hence $P_s^\lambda \in \limsup_D F(A_n)$.

Let $P_s^\lambda \in \limsup_D F(A_n)$. Then there exists $P_x^\lambda \in \mu$ such that $F(P_x^\lambda) = P_s^\lambda$. Let U and V be q-neighborhoods of P_x^λ and $F(P_x^\lambda)$, respectively. Since F is a weak open map, $\sigma(F(U)) > 0$. By using F is one-to-one, $\lambda + F(U)(s) = \lambda + U(x) > \mu(x) = F(\mu)(s) = \nu(s)$. Hence, $F(U)$ is a q-neighborhood of P_s^λ. Therefore, $V \wedge F(U)$ is a q-neighborhood of P_s^λ. On the other hand, since $P_s^\lambda \in \limsup_D F(A_n)$, for a given $n_0 \in D$, there exists $n \in D$ such that $n \geq n_0$ and $[V \wedge F(U)]qF(A_n)[\nu]$. Since $[V \wedge F(U)](t) + F(A_n)(t) > \nu(t)$ for some $t \in S$ and F is one-to-one, there exists unique $y \in X$ such that $F(y,t) = \mu(y)$ and $F^{-1}(V \wedge F(U))(y) + A_n(y) = (V \wedge F(U))(t) + F(A_n)(t) > \nu(t) = F(\mu)(t) = \mu(y)$. Since F is one-to-one, $F^{-1}(V \wedge F(U)) = F^{-1}(V) \wedge U$. Therefore $[U \wedge F^{-1}(V)]qA_n[\mu]$, which implies that $UqA_n[\mu]$. Hence, $P_x^\lambda \in \limsup_D(A_n)$ and $F(P_x^\lambda) \in F(\limsup_D(A_n))$. Thus, $F(\limsup_D(A_n)) = \limsup_D F(A_n)$. □

Theorem 4.7. *Let $F : \mu \to \nu$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$. If F is q_n-weakly smooth fuzzy continuous and weak open, then*

$$F(\liminf_D(A_n)) = \liminf_D F(A_n),$$

for every fuzzy net $\{A_n : n \in D\}$ in μ.

Proof. This is similar to the proof of Theorem 4.6. □

Theorem 4.8. *Let $F : \mu \to \nu$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$. If F is q_n-weakly smooth fuzzy continuous on μ and weak open, then F is net 1 continuous on μ.*

Proof. Let us assume that F be q_n-weakly smooth fuzzy continuous and weak open. If $\{A_n : n \in D\}$ converges to A in μ, then $\limsup_D(A_n) = \liminf_D(A_n) = A$ and $F(\limsup_D(A_n)) = F(\liminf_D(A_n)) = F(A)$. Then by Theorems 4.6, 4.7, we get $\limsup_D F(A_n) = \liminf_D F(A_n) = F(A)$. Therefore, $\{F(A_n) : n \in D\}$ converges to $F(A)$. This completes the proof of this theorem. □

The following theorems hold trivially from Theorem 4.8.

Theorem 4.9. *Let $F : \mu \to \nu$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$. If F is smooth fuzzy continuous on μ and weak open, then F is net 1 continuous on μ.*

Theorem 4.10. *Let $F : \mu \to \nu$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$. If F is weakly smooth fuzzy continuous and weak open on μ, then F is net 1 continuous on μ.*

The statements of Theorems 4.8, 4.9, 4.10 are not true if F is not one-to-one or F is not a weak open map. The following counterexamples justify our statement.

Counterexample 4.11. Let $X = \{x, y, z\}$, $S = \{s, t\}$, $\mu_{[x,y,z]}^{[0.8,0.7,0.6]}$, $\nu_{[s,t]}^{[0.8,0.6]}$. Define two fuzzy subsets $U_1 \leq \mu$ and $V_1 \leq \nu$ by $U_1{}_{[x,y,z]}^{[0.1,0.1,0.3]}$, $V_1{}_{[s,t]}^{[0.1,0.3]}$.
We define $\tau : \Im_\mu \to I$ by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.6, & U = U_1, \\ 0, & \text{otherwise} \end{cases}$$

and $\sigma : \Im_\nu \to I$ by

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.6, & V = V_1, \\ 0, & \text{otherwise} \end{cases},$$

Let a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x, s) = 0.8, F(x, t) = 0, F(y, s) = 0.7, F(y, t) = 0, F(z, s) = 0, F(z, t) = 0.6.$$

Therefore, F is not one-to-one and $F(\mu)_{[s,t]}^{[0.8,0.6]} = \nu$. Clearly, $F^{-1}(V_1) = U_1$, $F^{-1}(0_S) = 0_X$ and $F^{-1}(\nu) = \mu$. Hence, F is smooth fuzzy continuous. Therefore, F is weakly smooth fuzzy continuous and q_n-weakly smooth fuzzy continuous. Consider a fuzzy net $(A_n)_{[x,y,z]}^{[0,0.6+\frac{1}{n+10},0]}$, $\forall n \in N$, where N is the set of all natural numbers. Hence, $U_1 q A_n[\mu], \forall n \in N$. Clearly $\limsup_N(A_n) = \liminf_N(A_n) = \mu$. Therefore, $\{A_n : n \in N\}$ converges to μ. Now, $F(A_n)_{[s,t]}^{[0.6+\frac{1}{n+10},0]}, \forall n \in N$ which is a fuzzy net in ν. Let $P_s^\lambda, P_t^\delta \in \nu$.
Case 1: $0.7 < \lambda \leq 0.8$ and $0.3 < \delta \leq 0.6$
Here, V_1 is a q-neighborhood of P_s^λ and P_t^δ but $V_1 \not{q} F(A_n)[\nu], \forall n \in N$. Therefore, P_s^λ and $P_t^\delta \notin \limsup_N F(A_n)$.
Case 2: $0 < \lambda \leq 0.7$ and $0 < \delta \leq 0.3$
In this case, the only q-neighborhood of P_s^λ and P_t^δ is ν, which is quasi-coincident with $F(A_n), \forall n \in N$.
Hence, $\limsup_N F(A_n)_{[s,t]}^{[0.7,0.3]} \neq F(\mu)$. Therefore, $\{F(A_n) : n \in N\}$ does not converge to ν.

Counterexample 4.12. Let $X = \{x, y\}$, $S = \{s, t\}$ and let $\mu_{[x,y]}^{[0.8,0.7]}$, $\nu_{[s,t]}^{[0.8,0.7]}$ and $U1_{[x,y]}^{[0.1,0.4]}$. If $\tau : \Im_\mu \to I$ is defined by

$$\tau(U) \quad = \quad \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.6, & U = U_1, \\ 0, & \text{otherwise} \end{cases}$$

and $\sigma : \Im_\nu \to I$ is defined by

$$\sigma(V) \quad = \quad \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0, & \text{otherwise} \ , \end{cases}$$

then obviously (μ, τ), (ν, σ) are smooth fuzzy topological spaces. Let a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x, s) = 0.8, F(x, t) = 0, F(y, s) = 0, F(y, t) = 0.7.$$

Clearly, F is one-to-one, $F(\mu)_{[s,t]}^{[0.8,0.7]} = \nu$, F is smooth fuzzy continuous, weakly smooth fuzzy continuous and q_n-weakly smooth fuzzy continuous. But F is not a weak open map, because $F(U_1)_{[s,t]}^{[0.1,0.4]}$. Consider a fuzzy net $(A_n)_{[x,y]}^{[\frac{1}{n+1}, \frac{1}{n+9}]}$, $\forall n \in N$, where N is the set of all natural numbers. Let P_x^λ and $P_y^\delta \in \mu$ be arbitrary.
Case 1: $0.7 < \lambda \leq 0.8$ and $0.3 < \delta \leq 0.7$. Since U_1 is a q-neighborhood of P_x^λ as well as a q-neighborhood of P_y^δ and $U_1 \not q A_n[\mu]$ for all $n \in N$, we obtain $P_x^\lambda,\ P_y^\delta \notin \limsup_N(A_n)$.
Case 2: $0 < \lambda \leq 0.7$ and $0 < \delta \leq 0.3$. In this case, the only q-neighborhood of P_x^λ and P_y^δ is μ which is quasi-coincident with $A_n, \forall n \in N$. Therefore, $\limsup_N(A_n) = \liminf_N(A_n) = A_{[x,y]}^{[0.7,0.3]}$. The only q-neighborhood of P_s^λ and P_t^δ is ν, which is quasi coincident with $F(A_n)_{[s,t]}^{[\frac{1}{n+1}, \frac{1}{n+9}]}$, $\forall n \in N$. Hence, $\limsup_N F(A_n) = \liminf_N F(A_n) = \nu \neq F(A)_{[s,t]}^{[0.7,0.3]}$. Therefore, $\{F(A_n)\}$ does not converge to $F(A)$.

Remark 4.13. The converse of the Theorem 4.8 holds without assuming that F is one-to-one and weak open.

Theorem 4.14. *Let $F : \mu \to \nu$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$. If F is net 1 continuous on μ, then F is q_n-weakly smooth fuzzy continuous on μ.*

Proof. Assume that F is net 1 continuous on μ. If F is not q_n-weakly smooth fuzzy continuous, then there is a q-neighborhood $V_1 \in \Im_\nu$ of $F(P_x^\lambda)$, for every q-neighborhood U_α of P_x^λ such that $F(U_\alpha) \nleq V_1$. Therefore, there exists $t_\alpha \in S$ such that $F(U_\alpha)(t_\alpha) > V_1(t_\alpha)$. For each $t_\alpha \in S$, by assumption, there exists $y_\alpha \in X$ such that $F(y_\alpha, t_\alpha) = \mu(y_\alpha)$. Now, consider a fuzzy net $\{A_n : n \in N\}$ which is defined as follows, $A_n(x) = C(x) = \begin{cases} \mu(y_\alpha) - V_1(t_\alpha), & x = y_\alpha \\ 0, & x \notin \{y_\alpha\}, \end{cases}$ $\forall x \in S$ and $\forall n \in N$, where N is the set of all natural numbers. Hence $\{A_n : n \in N\}$ is a constant fuzzy net. Then by Theorem 4.5, $\lim_N(A_n) = \overline{C} = A$. Here, $\{F(A_n) : n \in N\}$ is a fuzzy net in

ν and

$$F(A_n)(s) \quad \leq \quad \begin{cases} \nu(t_\alpha) - V_1(t_\alpha), & s = t_\alpha \\ 0, & \text{otherwise.} \end{cases}$$

For each U_β, we have $F(U_\beta)(t_\beta) > V(t_\beta)$, for some $t_\beta \in S$. Therefore, there exists at least one $y_\beta \in X$ such that $U_\beta(y_\beta) > V_1(t_\beta)$. Since $(A_n)(y_\beta) = \mu(y_\beta) - V_1(t_\beta) > \mu(y_\beta) - U_\beta(y_\beta)$, we have $A_n(y_\beta) + U_\beta(y_\beta) > \mu(y_\beta)$ and hence $(A_n)qU_\beta[\mu], \forall n \in N$. Similarly $A_n qU_\alpha[\mu]$, for every q-neighborhood U_α of $P_x^\lambda \Rightarrow P_x^\lambda \in A \Rightarrow F(P_x^\lambda) \in F(A)$. Next, we claim that $F(P_x^\lambda) \notin \limsup_N F(A_n)$. Let $r \in S$ be arbitrary. If there exists $p \in X$ such that $F(p,r) > 0$, then $V_1(r) + F(A_n)(r) = \nu(r)$. If $F(p,r) = 0$, $\forall p \in X$, then $F(A_n)(r) = 0$. Therefore, $V_1 \not{q} F(A_n)[\nu], \forall n \in N \Rightarrow F(P_x^\lambda) \notin \limsup_N F(A_n) \Rightarrow \{F(A_n) : n \in N\}$ does not converge to $F(A)$, which is a contradiction. Thus, F is q_n-weakly smooth fuzzy continuous on μ. □

Theorem 4.15. *Let $F : \mu \to \nu$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$ and let μ be a positive minimum smooth fuzzy topological space. If F is net 1 continuous on μ, then F is weakly smooth fuzzy continuous on μ.*

Proof. By Theorem 4.14 and Remark 3.21 the proof follows. □

The statement of the above theorem is not true if F is not one-to-one as shown in the following counterexample.

Counterexample 4.16. Let $X = \{x, y\}$, $S = \{s, t\}$, $\mu_{[x,y]}^{[0.5,0.6]}$, $\nu_{[s,t]}^{[0.6,0]}$. Define two fuzzy subsets $U_1 \leq \mu$ and $V_1 \leq \nu$ by $U_1{}_{[x,y]}^{[0.1,0.2]}$, $V_1{}_{[s,t]}^{[0.2,0]}$.
Let $\tau : \Im_\mu \to I$ be defined by

$$\tau(U) \quad = \quad \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.7, & U = U_1, \\ 0, & \text{otherwise} \end{cases}$$

and let $\sigma : \Im_\nu \to I$ be defined by

$$\sigma(V) \quad = \quad \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.6, & V = V_1, \\ 0, & \text{otherwise .} \end{cases}$$

Let a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x,s) = 0.5, F(x,t) = 0, F(y,s) = 0.6, F(y,t) = 0.$$

Here, F is not one-to-one and $F(\mu)_{[s,t]}^{[0.6,0]} = \nu$. Since $F^{-1}(V_1)_{[x,y]}^{[0.2,0.2]}$, F is not weakly smooth fuzzy continuous. Let $\{A_n : n \in D\}$ be a fuzzy net converging to A in μ. We claim that $\{F(A_n) : n \in D\}$ converges to $F(A)$ in ν. Let $P_s^\lambda \in F(A)$. Clearly we have P_x^λ, P_y^λ in A and $F(P_x^\lambda) = F(P_y^\lambda) = P_s^\lambda$.
Case 1: $0.4 < \lambda \leq 0.6$
Here, V_1 and ν are the only q-neighborhoods of P_s^λ. Since $P_y^\lambda \in \liminf_D(A_n)$ and U_1 is a q-neighborhood of P_y^λ, there exists $n_0 \in D$ such that $A_n qU_1[\mu], \forall n \geq n_0$. Therefore, we have $A_n(x) + U_1(x) > \mu(x)$ or $A_n(y) + U_1(y) > \mu(y)$. Hence $A_n(x) > 0.4$ or $A_n(y) > 0.4$, which implies that $F(A_n)(s) > 0.4$. Since $V_1(s) + F(A_n)(s) >$

$0.2 + 0.4 = 0.6 = \nu(s)$, we get $V_1 qF(A_n)[\nu]$ and $\nu qF(A_n)[\nu], \forall n \in D$.
Case 2: $0 < \lambda \leq 0.4$
The only q-neighborhood of P_s^λ is ν and $\nu qF(A_n)[\nu], \forall n \in D$. Therefore, $P_s^\lambda \in \liminf_D F(A_n)$ and hence $F(A) \leq \liminf_D F(A_n)$.

Next we assume that $P_s^\lambda \in \limsup_D F(A_n)$.
Case 1: $0.4 < \lambda \leq 0.6$. We note that, V_1 is a q-neighborhoods of P_s^λ. Therefore, for a given $n_0 \in D$, there exists $n \in D$ such that $n \geq n_0$ and $V_1 qF(A_n)[\nu]$. Since $V_1(t) = 0$, we have $V_1(s) + F(A_n)(s) > \nu(s)$, which implies that $F(A_n)(s) > 0.4$. Therefore, we get $A_n(x) > 0.4$ or $A_n(y) > 0.4$ and hence $U_1 qA_n[\mu]$. Clearly, $\mu qA_n[\mu], \forall n \geq n_0$. Since U_1 and μ are the q-neighborhoods of P_y^λ, we get $P_y^\lambda \in \limsup_D(A_n) = A$. Thus, $P_s^\lambda = F(P_y^\lambda) \in F(A)$.
Case 2: $0 < \lambda \leq 0.4$. In this case, the only q-neighborhood of P_y^λ is μ, which is quasi-coincident with A_n, $\forall n \in D$. Therefore, $P_y^\lambda \in \limsup_D(A_n) = A$ and hence $F(P_y^\lambda) \in F(A)$. Therefore, $F(A) \leq \liminf_D F(A_n) \leq \limsup_D F(A_n) \leq F(A)$. Thus $\{F(A_n) : n \in D\}$ converges to $F(A)$.

The converse of the Theorem 4.9 is not true.

Counterexample 4.17. Let $X = \{x, y\}$, $S = \{s, t\}$, $\mu^{[0.5,0.6]}_{[x,y]}$, $\nu^{[0.5,0.6]}_{[s,t]}$. Define two fuzzy subsets $U_1 \leq \mu$ and $V_1 \leq \nu$ by $U_1{}^{[0.1,0]}_{[x,y]}$, $V_1{}^{[0.1,0]}_{[s,t]}$.
If $\tau : \mathfrak{I}_\mu \to I$ is defined by

$$\tau(U) \quad = \quad \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.4, & U = U_1, \\ 0, & \text{otherwise} \end{cases}$$

and $\sigma : \mathfrak{I}_\nu \to I$ is defined by

$$\sigma(V) \quad = \quad \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.6, & V = V_1, \\ 0, & \text{otherwise} \end{cases},$$

then obviously (μ, τ), (ν, σ) are smooth fuzzy topological spaces. Let a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x, s) = 0.5, F(x, t) = 0, F(y, s) = 0, F(y, t) = 0.6.$$

Clearly, F is one-to-one and $F(\mu)^{[0.5,0.6]}_{[s,t]} = \nu$ and $F^{-1}(V_1)^{[0.1,0.1]}_{[x,y]} = U_1$. But $\tau(F^{-1}(V_1)) = 0.4 < \sigma(V_1)$. Hence, F is not smooth fuzzy continuous. Consider a fuzzy net $\{A_n : n \in D\}$ converging to A in μ. We claim that $\{F(A_n) : n \in D\}$ converges to $F(A)$ in ν. Let $P_s^\lambda \in F(A)$. Then there exists P_x^λ in A such that $F(P_x^\lambda) = P_s^\lambda$.
Case 1: $0.4 < \lambda \leq 0.5$. Here, V_1 and ν are the only q-neighborhoods of P_s^λ. Since $P_x^\lambda \in \liminf_D(A_n)$ and U_1 is the q-neighborhood of P_x^λ, there exists $n_0 \in D$, such that $A_n qU_1[\mu], \forall n \geq n_0$. Since $U_n(y) = 0$, we have $A_n(x) + U_1(x) > \mu(x)$ and we get $V_1(s) + F(A_n)(s) = F(U_1)(x) + F(A_n)(s) \geq U_1(x) + A_n(x) > \mu(x) = F(\mu)(s) = \nu(s)$. Hence, $V_1 qF(A_n)[\nu]$. Obviously, we have, $\nu qF(A_n)[\nu], \forall n \in D$.
Case 2: $0 < \lambda \leq 0.4$. The only q-neighborhood of P_s^λ is ν. Clearly, $\nu qF(A_n)[\nu], \forall n \in$

D. Therefore, $P_s^\lambda \in \liminf_D F(A_n)$. The only q-neighborhood of P_t^δ is ν, which is quasi-coincident with all $F(A_n)$. Hence, $F(A) \leq \liminf_D F(A_n)$.

Let $P_s^\lambda, P_t^\delta \in \limsup_D F(A_n)$ be arbitrary. Then there exist $P_x^\lambda, P_y^\delta \in \mu$ such that $F(P_x^\lambda) = P_s^\lambda$ and $F(P_y^\delta) = P_t^\delta$.

Case 1: $0.4 < \lambda \leq 0.5$. In this case, V_1 and ν are the q-neighborhoods of P_s^λ. Therefore, for each $n_0 \in D$, there exists $n \in D$ such that $n \geq n_0$ and $V_1 q F(A_n)[\nu]$. Since F is one-to-one, we have $V_1(s) + F(A_n)(s) > \nu(s)$, which implies that $U_1(x) + A_n(x) = V_1(s) + F(A_n)(s) > \nu(s) = \mu(x)$. Clearly $\mu q A_n[\mu]$. Thus, $P_x^\lambda \in \limsup_D(A_n)$.

Case 2: $0 < \lambda \leq 0.4$. In this case, the only q-neighborhood of P_x^λ is μ which is quasi-coincident with all A_n. The only q-neighborhood of P_y^λ is μ, which is quasi-coincident with all A_n. Finally, we get that $P_x^\lambda, P_y^\delta \in \limsup_D(A_n) = A$. Since $P_s^\lambda = F(P_x^\lambda) \in F(A)$ and $P_t^\delta = F(P_y^\lambda) \in F(A)$, we conclude that $F(A) \leq \liminf_D(A_n) \leq \limsup_D(A_n) \leq F(A)$. Therefore, $\{F(A_n : n \in D)\}$ converges to $F(A)$.

5. Convergence of net of fuzzy points

Definition 5.1. (Cf. [27]) Let $\zeta = \{\zeta_\lambda : \lambda \in D\}$ be a fuzzy net in μ. ζ is said to be quasi-coincident with A if for each $\lambda \in D$, ζ_λ is quasi-coincident with A. ζ is said to be eventually quasi-coincident with A if there is $m \in D$ such that ζ_λ is quasi-coincident with A, for every $\lambda \in D$ with $\lambda \geq m$.

Definition 5.2. (Cf. [27]) A fuzzy net $\zeta = \{\zeta_\lambda : \lambda \in D\}$ in a smooth fuzzy topological space (μ, τ) is said to converge to a fuzzy point ξ if ζ is eventually quasi-coincident with each q-neighborhood of ξ.

Lemma 5.3. *(Cf. [27]) In a smooth fuzzy topological space (μ, τ), a fuzzy point $P_x^\lambda \in \overline{A}$ if and only if there is a fuzzy net ζ in A such that ζ converges to P_x^λ.*

Definition 5.4. A fuzzy proper function $F : \mu \to \nu$ is said to be net 2 continuous on μ if the net $\{F(\zeta_n) : n \in D\}$ of fuzzy points in ν converges to a fuzzy point $F(\xi)$, whenever a net $\{\zeta_n : n \in D\}$ of fuzzy points in μ converges to a fuzzy point ξ in μ.

Theorem 5.5. *Let $F : \mu \to \nu$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$. If F is q_n-weakly smooth fuzzy continuous on μ, then F is net 2 continuous on μ.*

Proof. Let us assume that F is q_n-weakly smooth fuzzy continuous on μ and let a fuzzy net $\{\zeta_n : n \in D\}$ converge to P_x^λ. If V is a q-neighborhood of $F(P_x^\lambda)$, using F is q_n-weakly smooth fuzzy continuous, there exists a q-neighborhood U of P_x^λ such that $F(U) \leq V$. Then by hypothesis, there exists $n_0 \in D$, such that $U q \zeta_n[\mu], \forall n \geq n_0$. Therefore, $U(z) + \zeta_n(z) > \mu(z)$, for some $z \in X$ and $\forall n \geq n_o$. For this $z \in X$, there exists unique $t \in S$ such that $F(z,t) = \mu(z)$. Since F is one-to-one, $V(t) + F(\zeta_n)(t) \geq F(U)(t) + F(\zeta_n)(t) = U(z) + \zeta_n(z) > \mu(z) = F(\mu)(t) = \nu(t)$. Hence, $\{F(\zeta_n) : n \in D\}$ is a fuzzy net in ν and it converges to $F(P_x^\lambda)$. □

The following theorems hold trivially from the above theorem.

Theorem 5.6. *Let $F : \mu \to \nu$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$. IF F is weakly smooth fuzzy continuous on μ, then F is net 2 continuous on μ.*

Theorem 5.7. *Let $F : \mu \to \nu$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$. If F is smooth fuzzy continuous on μ, then F is net 2 continuous on μ.*

If F is not one-to-one, then the statements of Theorems 5.5, 5.6, 5.7 are not true. The following counterexample justifies our statement.

Counterexample 5.8. Let $X = \{x, y, z\}$, $S = \{s, t\}$ and let $\mu^{[0.8,0.7,0.6]}_{[x,y,z]}$, $\nu^{[0.8,0.6]}_{[s,t]}$, $U_1{}^{[0.1,0.1,0.3]}_{[x,y,z]}$, $V_1{}^{[0.1,0.3]}_{[s,t]}$.
If $\tau : \Im_\mu \to I$ is defined by

$$\tau(U) \quad = \quad \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.7, & U = U_1, \\ 0, & \text{otherwise} \end{cases}$$

and $\sigma : \Im_\nu \to I$ is defined by

$$\sigma(V) \quad = \quad \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.6, & V = V_1, \\ 0, & \text{otherwise }, \end{cases}$$

then obviously (μ, τ), (ν, σ) are smooth fuzzy topological spaces. Let a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x, s) = 0.8, F(x, t) = 0, F(y, s) = 0.7, F(y, t) = 0, F(z, s) = 0, F(z, t) = 0.6.$$

Clearly, F is one-to-one and $F(\mu)^{[0.8,0.6]}_{[s,t]} = \nu$. Here, $F^{-1}(V_1) = U_1$ and $\tau(F^{-1}(V_1)) = 0.7 > 0.6 = \sigma(V_1)$. Hence F is smooth fuzzy continuous which implies that F is weakly smooth fuzzy continuous and F is q_n-weakly smooth fuzzy continuous. Consider a fuzzy net $(\zeta_n)^{[0,0.6+\frac{1}{n+10},0]}_{[x,y,z]}$, $\forall n \in N$, where N is the set of all natural numbers. Here, the q-neighborhoods of $P^{0.75}_x$ are U_1 and μ. Clearly, μ and U_1 are quasi-coincident with $\zeta_n, \forall n \in N$. Therefore, $\{\zeta_n\}$ converge to $P^{0.75}_x$. Now, the fuzzy net in ν is $\left\{ F(\zeta_n)^{[0.6+\frac{1}{n+10},0]}_{[s,t]} : n \in N \right\}$. We note that, V_1 is a q-neighborhood of $F(P^{0.75}_x)$. But $V_1(s) + F(\zeta_n)(s) = 0.1 + 0.6 + \frac{1}{1+10} < 0.8, \forall n \in N$. Hence $\{F(\zeta_n)\}$ does not converge to $F(P^{0.75}_x)$.

The converse of the Theorem 5.5 holds without assuming one-to-one.

Theorem 5.9. *Let $F : \mu \to \nu$ be a fuzzy proper function such that $\nu = F(\mu)$. If F is net 2 continuous on μ, then F is q_n-weakly smooth fuzzy continuous on μ.*

Proof. Let us assume that F is net 2 continuous on μ. Suppose if F is not q_n-weakly smooth fuzzy continuous, then there is q-neighborhood $V_1 \in \Im_\nu$ of $F(P^\lambda_x)$ such that $F(U_\alpha) \nleq V_1$, for every q-neighborhood U_α of P^λ_x. Therefore there exists $t_\alpha \in S$ such that $F(U_\alpha)(t_\alpha) > V_1(t_\alpha)$. Hence, there exists $y_\alpha \in X$ such that $U_\alpha(y_\alpha) > V_1(t_\alpha)$. If $D = \{U_\alpha : U_\alpha \text{ is a } q\text{-neighborhood of } P^\lambda_x\}$, then $(D, \leq)$ is a directed set. Define $\zeta_{U_\alpha}(y_\alpha) = \mu(y_\alpha) - V_1(t_\alpha)$. Then, $\{\zeta_{U_\alpha} : U_\alpha \in D\}$ is a fuzzy net of fuzzy points in μ. Now, we have $\zeta_{U_\alpha}(y_\alpha) = \mu(y_\alpha) - V_1(t_\alpha) > \mu(y_\alpha) - U_\alpha(y_\alpha)$, for every $U_\alpha \in D$. Therefore, $\zeta_{U_\alpha} q U_\alpha[\mu]$. If U_k is a q-neighborhood of P^λ_x, then $\zeta_{U_\alpha} q U_k[\mu]$, for every $U_\alpha \leq U_k$. Hence, $\{\zeta_{U_\alpha} : U_\alpha \in D\}$ converges to P^λ_x. Clearly $\{F(\zeta_{U_\alpha}) : U_\alpha \in D\}$ is a fuzzy net in ν and $F(U_\alpha)(t_\alpha) = \mu(y_\alpha) - V_1(t_\alpha) \leq \nu(t_\alpha) - V_1(t_\alpha)$. Therefore, $F(\zeta_{U_\alpha}) \not{q} V_1[\nu], \forall U_\alpha \in D \Rightarrow \{F(\zeta_{U_\alpha}) : U_\alpha \in D\}$ does not converge to $F(P_x^\lambda)$, which is a contradiction. Thus, F is q_n-weakly smooth fuzzy continuous. □

The converse of the Theorem 5.6, holds for the positive minimum smooth fuzzy topological space (μ, τ).

Theorem 5.10. *Let $F : \mu \to \nu$ be a one-to-one fuzzy proper function such that $\nu = F(\mu)$ and let (μ, τ) be a positive minimum smooth fuzzy topological space. If F is net 2 continuous on μ, then F is weakly smooth fuzzy continuous.*

Proof. Let $\mathscr{F}_\sigma(V) > 0$ and let $P_z^\lambda \in \overline{F^{-1}(V)}$. Then by Lemma 5.3, there exists a fuzzy net $\{\zeta_n : n \in D\}$ in $F^{-1}(V)$ converging to P_z^λ. Then by hypothesis, $\{F(\zeta_n) : n \in D\}$ is a fuzzy net in V and it converges to $F(P_z^\lambda)$. Therefore, $F(P_z^\lambda) \in \overline{V} = V$. Hence, $P_z^\lambda \in F^{-1}(V)$. Thus, we conclude that F is weakly smooth fuzzy continuous. □

The statement of the above theorem is not true when F is not one-to-one.

Counterexample 5.11. Let $X = \{x, y\}$, $S = \{s, t\}$, $\mu_{[x,y]}^{[0.6,0.7]}$, $\nu_{[s,t]}^{[0.7,0]}$. Define two fuzzy subsets $U_1 \leq \mu$ and $V_1 \leq \nu$ by ${U_1}_{[x,y]}^{[0.2,0.3]}$, ${V_1}_{[s,t]}^{[0.3,0]}$.
Let $\tau : \Im_\mu \to I$ be defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.7, & U = U_1, \\ 0, & \text{otherwise} \end{cases}$$

and let $\sigma : \Im_\nu \to I$ be defined by

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.6, & V = V_1, \\ 0, & \text{otherwise .} \end{cases}$$

Let a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x, s) = 0.6, F(x, t) = 0, F(y, s) = 0.7, F(y, t) = 0.$$

Clearly, F is not one-to-one and $F(\mu)_{[s,t]}^{[0.7,0]} = \nu$. Since $F^{-1}(V_1)_{[x,y]}^{[0.3,0.3]}$, F is not weakly smooth fuzzy continuous. First, we show that if a fuzzy net $\{\zeta_n : n \in D\}$ converges to P_x^λ in μ, then $\{F(\zeta_n) : n \in D\}$ converges to $F(P_x^\lambda)$ in ν.
Case 1: $0.4 < \lambda \leq 0.6$
In this case, the q-neighborhoods of $F(P_x^\lambda) = P_s^\lambda$ are V_1 and ν. Since $\{\zeta_n : n \in D\}$ converges to P_x^λ and U_1 is a q-neighborhood of P_x^λ, there exists $n_0 \in D$ such that $\zeta_n q(U_1)[\mu], \forall n \geq n_0$. Therefore, we have $\zeta_n(x) + U_1(x) > \mu(x)$ or $\zeta_n(y) + U_1(y) > \mu(y)$. Hence $\zeta_n(x) > 0.4$ or $\zeta_n(y) > 0.4$, which implies that $F(\zeta_n)(s) > 0.4$. From these observations, we have $V_1(s) + F(\zeta_n)(s) > 0.3 + 0.4 = 0.7 = \nu(s)$. Hence, $V_1 q F(\zeta_n)[\nu]$. Clearly, $F(\zeta_n) q \nu, \forall n \in D$.
Case 2: $0 < \lambda \leq 0.4$
Since the only q-neighborhood of P_s^λ is ν and $\nu q F(\zeta_n)[\nu], \forall n \in D$, clearly $\{F(\zeta_n) : n \in D\}$ converges to $F(P_x^\lambda)$.

Similarly, if any $\{\zeta_n : n \in D\}$ converges to P_y^λ in μ, then $\{F(\zeta_n) : n \in D\}$ converges to $F(P_y^\lambda)$ in ν.

The converse of the Theorem 5.7 is not true.

Counterexample 5.12. Let $X = \{x, y\}$, $S = \{s, t\}$, $\mu_{[x,y]}^{[0.7,0.8]}$, $\nu_{[s,t]}^{[0.7,0.8]}$, ${U_1}_{[x,y]}^{[0.3,0]}$ and ${V_1}_{[s,t]}^{[0.3,0]}$.
If $\tau : \Im_\mu \to I$ is defined by

$$\tau(U) \quad = \quad \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0.4, & U = U_1, \\ 0, & \text{otherwise} \end{cases}$$

and $\sigma : \Im_\nu \to I$ is defined by

$$\sigma(V) \quad = \quad \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.6, & V = V_1, \\ 0, & \text{otherwise} \end{cases},$$

then (μ, τ) and (ν, σ) are smooth fuzzy topological spaces. Let a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x, s) = 0.7, F(x, t) = 0, F(y, s) = 0, F(y, t) = 0.8.$$

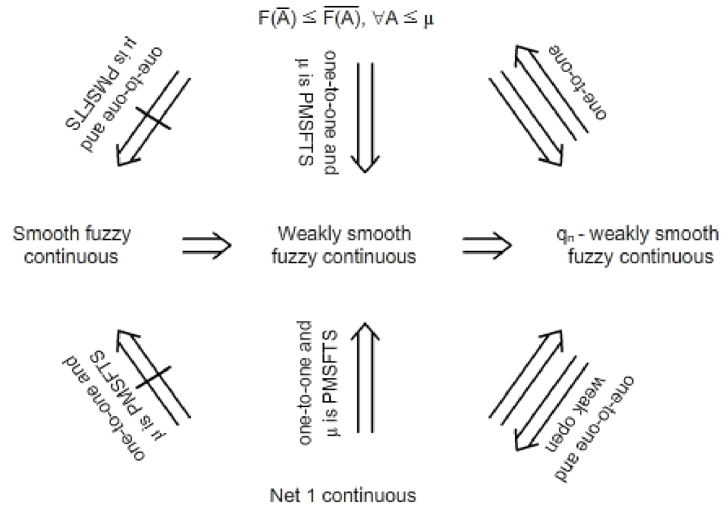
Clearly F is one-to-one and $F(\mu)_{[s,t]}^{[0.7,0.8]}$. Since, $\tau(F^{-1}(V_1)) = 0.4 < 0.6 = \sigma(V_1)$, F is not smooth fuzzy continuous. Assume that $\{\zeta_n : n \in D\}$ is a fuzzy net in μ converging to P_x^λ. We claim that $\{F(\zeta_n) : n \in D\}$ converges to $F(P_x^\lambda)$ in ν.
Case 1: $0.4 < \lambda \leq 0.7$. The q-neighborhoods of $F(P_x^\lambda) = P_s^\lambda$ are V_1 and ν only. Clearly, $F(\zeta_n)q\nu, \forall n \in D$. Since net $\{\zeta_n : n \in D\}$ of fuzzy points converges to P_x^λ and U_1 is a q-neighborhood of P_x^λ, there exists $n_0 \in D$ such that $\zeta_n q U_1[\mu], \forall n \geq n_0$. Since $U_1(y) = 0$, $\zeta_n(x) + U_1(x) > \mu(x)$. Hence, $\zeta_n(x) > 0.4$, which implies that $F(\zeta_n)(s) > 0.4$. Therefore, $V_1(s) + F(\zeta_n)(s) > 0.3 + 0.4 = 0.7 = \nu(s)$. Thus, $V_1 q F(s_\lambda)[\nu]$.
Case 2: $0 < \lambda \leq 0.4$.
The only q-neighborhood of P_s^λ is ν and $\nu q F(\zeta_n)[\nu], \forall n \in D$. Therefore, $\{F(\zeta_n) : n \in D\}$ converges to $F(P_x^\lambda)$. Similarly, one can prove that if any $\{\zeta_n : n \in D\}$ in μ converges to P_y^λ, then $\{F(\zeta_n) : n \in D\}$ converges to $F(P_y^\lambda)$.

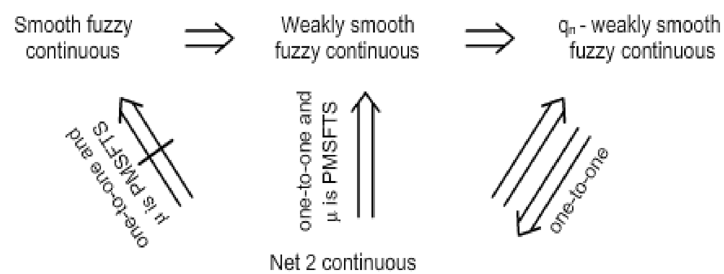
6. Conclusion

In this article, we have discussed various notions of continuity and also find the relations among them. Finally, the results obtained in sections 3, 4 and 5 are summarized in the following diagrams.

$$\text{Smooth fuzzy continuous} \underset{1-1}{\overset{1-1}{\rightleftarrows}} \mathscr{F}_\tau(F^{-1}(V)) \geq \mathscr{F}_\sigma(V),\ \forall V \in \Im_\nu.$$

$$\text{Weakly smooth fuzzy continuous} \underset{1-1}{\overset{1-1}{\rightleftarrows}} \mathscr{F}_\tau(F^{-1}(V)) > 0 \text{ if } \mathscr{F}_\sigma(V) > 0,\ \forall V \in \Im_\nu.$$





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