Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 6, No. 3, (November 2013), pp. 677–687
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Complete interval-valued fuzzy graphs

Hossein Rashmanlou, Young Bae Jun

Received 29 October 2012; Revised 4 January 2013; Accepted 8 April 2013

Abstract. In this paper, we define three new operation on interval-valued fuzzy graphs; namely direct product, semi strong product and strong product. Likewise, We give sufficient conditions for each one of them to be complete and show that if any of these products is complete, then at least one factor is a complete interval-valued fuzzy graph.

2010 AMS Classification: 03E72

Keywords: Interval-valued fuzzy graph, Complete interval-valued fuzzy graph.

Corresponding Author: Hossein Rashmanlou (hrashmanlou@yahoo.com)

1. Introduction

Graph theory has several interesting application in system analysis, operation research and economics. Since most of the time the aspects of graph problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy logic. The concept of fuzzy relation which has a widespread application in pattern recognition was introduced by Zadeh [17] in his Landmark Paper "Fuzzy sets" in 1965. Fuzzy graph and several fuzzy analogs of graph theoretic concepts were first introduced by Rosenfeld [13] in 1975. Mordeson and Peng [9] defined the concept of complement of fuzzy graph and studied some operations on it. In [14], the definition of complement of a fuzzy graph was modified. Moreover some properties of self-complementary fuzzy graphs and the complement of the operations of union, join and composition of fuzzy graphs that were introduced in [9] were studied. Hawary in [7] defined complete fuzzy graphs and gave three new operations on it.

In 1975, Zadeh [18] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets [17] in which the values of the membership degrees are intervals of numbers instead of the numbers. Akram, Feng, Sarwar and Jun [3] defined certain types of vague graphs. In 2011 Akram and Dudek [1] defined interval-valued fuzzy graphs and study some operations on it. Also they studied Intuitionistic fuzzy hypergraphs with applications [6]. Akram and Davvaz discussed the properties of strong intuitionistic fuzzy graphs and they introduced the concept of intuitionistic fuzzy line graphs [2]. Akram [4] defined bipolar fuzzy graphs and studied Interval-valued fuzzy line graphs [5]. Talebi and Rashmanlou [15] studied properties of isomorphism and complement on interval-valued fuzzy graphs. Likewise, they defined isomorphism on vague graphs [16].The concept of weak isomorphism, co-weak isomorphism and isomorphism between fuzzy graphs were introduced by K. R. Bhutani in [8]. In this paper, we provide three new operations on interval-valued fuzzy graphs; namely direct product, semi strong product and strong product. We give sufficient conditions for each one of them to be complete. For the notations not mentioned in the paper, the readers are referred to [5]∼[14].

2. Preliminaries

Definition 2.1. A fuzzy graph with V as the underlying set is a pair $G : (\sigma, \mu)$ where $\sigma : V \to [0,1]$ is a fuzzy subset and $\mu : V \times V \to [0,1]$ is a fuzzy relation on σ such that $\mu(x,y) \le \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ stands for minimum. The underlying crisp graph of G is denoted by $G^* : (\sigma^*, \mu^*)$ where
$\sigma^* = sup\ p(\sigma) = \{x \in V : \sigma(x) > 0\}$ and
$\mu^* = sup\ p(\mu) = \{(x,y) \in V \times V : \mu(x,y) > 0\}$. $H = (\sigma', \mu')$ is a fuzzy subgraph of G if there exists $X \subseteq V$ such that, $\sigma' : X \to [0,1]$ is a fuzzy subset and
$\mu' : X \times X \to [0,1]$ is a fuzzy relation on σ' such that $\mu(x,y) \le \sigma(x) \wedge \sigma(y)$ for all $x, y \in X$.

Definition 2.2. A fuzzy graph $G : (\sigma, \mu)$ is complete if $\mu(x,y) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$.

Definition 2.3. Two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$ are isomorphic if there exists a bijection $h : V_1 \to V_2$ such that $\sigma_1(x) = \sigma_2(h(x))$ and $\mu_1(x,y) = \mu_2(h(x), h(y))$ for all $x, y \in V_1$.

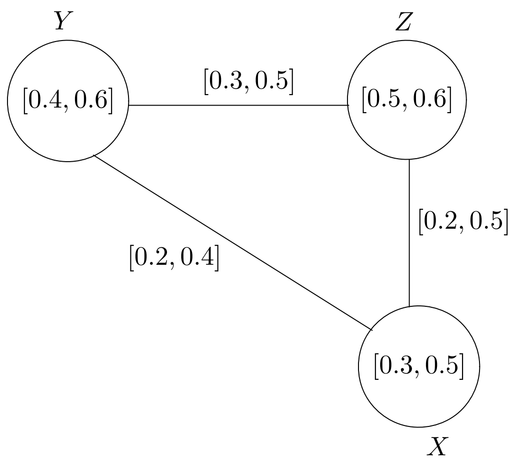
Definition 2.4. By an interval-valued fuzzy graph of a graph $G^* = (V, E)$ we mean a pair $G = (A, B)$, where $A = [\mu_{A^-}, \mu_{A^+}]$ is an interval-valued fuzzy set on V and $B = [\mu_{B^-}, \mu_{B^+}]$ is an interval-valued fuzzy relation on E, such that:

$$\mu_{B^-}(xy) \le \min(\mu_{A^-}(x), \mu_{A^-}(y)), \ \ \mu_{B^+}(xy) \le \min(\mu_{A^+}(x), \mu_{A^+}(y)) \quad for\ all\ xy \in E.$$

We call A the interval-valued fuzzy vertex set of V, B the interval-valued fuzzy edge set of E, respectively. Note that B is a symmetric interval-valued fuzzy relation on A. We use the notation xy for an element of E.

Example 2.5. Consider a graph $G^* = (V, E)$ such that
$V = \{x, y, z\}$, $E = \{xy, yz, zx\}$. Let A be an interval-valued fuzzy set of V and let B be an interval-valued fuzzy set of $E \le V \times V$ defined by

$$\begin{aligned} A &= \Big\langle \Big(\frac{x}{0.3}, \frac{y}{0.4}, \frac{z}{0.5}\Big), \Big(\frac{x}{0.5}, \frac{y}{0.6}, \frac{z}{0.6}\Big) \Big\rangle, \\ B &= \Big\langle \Big(\frac{xy}{0.2}, \frac{yz}{0.3}, \frac{zx}{0.2}\Big), \Big(\frac{xy}{0.4}, \frac{yz}{0.5}, \frac{zx}{0.5}\Big) \Big\rangle. \end{aligned}$$



By routine computations, it is easy to see that $G = (A, B)$ is an interval-valued fuzzy graph of G^*.

Definition 2.6. The complement of an interval-valued fuzzy graph $G : (A, B)$ of a graph $G^* : (V, E)$ is an interval-valued fuzzy graph $\overline{G} : (\overline{A}, \overline{B})$ of $\overline{G^*} : (V, V \times V)$, where $\overline{A} = A = [\mu_{A^-}, \mu_{A^+}]$ and $\overline{B} = [\overline{\mu_{B^-}}, \overline{\mu_{B^+}}]$ is defined by

$$\begin{aligned}\overline{\mu_{B^-}}(x, y) &= \min(\mu_{A^-}(x), \mu_{A^-}(y)) - \mu_{B^-}(xy) \quad \forall\ x, y \in V,\\ \overline{\mu_{B^+}}(x, y) &= \min(\mu_{A^+}(x), \mu_{A^+}(y)) - \mu_{B^+}(xy) \quad \forall\ x, y \in V.\end{aligned}$$

Definition 2.7. An interval-valued fuzzy graph G is said to be a self complementary interval-valued fuzzy graph if $G \cong \overline{G}$.

Definition 2.8. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two interval-valued fuzzy graphs. A homomorphism $f : G_1 \to G_2$ is a mapping $f : V_1 \to V_2$ such that:
(a) $\mu_{A_1^-}(x_1) \leq \mu_{A_2^-}(f(x_1)),\ \ \mu_{A_1^+}(x_1) \leq \mu_{A_2^+}(f(x_1))$
(b) $\mu_{B_1^-}(x_1y_1) \leq \mu_{B_2^-}(f(x_1)f(y_1)),\ \ \mu_{B_1^+}(x_1y_1) \leq \mu_{B_2^+}(f(x_1)f(y_1))$
$for\ all\ x_1 \in V_1,\ x_1y_1 \in E_1$.
A bijective homomorphism with the property
(c) $\mu_{A_1^-}(x_1) = \mu_{A_2^-}(f(x_1)),\ \ \mu_{A_1^+}(x_1) = \mu_{A_2^+}(f(x_1))$
is called a weak isomorphism.
A bijective homomorphism $f : G_1 \to G_2$ such that
(d) $\mu_{B_1^-}(x_1y_1) = \mu_{B_2^-}(f(x_1)f(y_1)),\ \ \mu_{B_1^+}(x_1y_1) = \mu_{B_2^+}(f(x_1)f(y_1))$
$for\ all\ x_1y_1 \in E_1$ is called a co-weak isomorphism.
A bijective mapping $f : G_1 \to G_2$ satisfying (c) and (d) is called an isomorphism.

Definition 2.9. The semi-strong product of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$, where we assume that $V_1 \cap V_2 = \phi$, is defined to be the fuzzy graph $G_1 \bullet G_2 : (\sigma_1 \bullet \sigma_2, \mu_1 \bullet \mu_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$ where

$$\begin{aligned}E = \{&(u, v_1)(u, v_2) : u \in V_1, (v_1, v_2) \in E_2\}\\ &\cup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2)E_2\},\end{aligned}$$

$(\sigma_1 \bullet \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$, for all $(u, v) \in V_1 \times V_2$,
$(\mu_1 \bullet \mu_2)((u, v_1)(u, v_2)) = \sigma_1(u) \wedge \mu_2(v_1, v_2)$ and
$(\mu_1 \bullet \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2)$.

Definition 2.10. The strong product of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$, where we assume that $V_1 \cap V_2 = \phi$, is defined to be the fuzzy graph $G_1 \otimes G_2 : (\sigma_1 \otimes \sigma_2, \mu_1 \otimes \mu_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$ where

$$\begin{aligned} E = \{&(u, v_1)(u, v_2) : u \in V_1, (v_1, v_2) \in E_2\} \\ &\cup \{(u_1, w)(u_2, w) : w \in V_2, (u_1, u_2) \in E_1\} \\ &\cup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}, \end{aligned}$$

$(\sigma_1 \otimes \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$, for all $(u, v) \in V_1 \times V_2$,
$(\mu_1 \otimes \mu_2)((u, v_1)(u, v_2)) = \sigma_1(u) \wedge \mu_2(v_1, v_2)$,
$(\mu_1 \otimes \mu_2)((u_1, w)(u_2, w)) = \sigma_2(w) \wedge \mu_1(u_1, u_2)$ and
$(\mu_1 \otimes \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2)$.

Definition 2.11. The direct product of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$, where we assume that $V_1 \cap V_2 = \phi$, is defined to be the fuzzy graph $G_1 \sqcap G_2 : (\sigma_1 \sqcap \sigma_2, \mu_1 \sqcap \mu_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$ where

$$E = \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\},$$

$(\sigma_1 \sqcap \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$, for all $(u, v) \in V_1 \times V_2$ and
$(\mu_1 \sqcap \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2)$.

3. Complete Interval valued fuzzy graphs

Definition 3.1. An interval-valued fuzzy graph $G = (A, B)$ is called complete if

$$\mu_{B^-}(xy) = \min(\mu_{A^-}(x), \mu_{A^-}(y)), \quad \mu_{B^+}(xy) = \min(\mu_{A^+}(x), \mu_{A^+}(y))$$

for all $xy \in E$.

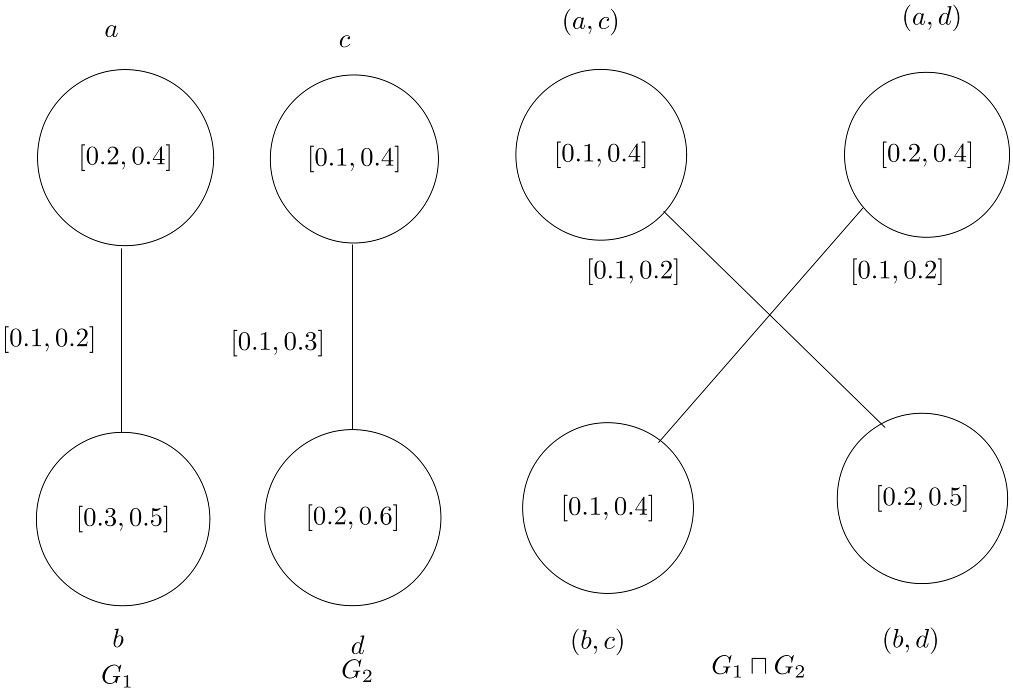
Definition 3.2. The direct product of two interval-valued fuzzy graphs $G_1 = (A_1, B_1)$ with crisp graph $G_1^* = (V_1, E_1)$ and $G_2 = (A_2, B_2)$ with crisp graph $G_2^* = (V_2, E_2)$, where we assume that $V_1 \cap V_2 = \phi$, is defined to be the interval-valued fuzzy graph $G_1 \sqcap G_2 : (\sigma_1 \sqcap \sigma_2, \mu_1 \sqcap \mu_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$ where:

$$E = \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\},$$

$$\begin{cases} (\mu_{A_1^-} \sqcap \mu_{A_2^-})(u, v) = \mu_{A_1^-}(u) \wedge \mu_{A_2^-}(v), & \textit{for all } (u, v) \in V_1 \times V_2 \\ (\mu_{A_1^+} \sqcap \mu_{A_2^+})(u, v) = \mu_{A_1^+}(u) \wedge \mu_{A_2^+}(v) \end{cases}$$

$$\begin{cases} (\mu_{B_1^-} \sqcap \mu_{B_2^-})((u_1, v_1)(u_2, v_2)) = \mu_{B_1^-}(u_1u_2) \wedge \mu_{B_2^-}(v_1v_2), \\ (\mu_{B_1^+} \sqcap \mu_{B_2^+})((u_1, v_1)(u_2, v_2)) = \mu_{B_1^+}(u_1u_2) \wedge \mu_{B_2^+}(v_1v_2). \end{cases}$$

Example 3.3. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be graphs such that $V_1 = \{a, b\}$, $V_2 = \{c, d\}$, $E_1 = \{ab\}$ and $E_2 = \{cd\}$. Consider two interval-value fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, and $G_1 \sqcap G_2$ as follows.



By a routine computation it is easy to see that $G_1 \sqcap G_2$ is an interval-value fuzzy graph.

Theorem 3.4. *If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are complete interval-valued fuzzy graphs, then $G_1 \sqcap G_2$ is complete.*

Proof. If $(u_1, v_1)(u_2, v_2) \in E$, then since G_1 and G_2 are complete we have

$$\begin{aligned}
(\mu_{B_1^-} \sqcap \mu_{B_2^-})((u_1, v_1)(u_2, v_2)) &= \mu_{B_1^-}(u_1u_2) \wedge \mu_{B_2^-}(v_1v_2) \\
&= \mu_{A_1^-}(u_1) \wedge \mu_{A_2^-}(u_2) \wedge \mu_{A_2^-}(v_1) \wedge \mu_{A_2^-}(v_2) \\
&= (\mu_{A_1^-} \sqcap \mu_{A_2^-})(u_1, v_1) \wedge (\mu_{A_1^-} \sqcap \mu_{A_2^-})(u_2, v_2). \\
(\mu_{B_1^+} \sqcap \mu_{B_2^+})((u_1, v_1)(u_2, v_2)) &= \mu_{B_1^+}(u_1u_2) \wedge \mu_{B_2^+}(v_1v_2) \\
&= \mu_{A_1^+}(u_1) \wedge \mu_{A_1^+}(u_2) \wedge \mu_{A_2^+}(v_1) \wedge \mu_{A_2^+}(v_2) \\
&= (\mu_{A_1^+} \sqcap \mu_{A_2^+})(u_1, v_1) \wedge (\mu_{A_1^+} \sqcap \mu_{A_2^+})(u_2, v_2).
\end{aligned}$$

□

Definition 3.5. The semi-strong product of two interval-valued fuzzy graphs $G_1 : (A_1, B_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (A_2, B_2)$ with crisp graph $G_2^* : (V_2, E_2)$, where we assume that $V_1 \cap V_2 = \phi$, is defined to be the interval-valued fuzzy graph $G_1 \bullet G_2 : (A_1 \bullet A_2, B_1 \bullet B_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$ where

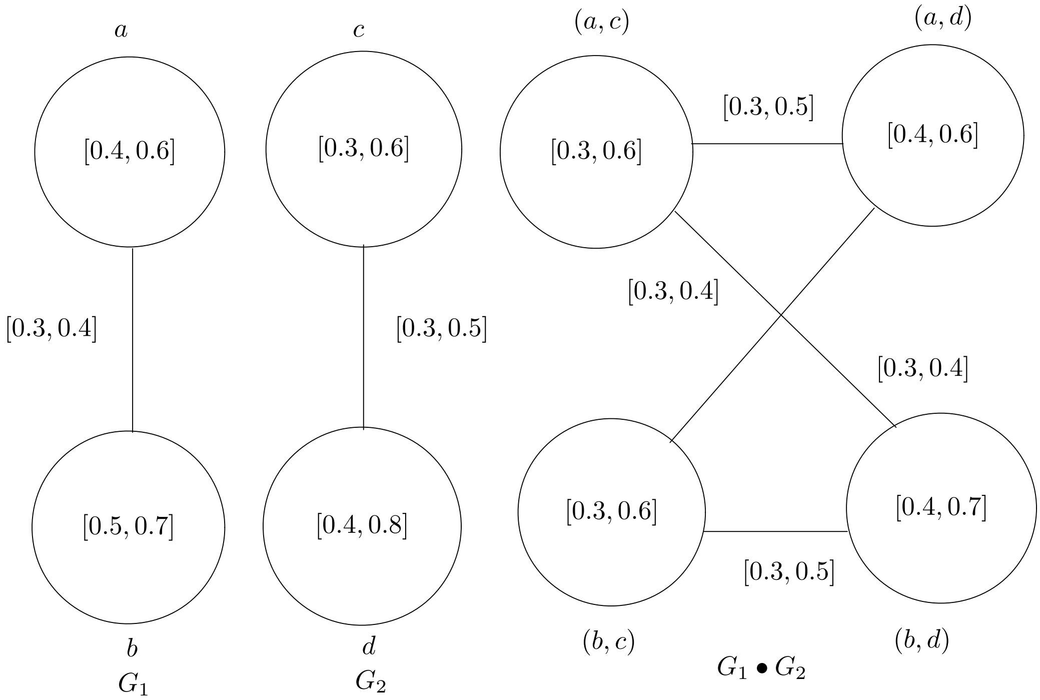
$$E = \{(u, v_1)(u, v_2) : u \in V_1, (v_1, v_2) \in E_2\} \\ \cup \{(u_1, v_1)(u_2, v_2) : (u_1, v_1) \in E_1, (v_1, v_2) \in E_2\},$$

$$(i) \quad \begin{cases} (\mu_{A_1^-} \bullet \mu_{A_2^-})(u, v) = \mu_{A_1^-}(u) \wedge \mu_{A_2^-}(v), & for\ all\ (u, v) \in V_1 \times V_2 \\ (\mu_{A_1^+} \bullet \mu_{A_2^+})(u, v) = \mu_{A_1^+}(u) \wedge \mu_{A_2^+}(v) \end{cases}$$

$$(ii) \quad \begin{cases} (\mu_{B_1^-} \bullet \mu_{B_2^-})((u, v_1), (u, v_2)) = \mu_{A_1^-}(u) \wedge \mu_{B_2^-}(v_1 v_2), & and \\ (\mu_{B_1^+} \bullet \mu_{B_2^+})((u, v_1), (u, v_2)) = \mu_{A_1^+}(u) \wedge \mu_{B_2^+}(v_1 v_2) \end{cases}$$

$$(iii) \quad \begin{cases} (\mu_{B_1^-} \bullet \mu_{B_2^-})((u_1, v_1), (u_2, v_2)) = \mu_{B_1^-}(u_1 u_2) \wedge \mu_{B_2^-}(v_1 v_2), \\ (\mu_{B_1^+} \bullet \mu_{B_2^+})((u_1, v_1), (u_2, v_2)) = \mu_{B_1^+}(u_1 u_2) \wedge \mu_{B_2^+}(v_1 v_2). \end{cases}$$

Example 3.6. In this exampleple we consider two interval-valued fuzzy graphs $G_1 = (A_1, B_1)$, $G_2 = (A_2, B_2)$ and $G_1 \bullet G_2$ as follows.



It is easy to show that $G_1 \bullet G_2$ is an interval-valued fuzzy graph.

Theorem 3.7. *If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are complete interval-valued fuzzy graphs, then $G_1 \bullet G_2$ is complete.*

Proof. If $(u,v_1)(u,v_2) \in E$, then

$$\begin{aligned}(\mu_{B_1^-} \bullet \mu_{B_2^-})((u,v_1)(u,v_2)) &= \mu_{A_1^-}(u) \wedge \mu_{B_2^-}(v_1v_2)\\ &= \mu_{A_1^-}(u) \wedge \mu_{A_2^-}(v_1) \wedge \mu_{A_2^-}(v_2)(since\ G_2\ is\ complete)\\ &= (\mu_{A_1^-} \bullet \mu_{A_2^-})(u,v_1) \wedge (\mu_{A_1^-} \bullet \mu_{A_2^-})(u,v_2).\end{aligned}$$

If $((u_1,v_1)(u_2,v_2)) \in E$, then since G_1 and G_2 are complete

$$\begin{aligned}(\mu_{B_1^-} \bullet \mu_{B_2^-})((u_1,v_1)(u_2,v_2)) &= \mu_{B_1^-}(u_1u_2) \wedge \mu_{B_2^-}(v_1v_2)\\ &= \mu_{A_1^-}(u_1) \wedge \mu_{A_2^-}(u_2) \wedge \mu_{A_2^-}(v_1) \wedge \mu_{A_2^-}(v_2)\\ &= (\mu_{A_1^-} \bullet \mu_{A_2^-})(u_1,v_1) \wedge (\mu_{A_1^-} \bullet \mu_{A_2^-})(u_2,v_2).\end{aligned}$$

Similarly we can show that

$$(\mu_{B_1^+} \bullet \mu_{B_2^+})((u,v_1)(u,v_2)) = (\mu_{A_1^+} \bullet \mu_{A_2^+})(u,v_1) \wedge (\mu_{A_1^+} \bullet \mu_{A_2^+})(u,v_2)$$

if $(u,v_1)(u,v_2) \in E$ and

$$(\mu_{B_1^+} \bullet \mu_{B_2^+})((u_1,v_1)(u_2,v_2)) = (\mu_{A_1^+} \bullet \mu_{A_2^+})(u_1,v_1) \wedge (\mu_{A_1^+} \bullet \mu_{A_2^+})(u_2,v_2)$$

if $(u_1,v_1)(u_2,v_2) \in E$. □

Definition 3.8. The strong product of two interval-valued fuzzy graphs $G_1 : (A_1, B_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (A_2, B_2)$ with crisp graph $G_2^* : (V_2, E_2)$, where we assume that $V_1 \cap V_2 = \phi$, is defined to be the interval-valued fuzzy graph $G_1 \otimes G_2 : (A_1 \otimes A_2, B_1 \otimes B_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$ where

$$\begin{aligned}E = &\{(u,v_1)(u,v_2) : u \in V_1, (v_1,v_2) \in E_2\}\\ &\cup \{(u_1,w)(u_2,w) : w \in V_2, (u_1,u_2) \in E_1\}\\ &\cup \{(u_1,v_1)(u_2,v_2) : (u_1,u_2) \in E_1, (v_1,v_2) \in E_2\},\end{aligned}$$

$$(i)\quad \begin{cases}(\mu_{A_1^-} \otimes \mu_{A_2^-})(u,v) = \mu_{A_1^-}(u) \wedge \mu_{A_2^-}(v), \quad for\ all\ (u,v) \in V_1 \times V_2\\ (\mu_{A_1^+} \otimes \mu_{A_2^+})(u,v) = \mu_{A_1^+}(u) \wedge \mu_{A_2^+}(v)\end{cases}$$

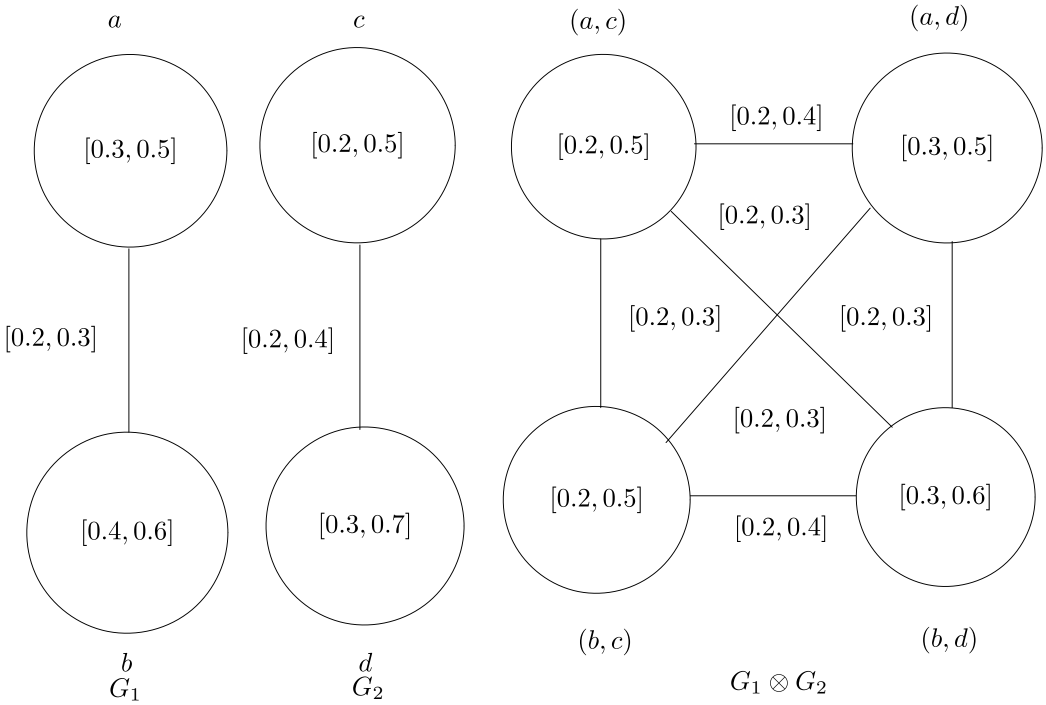
$$(ii)\quad \begin{cases}(\mu_{B_1^-} \otimes \mu_{B_2^-})((u,v_1),(u,v_2)) = \mu_{A_1^-}(u) \wedge \mu_{B_2^-}(v_1v_2),\\ (\mu_{B_1^+} \otimes \mu_{B_2^+})((u,v_1),(u,v_2)) = \mu_{A_1^+}(u) \wedge \mu_{B_2^+}(v_1v_2)\end{cases}$$

$$(iii)\quad \begin{cases}(\mu_{B_1^-} \otimes \mu_{B_2^-})((u_1,w),(u_2,w)) = \mu_{A_2^-}(w) \wedge \mu_{B_1^-}(u_1u_2) and\\ (\mu_{B_1^+} \otimes \mu_{B_2^+})((u_1,w),(u_2,w)) = \mu_{A_2^+}(w) \wedge \mu_{B_1^+}(u_1u_2)\end{cases}$$

$$(iv)\quad \begin{cases}(\mu_{B_1^-} \otimes \mu_{B_2^-})((u_1,v_1),(u_2,v_2)) = \mu_{B_1^-}(u_1u_2) \wedge \mu_{B_2^-}(v_1v_2)\\ (\mu_{B_1^+} \otimes \mu_{B_2^+})((u_1,v_1),(u_2,v_2)) = \mu_{B_1^+}(u_1u_2) \wedge \mu_{B_2^+}(v_1v_2)\end{cases}$$

Example 3.9. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be graphs such that $V_1 = \{a, b\}$, $V_2 = \{c, d\}$, $E_1 = \{ab\}$ and $E_2 = \{cd\}$. Consider two interval-valued fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, where $A_1 = \left\langle (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.5}, \frac{b}{0.6}) \right\rangle$, $B_1 = \left\langle \frac{ab}{0.2}, \frac{ab}{0.3}, \right\rangle$, $A_2 = \left\langle (\frac{c}{0.2}, \frac{d}{0.3}), (\frac{c}{0.5}, \frac{d}{0.7}) \right\rangle$, $B_2 = \left\langle \frac{cd}{0.2}, \frac{cd}{0.4}, \right\rangle$. Then, it is not difficult to verify the following statements:

$(\mu_{A_1^-} \otimes \mu_{A_2^-})(a,c) = 0.2, (\mu_{A_1^-} \otimes \mu_{A_2^-})(a,d) = 0.3, (\mu_{A_1^+} \otimes \mu_{A_2^+})(b,c) = 0.2,$
$(\mu_{A_1^-} \otimes \mu_{A_2^-})(b,d) = 0.3,$
$(\mu_{A_1^+} \otimes \mu_{A_2^+})(a,d) = 0.5, \mu_{A_1^+} \otimes \mu_{A_2^+})(a,d) = 0.5, (\mu_{A_1^+} \otimes \mu_{A_2^+})(b,c) = 0.5,$
$(\mu_{A_1^+} \otimes \mu_{A_2^+})(b,d) = 0.6,$
$(\mu_{B_1^-} \otimes \mu_{B_2^-})((a,c)(a,d)) = 0.2, (\mu_{B_1^+} \otimes \mu_{B_2^+})((a,c)(a,d)) = 0.4,$
$(\mu_{B_1^-} \otimes \mu_{B_2^-})((a,c)(b,c)) = 0.2, (\mu_{B_1^+} \otimes \mu_{B_2^+})((a,c)(b,c)) = 0.3,$
$(\mu_{B_1^-} \otimes \mu_{B_2^-})((a,c)(b,d)) = 0.2, (\mu_{B_1^+} \otimes \mu_{B_2^+})((a,d)(b,d)) = 0.3,$
$(\mu_{B_1^-} \otimes \mu_{B_2^-})((b,c)(b,d)) = 0.2, (\mu_{B_1^+} \otimes \mu_{B_2^+})((b,c)(b,d)) = 0.4,$
$(\mu_{B_1^-} \otimes \mu_{B_2^-})((a,c)(b,d)) = 0.2, (\mu_{B_1^+} \otimes \mu_{B_2^+})((a,c)(b,d)) = 0.3,$
$(\mu_{B_1^-} \otimes \mu_{B_2^-})((a,d)(b,c)) = 0.2, (\mu_{B_1^+} \otimes \mu_{B_2^+})((a,d)(b,c)) = 0.3.$



G_1 G_2 $G_1 \otimes G_2$

By a routine computation, it is easy to see that $G_1 \otimes G_2$ is an interval-value fuzzy graph.

Theorem 3.10. *If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are complete interval-valued fuzzy graphs, then $G_1 \otimes G_2$ is complete.*

Proof. If $(u,v_1)(u,v_2) \in E$, then

$$(\mu_{B_1^-} \otimes \mu_{B_2^-})((u,v_1)(u,v_2)) = \mu_{A_1^-}(u) \wedge \mu_{B_2^-}(v_1v_2)$$
$$= \mu_{A_1^-}(u) \wedge \mu_{A_2^-}(v_1) \wedge \mu_{A_2^-}(v_2) \text{ (since } G_2 \text{ is complete)}$$
$$= (\mu_{A_1^-} \otimes \mu_{A_2^-})(u,v_1) \wedge (\mu_{A_1^-} \otimes \mu_{A_2^-})(u,v_2).$$
$$(\mu_{B_1^+} \otimes \mu_{B_2^+})((u,v_1)(u,v_2)) = \mu_{A_1^+}(u) \wedge \mu_{B_2^+}(v_1v_2)$$
$$= \mu_{A_1^+}(u) \wedge \mu_{A_2^+}(v_1) \wedge \mu_{A_2^+}(v_2) \text{ (since } G_2 \text{ is complete)}$$
$$= (\mu_{A_1^+} \otimes \mu_{A_2^+})(u,v_1) \wedge (\mu_{A_1^+} \otimes \mu_{A_2^-})(u,v_2).$$

If $(u_1,w)(u_2,w) \in E$, then

$$(\mu_{B_1^-} \otimes \mu_{B_2^-})((u_1,w)(u_2,w)) = \mu_{A_2^-}(w) \wedge \mu_{B_1^-}(u_1u_2)$$
$$= \mu_{A_2^-}(w) \wedge \mu_{A_1^-}(u_1) \wedge \mu_{A_1^-}(u_2) \text{ (since } G_1 \text{ is complete)}$$
$$= (\mu_{A_1^-} \otimes \mu_{A_2^-})(u_1,w) \wedge (\mu_{A_1^-} \otimes \mu_{A_2^-})(u_2,w).$$

Similarly we can show that

$$(\mu_{B_1^+} \otimes \mu_{B_2^+})((u_1,w)(u_2,w)) = (\mu_{A_1^+} \otimes \mu_{A_2^+})(u_1,w) \wedge (\mu_{A_1^+} \otimes \mu_{A_2^+})(u_2,w).$$

if $(u_1,v_1)(u_2,v_2) \in E$, then since G_1 and G_2 are complete

$$\begin{aligned}
(\mu_{B_1^-} \otimes \mu_{B_2^-})((u_1,v_1)(u_2,v_2)) &= \mu_{B_1^-}(u_1u_2) \wedge \mu_{B_2^-}(v_1v_2)\\
&= \mu_{A_1^-}(u_1) \wedge \mu_{A_1^-}(u_2) \wedge \mu_{A_2^-}(v_1) \wedge \mu_{A_2^-}(v_2)\\
&= (\mu_{A_1^-} \otimes \mu_{A_2^-})(u_1,v_1) \wedge (\mu_{A_1^-} \otimes \mu_{A_2^-})(u_2,v_2).\\
(\mu_{B_1^+} \otimes \mu_{B_2^+})((u_1,v_1)(u_2,v_2)) &= \mu_{B_1^+}(u_1u_2) \wedge \mu_{B_2^+}(v_1v_2)\\
&= \mu_{A_1^+}(u_1) \wedge \mu_{A_2^+}(u_2) \wedge \mu_{A_2^+}(v_1) \wedge \mu_{A_2^+}(v_2)\\
&= (\mu_{A_1^+} \otimes \mu_{A_2^+})(u_1,v_1) \wedge (\mu_{A_1^+} \otimes \mu_{A_2^+})(u_2,v_2).
\end{aligned}$$

Hence, $G_1 \otimes G_2$ is complete. □

Theorem 3.11. *If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are interval-valued fuzzy graphs such that $G_1 \sqcap G_2$ is complete, then at least G_1 or G_2 must be complete.*

Proof. Suppose that G_1 and G_2 are not complete. Then there exists at least one $(u_1,v_1) \in E_1$ and $(u_2,v_2) \in E_2$ such that

$$\begin{cases} \mu_{B_1^-}(u_1v_1) < \mu_{A_1^-}(u_1) \wedge \mu_{A_1^-}(v_1) \\ \mu_{B_1^+}(u_1v_1) < \mu_{A_1^+}(u_1) \wedge \mu_{A_1^+}(v_1) \end{cases} \quad \textit{and} \quad \begin{cases} \mu_{B_2^-}(u_2v_2) < \mu_{A_2^-}(u_2) \wedge \mu_{A_2^-}(v_2) \\ \mu_{B_2^+}(u_2v_2) < \mu_{A_2^+}(u_2) \wedge \mu_{A_2^+}(v_2). \end{cases}$$

Now

$$\begin{aligned}
(\mu_{B_1^-} \sqcap \mu_{B_2^-})((u_1,v_1)(u_2,v_2)) &= \mu_{B_1^-}(u_1u_2) \wedge \mu_{B_2^-}(v_1v_2)\\
&< \mu_{A_1^-}(u_1) \wedge \mu_{A_1^-}(u_2) \wedge \mu_{A_2^-}(v_1) \wedge \mu_{A_2^-}(v_2)\\
&\quad \text{(since } G_1 \text{ and } G_2 \text{ are not complete).}
\end{aligned}$$

Similarly

$$(\mu_{B_1^+} \sqcap \mu_{B_2^+})((u_1,v_1)(u_2,v_2)) < \mu_{A_1^+}(u_1) \wedge \mu_{A_2^+}(u_2) \wedge \mu_{A_2^+}(v_1) \wedge \mu_{A_2^+}(v_2).$$

But

$$(\mu_{A_1^-} \sqcap \mu_{A_2^-})((u_1, v_1) = \mu_{A_1^-}(u_1) \wedge \mu_{A_2^-}(v_1)$$

and $(\mu_{A_1^-} \sqcap \mu_{A_2^-})((u_2, v_2) = \mu_{A_1^-}(u_2) \wedge \mu_{A_2^-}(u_2)$. Thus

$$\begin{aligned}
&(\mu_{A_1^-} \sqcap \mu_{A_2^-})(u_1, v_1) \wedge (\mu_{A_1^-} \sqcap \mu_{A_2^-})(u_2, v_2)\\
&= \mu_{A_1^-}(u_1) \wedge \mu_{A_1^-}(u_2) \wedge \mu_{A_2^-}(v_1) \wedge \mu_{A_2^-}(v_2)\\
&> (\mu_{B_1^-} \sqcap \mu_{B_2^-})((u_1, v_1)(u_2, v_2)).
\end{aligned}$$

Similarly we can show that

$$(\mu_{A_1^+} \sqcap \mu_{A_2^+})(u_1, v_1) \wedge (\mu_{A_1^+} \sqcap \mu_{A_2^+})(u_2, v_2) > (\mu_{B_1^+} \sqcap \mu_{B_2^+})((u_1, v_1)(u_2, v_2)).$$

Hence, $G_1 \sqcap G_2$ is not complete, a contradiction. □

The next result can be proved in a similar manner as in the preceding theorem.

Theorem 3.12. *If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are interval-valued fuzzy graphs such that $G_1 \bullet G_2$ or $G_1 \otimes G_2$ is complete, then at least G_1 or G_2 must be complete.*

4. Conclusions

Graph theory is an extremely useful tool in solving the combinatorial problems in different areas including geometry, algebra, number theory, topology, operations research, optimization, and computer science. In this paper, we provide three new operation on interval-valued fuzzy graphs; namely direct product, semi strong product and strong product. We give sufficient conditions for each one of them to be complete and we show that if any of these products is complete, then at least one factor is a complete interval-valued fuzzy graph.

Acknowledgements. The authors are thankful to the referees for their valuable comments and suggestions.

References

[1] M. Akram and W. A. Dudek, Interval-valued fuzzy graphs, Comput. Math. Appl. 61 (2011) 289–299.
[2] M. Akram and B. Davvaz, Strong intuitionistic fuzzy graphs, Filomat 26(1) (2012) 177–196.
[3] M. Akram, F. Feng, S. Sarwar and Y. B. Jun, Certain types of vague graphs, Scientic Bulletin. Series A- Applied Mathematics and Physics No. 2, (2013) pp. 15.
[4] M. Akram, Bipolar fuzzy graphs, Inform. Sci. 181 (2011) 5548–5564.
[5] M. Akram, Interval-valued fuzzy line graphs, Neural Computing & Applications 21 (2012) 145–150.
[6] M. Akram and W. A. Dudek, Intuitionistic fuzzy hypergraphs with applications, Inform. Sci. 218 (2013) 182–193.
[7] T. AL-Hawary, Complete fuzzy graphs, International J. Math. Combin. 4 (2011) 26–34.
[8] K. Bohutani, On automorphism of fuzzy graphs, Pattern Recognition Latter 9 (1988) 159–162.
[9] J. N. Mordeson and C. S. Peng, Operations on fuzzy graphs, Inform. Sci. 79 (1994) 159–170.
[10] A. Nagoor Gani and J. Malarvizhi, Isomorphism on fuzzy graphs, Int. J. comp. and Math. Sci, Vol. 2, 4(2008), 190-196.
[11] A. Nagoor Gani and J. Malarvizhi, Isomorphism properties on strong fuzzy graphs, Int. J. Comput. Math. Sci. 2(4) (2008) 200–206.
[12] A. Nagoor Gani and K. Kadha, On regular fuzzy graphs, J. Physical sciences 12 (2008) 33–40.

[13] A. Rosendfeld, Fuzzy graphs, in L. A. Zadeh, K. S. Fu, K. Tanaka and M. Shirmura (Eds), Fuzzy and their Applications to cognitive and Decision processes, Academic press, New York (1975) 77–95.
[14] M. S. Sunitha and A. Vijaya Kumar, Complement of a fuzzy graph, Indian J. Pure Appl. Math. 33(9) (2002) 1451–1464.
[15] A. A. Talebi and H. Rashmanlou, Isomorphismon on interval-valued fuzzy graphs, Ann. Fuzzy Math. Inform. 6(1) (2013) 47–58.
[16] A. A. Talebi, H. Rashmanlou and N. Mehdipoor, Isomorphismon on vague graphs, Ann. Fuzzy Math. Inform. (in press).
[17] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[18] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning. I. Information Sci. 8 (1975) 199–249.

Hossein Rashmanlou (hrashmanlou@yahoo.com)
Department of Mathematics, University of mazandaran, Babolsar, Iran

Young Bae Jun (skywine@gmail.com)
Department of Mathematics Education, Gyeongsang National University, Jinju 660-701, Korea