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Fuzzy cone metric spaces and fixed point theorems of contractive mappings

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Abstract. In this paper, an idea of fuzzy cone metric space is introduced. Some basic definitions viz. convergence of sequence, Cauchy sequence, closedness, completeness etc are given. Some fixed point theorems of contractive mappings on fuzzy cone metric spaces are established.

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1. Introduction

Since the introduction of fuzzy set theory by L. A. Zadeh [16] in 1965, fuzzy logic has become an important area of research in various branches of mathematics such as metric and topological spaces, automata theory, optimization, control theory etc. Fuzzy set theory also found applications for modeling uncertainty and vagueness in various fields of science and engineering. George and Veeramoni [6], Kramosil and Michalek [8] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have important applications in quantum particle physics particularly in connections with both string and E-infinity theory.

On the other hand, a number of generalizations of metric spaces have been done and one such generalization is generalized metric space or D-metric space initiated by Dhage [4] in 1992. Many other authors viz. Sedghi et al. [14] made a significant contribution in fixed point theory of D*-metric space which is another generalization of D-metric space. Recently Sedghi et al. [13] introduced the concept of $\mathbb{M}$-fuzzy metric space which is a generalization of fuzzy metric space due to George & Veeramoni [6]. The author [2] of this paper modify the definition of $\mathbb{M}$-fuzzy metric space and achieve two decomposition theorems. The idea of cone metric space is relatively new which is introduced by H. Long-Guang et al. [9] and it is a generalization of classical metric space. In such space, authors have considered a real Banach space as the range set of the cone metric. On cone metric space, many papers have appeared by different authors (for references please see [1, 3, 10, 12, 15]).

In this paper, idea of fuzzy cone metric space is introduced and some basic definitions are given. Here the range of fuzzy cone metric is considered as $E^*(I)$ where E is a given real Banach space and $E^*(I)$ denotes the set of all non-negative fuzzy real numbers defined on E. It is seen that fuzzy cone metric space is a generalization of Kaleva & Seikkala [7] type fuzzy metric space (when L = min and U = max). Some fixed point theorems have been established in such spaces.

The organization of the paper is as follows:
Section 1 comprises some preliminary results which are used in this paper.
Definition of fuzzy cone metric space and some basic properties are discussed in Section 2. In Section 3, some fixed point theorems for contractive mappings are established.

2. Preliminaries

A fuzzy number is a mapping $x : R \rightarrow [0 , 1]$ over the set R of all reals. A fuzzy number x is convex if $x(t) \geq min\,(x(s) , x(r))$ where $s \leq t \leq r$. The α-level set of a fuzzy real number η denoted by $[\eta]_\alpha$ and defined as $[\eta]_\alpha = \{t \in R : \eta(t) \geq \alpha\}$. If there exists a $t_0 \in R$ such that $x(t_0) = 1$, then x is called normal. For $0 < \alpha \leq 1$, α-level set of an upper semi continuous convex normal fuzzy number (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha , b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible. When $a_\alpha = -\infty$, for instance, then $[a_\alpha , b_\alpha]$ means the interval $(-\infty , b_\alpha]$. Similar is the case when $b_\alpha = +\infty$. A fuzzy number x is called non-negative if $x(t) = 0, \forall t < 0$. Kaleva (Felbin) denoted the set of all convex, normal, upper semicontinuous fuzzy real numbers by E (R(I)) and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by $G(R^*(I))$.

A partial ordering " $\preceq$ " in E is defined by $\eta \preceq \delta$ if and only if $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq b^2_\alpha$ for all $\alpha \in (0 , 1]$ where $[\eta]_\alpha = [a^1_\alpha , b^1_\alpha]$ and $[\delta]_\alpha = [a^2_\alpha , b^2_\alpha]$. The strict inequality in E is defined by $\eta \prec \delta$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$ for each $\alpha \in (0 , 1]$.

According to Mizumoto and Tanaka [11] , the arithmetic operations $\oplus$, $\ominus$, $\odot$ on $E \times E$ are defined by

$$(x \oplus y)(t) = Sup_{s\in R} min\,\{x(s) , y(t-s)\}, \quad t \in R$$
$$(x \ominus y)(t) = Sup_{s\in R} min\,\{x(s) , y(s-t)\}, \quad t \in R$$
$$(x \odot y)(t) = Sup_{s\in R, s\neq 0} min\,\{x(s) , y(\tfrac{t}{s})\}, \quad t \in R$$

Proposition 2.1 ([11]). *Let $\eta , \delta \in E(R(I))$ and $[\eta]_\alpha = [a^1_\alpha , b^1_\alpha]$, $[\delta]_\alpha = [a^2_\alpha , b^2_\alpha]$, $\alpha \in (0 , 1]$. Then*

$$[\eta \bigoplus \delta]_\alpha = [a^1_\alpha + a^2_\alpha , b^1_\alpha + b^2_\alpha]$$
$$[\eta \ominus \delta]_\alpha = [a^1_\alpha - b^2_\alpha , b^1_\alpha - a^2_\alpha]$$
$$[\eta \odot \delta]_\alpha = [a^1_\alpha a^2_\alpha , b^1_\alpha b^2_\alpha]$$

Definition 2.2 ([7]). A sequence $\{\eta_n\}$ in E is said to be convergent and converges to η denoted by $\lim_{n\rightarrow\infty} \eta_n = \eta$ if $\lim_{n\rightarrow\infty} a^n_\alpha = a_\alpha$ and $\lim_{n\rightarrow\infty} b^n_\alpha = b_\alpha$ where $[\eta_n]_\alpha = [a^n_\alpha, b^n_\alpha]$ and $[\eta]_\alpha = [a_\alpha, b_\alpha]$ $\forall \alpha \in (0, 1]$.

Note 2.3 ([7]). *If $\eta, \delta \in G(R^*(I))$ then $\eta \oplus \delta \in G(R^*(I))$.*

Note 2.4 ([7])**.** *For any scalar t, the fuzzy real number $t\eta$ is defined as $t\eta(s) = 0$ if t=0 otherwise $t\eta(s) = \eta(\frac{s}{t})$.*

Following is the definition of fuzzy metric space introduced by Kaleva et al [6].

Definition 2.5 ([7])**.** Let X be a nonempty set, d be a mapping from $X \times X$ to G and let the mappings $L, U : [0\ ,\ 1] \times [0\ ,\ 1] \to [0\ ,\ 1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0\ ,\ 0) = 0$ and $U(1\ ,\ 1) = 1$. Denote $[d(x,y)]_\alpha = [\lambda_\alpha(x,y)\ ,\ \rho_\alpha(x,y)]$ for $x \in X,\ 0 < \alpha \leq 1$. The quadruple (X, d, L, U) is called a fuzzy metric space and d is a fuzzy metric if

(i) $d(x,y) = \bar{0}$ if and only if $x = y$;

(ii)$d(x,y) = d(y,x),\ \forall x, y \in X$;

(iii) for all $x, y, z \in X$,

(a) $d(x,y)(s+t) \geq L(d(x,z)(s)\ ,\ d(z,y)(t))$ whenever $s \leq \lambda_1(x,z),\ t \leq \lambda_1(z,y)$ and $s+t \leq \lambda_1(x,y)$.

(b) $d(x,y)(s+t) \leq U(d(x,z)(s)\ ,\ d(z,y)(t))$ whenever $s \geq \lambda_1(x,z),\ t \geq \lambda_1(z,y)$ and $s+t \geq \lambda_1(x,y)$.

Remark 2.6 ([7])**.** Kaleva et al. proved that, if $L = \bigwedge$(Min) and $U = \bigvee$(Max) then the triangle inequality (iii) in the Definition 1. 1 is equivalent to

$$d(x\ ,\ y) \preceq d(x\ ,\ z) \bigoplus d(z\ ,\ y).$$

Further λ_α and ρ_α are crisp metrices on X for each $\alpha \in (0\ ,\ 1]$.

Definition of fuzzy norm on a linear space as introduced by C. Felbin is given below:

Definition 2.7 ([5])**.** Let X be a vector space over R. Let $||\ ||\ : X \to R^*(I)$ and let the mappings $L, U : [0\ ,\ 1] \times [0\ ,\ 1] \to [0\ ,\ 1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0\ ,\ 0) = 0$ and $U(1\ ,\ 1) = 1$. Write $[||x||]_\alpha = [||x||^1_\alpha\ ,\ ||x||^2_\alpha]$ for $x \in X,\ 0 < \alpha \leq 1$ and suppose for all $x \in X,\ x \neq \underline{0}$, there exists $\alpha_0 \in (0\ ,\ 1]$ independent of x such that for all $\alpha \leq \alpha_0$,

(A) $||x||^2_\alpha < \infty$

(B) $\inf||x||^1_\alpha > 0$.

The quadruple $(X\ ,\ ||\ ||,\ L\ ,\ U)$ is called a fuzzy normed linear space and $||\ ||$ is a fuzzy norm if

(i) $||x|| = \bar{0}$ if and only if $x = \underline{0}$;

(ii)$||rx|| = |r|||x||,\ x \in X,\ r \in R$;

(iii) for all $x, y \in X$,

(a) whenever $s \leq ||x||^1_1,\ t \leq ||y||^1_1$ and $s+t \leq ||x+y||^1_1$, $||x+y||(s+t) \geq L(||x||(s)\ ,\ ||y||(t))$,

(b) whenever $s \geq ||x||^1_1,\ t \geq ||y||^1_1$ and $s+t \geq ||x+y||^1_1$, $||x+y||(s+t) \leq U(||x||(s)\ ,\ ||y||(t))$

Remark 2.8 ([5])**.** Felbin proved that, if $L = \bigwedge$(Min) and $U = \bigvee$(Max) then the triangle inequality (iii) in the Definition 1. 1 is equivalent to

$$||x+y|| \preceq ||x|| \bigoplus ||y||.$$

Further $||\ ||^i_\alpha;\ i = 1,2$ are crisp norms on X for each $\alpha \in (0\ ,\ 1]$.

Definition 2.9 ([9])**.** Let E be a real Banach space and P be a subset of E. P is called a cone if

(i) P is closed, nonempty and $P \neq \{\underline{0}\}$;

(ii) $a, b \in R,\ a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;

(iii) $x \in P$ and $-x \in P\ \Rightarrow x = \underline{0}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ iff $y - x \in P$. On the other hand $x < y$ indicates that $x \leq y$ but $x \neq y$ while $x << y$ will stand for $y - x \in$IntP where IntP denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, with $0 \leq x \leq y$ implies $||x|| \leq K||y||$. The least positive number satisfying above is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq \ldots\ldots.. \leq x_n \leq \ldots. \leq y$ for some $y \in E$, then there is $x \in E$ such that $||x_n - x|| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if every decreasing sequence which is bounded below is convergent. It is clear that a regular cone is a normal cone.

In the following we always assume that E is a real Banach space, P is a cone in E with IntP $\neq \phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.10 ([9])**.** Let X be a nonempty set. Suppose the mapping $d:\ X \times X \to E$ satisfies

(d1) $\underline{0} \leq d(x, y)\ \forall x, y \in X$ and $d(x, y) = \underline{0}$ iff $x = y$;

(d2) $d(x, y) = d(y, x)\ \ \forall x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)\ \ \forall x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

3. Fuzzy cone metric spaces

In this section an idea of fuzzy cone metric space is introduced.

Definition 3.1. Let $(E, ||\ ||)$ be a fuzzy real Banach space where $||\ || : E \to R^*(I)$. Denote the range of $||\ ||$ by $E^*(I)$. Thus $E^*(I) \subset R^*(I)$.

Definition 3.2. A member $\eta \in R^*(I)$ is said to be an interior point if $\exists r > 0$ such that $S(\eta, r) = \{\delta \in R^*(I)\ \ : \eta \ominus \delta \prec \bar{r}\} \subset R^*(I)$. Set of all interior points of $R^*(I)$ is called interior of $R^*(I)$.

Definition 3.3. A subset of F of $E^*(I)$ is said to be fuzzy closed if for any sequence $\{\eta_n\}$ such that $\lim\limits_{n\to\infty} \eta_n = \eta$ implies $\eta \in F$.

Definition 3.4. A subset P of $E^*(I)$ is called a fuzzy cone if

(i) P is fuzzy closed, nonempty and $P \neq \{\bar{0}\}$;

(ii) $a, b \in R,\ a, b \geq 0,\ \eta, \delta \in P \Rightarrow a\eta \oplus b\delta \in P$;

Given a fuzzy cone $P \subset E^*(I)$, define a partial ordering $\leq$ with respect to P by $\eta \leq \delta$ iff $\delta \ominus \eta \in P$ and $\eta < \delta$ indicates that $\eta \leq \delta$ but $\eta \neq \delta$ while $\eta << \delta$ will stand for $\delta \ominus \eta \in$IntP where IntP denotes the interior of P.

The fuzzy cone P is called normal if there is a number $K > 0$ such that for all $\eta, \delta \in E^*(I)$, with $\bar{0} \leq \eta \leq \delta$ implies $\eta \preceq K\delta$. The least positive number satisfying above is called the normal constant of P. The fuzzy cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{\eta_n\}$ is a sequence such that $\eta_1 \leq \eta_2 \leq .... \leq \eta_n \leq .... \leq \eta$ for some $\eta \in E^*(I)$, then there is $\delta \in E^*(I)$ such that $\eta_n \to \delta$ as $n \to \infty$. Equivalently, the fuzzy cone P is regular if every decreasing sequence which is bounded below is convergent. It is clear that a regular fuzzy cone is a normal fuzzy cone.

In the following we always assume that E is a fuzzy real Banach space, P is a fuzzy cone in E with IntP $\neq \phi$ and $\leq$ is a partial ordering with respect to P.

Definition 3.5. Let X be a nonempty set. Suppose the mapping $d: \ X \times X \to E^*(I)$ satisfies

(Fd1) $\bar{0} \leq d(x,y)\ \forall x,y \in X$ and $d(x,y) = \bar{0}$ iff $x = y$;

(Fd2) $d(x,y) = d(y,x) \ \ \forall x,y \in X$;

(Fd3) $d(x,y) \leq d(x,z) \oplus d(z,y) \ \ \forall x,y,z \in X$.

Then d is called a fuzzy cone metric and (X,d) is called a fuzzy cone metric space.

Note 3.6. *Fuzzy cone metric space is a generalized fuzzy metric space. For, choose* $E = R$ *and* $P = \{\eta \in E^*(I): \ \eta \succeq \bar{0}\}$ *and partial ordering* $\leq$ *as* $\preceq$ *then* (X,d) *is a Kaleva & Seikkala type fuzzy metric space when L= min and U=Max.*

Example 3.7. Let $(E, ||\ ||')$ be a Banach space. Define $||\ ||: \ E \to R^*(I)$ by

$$||x||(t) = \begin{cases} 1 & \text{if } t\ > ||x||' \\ 0 & \text{if } t \leq \ ||x||' \end{cases}$$

Then $[||x||]_\alpha = [||x||'\ ,\ ||x||'] \ \ \forall \alpha \in (0,1]$. It is easy to verify that,

(i) $||x|| = \bar{0}$ iff $x = \underline{0}$

(ii) $||rx|| = |r|||x||$ (iii) $||x+y|| \preceq ||x|| \oplus ||y||$.

Thus $(E, ||\ ||)$ is a fuzzy normed linear space (when L=min and U=max). Let $\{x_n\}$ be a Cauchy sequence in $(E, ||\ ||)$ So, $\lim_{m,n\to\infty} ||x_n - x_m|| = \bar{0}$.
$\Rightarrow \lim_{m,n\to\infty} ||x_n - x_m|| = 0 \Rightarrow \{x_n\}$ be a Cauchy sequence in $(E, ||\ ||')$. Since $(E, ||\ ||')$ is complete, $\exists x \in E$ such that $\lim_{m,n\to\infty} ||x_n - x||' = 0$. i. e. $\lim_{n\to\infty} ||x_n - x|| = \bar{0}$. Thus $(E, ||\ ||)$ is a real fuzzy Banach space. Define $P = \{\eta \in E^*(I)\ : \eta \succeq \bar{0}\}$.
(i) P is fuzzy closd.
For, consider a sequence $\{\delta_n\}$ in P such that $\lim_{n\to\infty} \delta_n \to \delta$.
i. e. $\lim_{n\to} \delta^1_{n,\alpha} = \delta^1_\alpha$ and
$\lim_{n\to} \delta^2_{n,\alpha} = \delta^2_\alpha$ where $[\delta_n]_\alpha = [\delta^1_{n,\alpha}\ ,\ \delta^2_{n,\alpha}]$ and $[\delta]_\alpha = [\delta^1_\alpha\ ,\ \delta^2_\alpha] \ \ \forall \alpha \in (0,1]$.
Now $\delta_n \succeq \bar{0} \ \ \forall n$.
So, $\delta^1_{n,\alpha} \geq 0$ and $\delta^2_{n,\alpha} \geq 0 \ \ \forall \alpha \in (0,1]$.
$\Rightarrow \lim_{n\to\infty} \delta^1_{n,\alpha} \geq 0$ and $\lim_{n\to\infty} \delta^2_{n,\alpha} \geq 0 \ \ \forall \alpha \in (0,1]$
$\Rightarrow \delta^1_\alpha \geq 0$ and $\delta^2_\alpha \geq 0 \ \ \forall \alpha \in (0,1]$
$\Rightarrow \delta \succeq \bar{0}$.
So $\delta \in P$. Hence P is fuzzy closed.
(ii) It is obvious that, $a,b \in R,\ a,b \geq 0\ \eta,\delta \in P \Rightarrow a\eta \oplus b\delta \in P$.
Thus P is a fuzzy cone in E.
Now choose the ordering $\leq$ as $\preceq$ and define $d: E \times E \to E^*(I)$ by $d(x,y) = ||x-y||$. Then it is easy to verify that d satisfies the conditions (Fd1) to (Fd3). Hence $(E\ ,\ d)$ is a fuzzy cone metric space.

Definition 3.8. Let (X,d) be a fuzzy cone metric space. Let$\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\bar{0} << ||c||$ there is a positive integer N such that for all $n > N,\ d(x_n, x) << ||c||$, then $\{x_n\}$ is said to be convergent and converges to x and x is called the limit of $\{x_n\}$. We denote it by $\lim\limits_{n\to\infty} x_n = x$.

Lemma 3.9. *Let (X, d) be a fuzzy cone metric space and P be a normal fuzzy cone with normal constant K. Let$\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x iff $d(x_n, x) \to \bar{0}$ as $n \to \infty$.*

Proof. First we suppose that $\{x_n\}$ converges to x. For every real $\epsilon > 0$, choose $c \in E$ with $\bar{0} << ||c||$ and $K||c|| \prec \bar{\epsilon}$.
Then $\exists$ a natural number N, such that $\forall n > N,\ d(x_n, x) << ||c||$.
So that when $n > N,\ \ d(x_n, x) \preceq K||c|| \prec \bar{\epsilon}$ (since P is normal).
i. e. $d^1_\alpha(x_n, x) < \epsilon$ and $d^2_\alpha(x_n, x) < \epsilon\ \forall n \in N,\ \forall\alpha \in (0, 1]$.
i. e. $\lim\limits_{n\to\infty} d^1_\alpha(x_n, x) = 0$ and $\lim\limits_{n\to\infty} d^2_\alpha(x_n, x) = 0\ \forall\alpha \in (0, 1]$.
i. e. $d(x_n, x) \to \bar{0}$ as $n \to \infty$.
Conversely, suppose that $d(x_n, x) \to \bar{0}$ as $n \to \infty$.
For, $c \in E$ with $\bar{0} << ||c||$, there is $\delta > 0$ such that $||x|| \prec \bar{\delta}$.
This implies that $||c|| \ominus ||x|| \in$ IntP.
For this δ there is a positive integer N such that $\forall n > N,\ d(x_n, x) \prec \bar{\delta}$.
Let $d(x_n, x) = ||y_n||$. So $||y_n|| \prec \bar{\delta}\ \ \forall n > N$.
i. e. $||c|| \ominus ||y_n|| \in$ IntP $\ \ \forall n > N$
$\Rightarrow ||y_n|| << ||c||\ \ \forall n > N$
$\Rightarrow d(x_n, x) << ||c||\ \ \forall n > N$
$\Rightarrow x_n \to x$ as $n \to \infty$. □

Lemma 3.10. *Let (X, d) be a fuzzy cone metric space and P be a normal fuzzy cone with normal constant K. Let$\{x_n\}$ be a sequence in X. If $\{x_n\}$ is convergent then its limit is unique.*

Proof. If possible suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$. Thus for any $c \in E$ with $\bar{0} << ||c||$, there exists a natural number N such that $\forall n > N,\ d(x_n, x) << ||c||$ and $d(y_n, y) << ||c||$.
We have $d(x, y) \leq d(x, x_n) \oplus d(x_n, y)\ \ \leq 2||c||$.
Hence $d(x, y) \preceq 2K||c||$.
Since c is arbitrary, we have $d(x, y) = \bar{0}$. i. e. $x = y$. □

Definition 3.11. Let (X, d) be a fuzzy cone metric space and $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $\bar{0} << ||c||$, there exists a natural number N such that $\forall m, n > N,\ d(x_n, x_m) << ||c||$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 3.12. Let (X, d) be a fuzzy cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete fuzzy cone metric space.

Lemma 3.13. *Let (X, d) be a fuzzy cone metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is convergent then it is a Cauchy sequence.*

Proof. Let $\{x_n\}$ converges to x. So for any $c \in E$ with $\bar{0} << ||c||$ there exists a natural number N such that $\forall m, n > N,\ d(x_n, x) << ||\frac{c}{2}||$ and $d(x_m, x) << ||\frac{c}{2}||$.
Hence $d(x_n, x_m) \leq d(x_n, x) \oplus d(x, x_m) << ||c||\ \ \forall m, n > N$.
Thus $\{x_n\}$ is a Cauchy sequence. □

Lemma 3.14. *Let (X, d) be a fuzzy cone metric space, P be a normal fuzzy cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence iff $d(x_n, x_m) \to \bar{0}$ as $m, n \to \infty$.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. For $\epsilon > 0$ choose $c \in E$ with $\bar{0} << ||c||$ and $K||c|| \prec \bar{\epsilon}$. Then there is a natural number N such that $\forall m, n > N,\ d(x_n, x_m) << ||c||$.
So that when $m, n > N,\ d(x_n, x_m) \preceq K||c|| \prec \bar{\epsilon}$ (since P is normal).
Since $\epsilon > 0$ is arbitrary, it follows that $d(x_n, x_m) \to \bar{0}$ as $m, n \to \infty$.
Conversely suppose that $d(x_n, x_m) \to \bar{0}$ as $m, n \to \infty$.
For $c \in E$ with $\bar{0} << ||c||$, there is $\delta > 0$ such that $||x|| \prec \bar{\delta}$.
i. e. $||c|| \ominus (||c|| \ominus ||x)|| \prec \bar{\delta}$ implies $||c|| \ominus ||x|| \in$ IntP. For this $\delta > 0$, there exists a natural number N such that $\forall m, n > N,\ d(x_n, x_m) \prec \bar{\delta}$.
i. e. $||z_{m,n}|| \prec \bar{\delta}$ if we write $d(x_n, x_m) = ||z_{m,n}||$ where $z_{m,n} \in E$.
$\Rightarrow ||c|| \ominus ||z_{m,n}|| \in IntP \quad \forall m, n > N$
$\Rightarrow ||z_{m,n}|| << ||c|| \quad \forall m, n > N$
$\Rightarrow d(x_m, x_n) << ||c|| \quad \forall m, n > N$
$\Rightarrow \{x_n\}$ is a Cauchy sequence. □

Lemma 3.15. *Let (X, d) be a fuzzy cone metric space , P be a normal fuzzy cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \to x,\ y_n \to y$ as $n \to \infty$. Then $d(x_n, y_n) \to Kd(x, y)$ as $n \to \infty$.*

Proof. For every $\epsilon > 0$, choose $c \in E$ with $\bar{0} << ||c||$ and $||c|| \prec (\frac{\bar{\epsilon}}{2k})$.
Since $x_n \to x,\ y_n \to y$ there is a natural number N such that $\forall n > N$,
$d(x_n, x) << ||c||$ and $d(y_n, y) << ||c||$.
We have $d(x_n, y_n) \leq d(x_n, x) \oplus d(x, y) \oplus d(y, y_n)$
i. e. $d(x_n, y_n) \leq d(x, y) \oplus 2||c|| \quad \forall n > N$.
Thus $d(x_n, y_n) \preceq K(d(x, y) \oplus 2||c||) \quad \forall n > N$.
$\Rightarrow d^1_\alpha(x_n, y_n) \leq Kd^1_\alpha(x, y) + 2K||c||^1_\alpha$ and $d^2_\alpha(x_n, y_n) \leq Kd^2_\alpha(x, y) + 2K||c||^2_\alpha \quad \forall n > N$.
$\Rightarrow d^1_\alpha(x_n, y_n) < Kd^1_\alpha(x, y) + \epsilon$ and $d^2_\alpha(x_n, y_n) < Kd^2_\alpha(x, y) + \epsilon \quad \forall n > N$
$\Rightarrow \lim_{n\to\infty} d^1_\alpha(x_n, y_n) = Kd^1_\alpha(x, y)$ and $\lim_{n\to\infty} d^2_\alpha(x_n, y_n) = Kd^2_\alpha(x, y)$
$\Rightarrow \lim_{n\to\infty} d(x_n, y_n) = Kd(x, y)$. □

Definition 3.16. Let (X, d) be a fuzzy cone metric space. If for any sequence $\{x_n\}$ in X, there exists a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is convergent in X, then X is called a sequentially compact fuzzy cone metric space.

4. Fixed point theorems in fuzzy cone metric spaces

In this Section some fixed point theorems of contractive mappings are established in fuzzy cone metric spaces.

Theorem 4.1. *Let (X, d) be a complete fuzzy cone metric space, P be a normal fuzzy cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition*
$d(Tx, Ty) \leq kd(x, y) \quad \forall x, y \in X$ where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X. For any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Choose $x_0 \in X$.
Set $x_1 = Tx_0,\ x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0, ...$
we have,
$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq kd(x_n, x_{n-1}) \leq k^2 d(x_{n-1}, x_{n-2}) \leq .... \leq k^n d(x_1, x_0)$.
So for $n > m$, we have
$d(x_n, x_m) \leq d(x_n, x_{n-1}) \oplus d(x_{n-1}, x_{n-2}) \oplus .... \oplus (x_{m+1}, x_m)$.
i. e. $d(x_n, x_m) \leq (k^{n-1} + k^{n-2} + ... + k^m)d(x_1, x_0)$.
i. e. $d(x_n, x_m) \leq \frac{k^m}{1-k} d(x_1, x_0)$.
This implies that $d(x_n, x_m) \preceq \frac{k^m}{1-k} Kd(x_1, x_0)$ (since P is normal)
$\Rightarrow \lim_{m,n\to\infty} d(x_n, x_m) = \bar{0}$.
$\Rightarrow \{x_n\}$ is a Cauchy sequence in X.
By completeness of X, there is $x^* \in X$ such that $x_n \to x^*$.
Now,
$d(Tx^*, x^*) \leq d(Tx_n, Tx^*) \oplus d(Tx_n, x^*) \leq kd(x_n, x^*) \oplus d(x_{n+1}, x^*)$
$\Rightarrow d(Tx^*, x^*) \preceq K\{kd(x_n, x^*) \oplus d(x_{n+1}, x^*)\}$
$\Rightarrow d(Tx^*, x^*) = \bar{0}$ (since $d(x_n, x^*) \to \bar{0},\ d(x_{n+1}, x^*) \to \bar{0}$ as $n \to \infty$)
$\Rightarrow Tx^* = x^*$
$\Rightarrow x^*$ is a fixed point of T.
Now if y^* is another fixed point of T then
$d(x^*, y^*) = d(Tx^*, Ty^*) \leq kd(x^*, y^*)$
$\Rightarrow d(x^*, y^*) \preceq Kkd(x^*, y^*)$
$\Rightarrow d(x^*, y^*) = \bar{0}$.
$\Rightarrow x^* = y^*$. □

Corollary 4.2. *Let (X, d) be a complete fuzzy cone metric space, P be a normal fuzzy cone with normal constant K. For $c \in E$ with $\bar{0} << ||c||$ and $x_0 \in X$, set $B(x_0, c) = \{x \in X : d(x_0, x) \leq ||c||\}$.*
Suppose the mapping $T : X \to X$ satisfies the contractive condition
$d(Tx, Ty) \leq kd(x, y)\ \ \forall x, y \in B(x_0, c)$ where $k \in [0, 1)$ is a constant and $d(Tx_0, x_0) \leq (1 - k)||c||$.
Then T has a unique fixed point in $B(x_0, c)$.

Proof. We only need to prove that $B(x_0, c)$ is complete and $Tx \in B(x_0, c)\ \ \forall x \in B(x_0, c)$.
Let $\{x_n\}$ be a Cauchy sequence in $B(x_0, c)$. Then $\{x_n\}$ is also a Cauchy sequence in X. By completeness of X, there is x in X such that $x_n \to x$.
We have, $d(x_0, x) \leq d(x_n, x_0) \oplus d(x_n, x) \leq d(x_n, x) \oplus ||c||$.
Since $x_n \to x$, thus $d(x_n, x) \to \bar{0}$ and hence $d(x_0, x) \leq ||c||$.
i. e. $x \in B(x_0, c)$. Hence $B(x_0, c)$ is complete.
Now, for every $x \in B(x_0, c)$,
$d(x_0, Tx) \leq d(Tx_0, x_0) \oplus d(Tx_0, Tx) \leq (1 - k)||c|| \oplus kd(x_0, x) \leq (1 - k)||c|| \oplus k||c||$
$\Rightarrow d(x_0, Tx) \leq ||c||$.
$\Rightarrow Tx \in B(x_0, c)$.
This completes the proof. □

Corollary 4.3. *Let (X, d) be a complete fuzzy cone metric space, P be a normal fuzzy cone with normal constant K. Suppose a mapping $T : X \to X$ satisfies for* *some positive integer n,* $d(T^n x, T^n y) \leq kd(x, y) \quad \forall x, y \in X$ *where* $k \in [0, 1)$ *is a constant. Then T has a unique fixed point in X.*

Proof. From Theorem 4. 1, it follows that T^n has a unique fixed point x^*.
But $T^n(Tx^*) = T(T^n x^*) = Tx^*$. So Tx^* is a fixed point of T^n.
Hence $Tx^* = x^*$ and thus x^* is a fixed point of T.
Since the fixed point of T is also a fixed point of T^n, so the fixed point of T is unique. □

Theorem 4.4. *Let* (X, d) *be a sequentially compact fuzzy cone metric space and P be a regular fuzzy cone. Suppose the mapping* $T : X \to X$ *satisfies the the contractive condition* $d(Tx, Ty) < d(x, y) \quad \forall x, y (x \neq y) \in X$
Then T has a unique fixed point.

Proof. Choose $x_0 \in X$.
Set $x_1 = Tx_0,\ x_2 = Tx_1 = T^2 x_0, ..., x_{n+1} = Tx_n = T^{n+1} x_0, ...$
If for some $n, x_{n+1} = x_n$ then x_n is a fixed point of T and the proof is complete.
So we assume that for all $n, x_{n+1} \neq x_n$.
Set $||y_n|| = d(x_{n+1}, x_n)$ where $y_n \in E$.
Then $||y_{n+1}|| = d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) < d(x_n, x_{n+1}) = ||y_n||$
Thus $||y_n||$ is a decreasing sequence bounded below by $\bar{0}$.
Since P is regular, there is $y \in E$ such that $||y|| \in E^*(I)$ and $||y_n|| \to ||y||$ as $n \to \infty$.
Since X is sequentially compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $x^* \in X$ such that $\{x_{n_i}\} \to x^*$.
We have, $d(Tx_{n_i}, Tx^*) < d(x_{n_i}, x^*),\ i = 1, 2, ...$
So, $d(Tx_{n_i}, Tx^*) \preceq Kd(x_{n_i}, x^*),\ i = 1, 2, ...$ where K is the normal constant of E.
i. e. $d^1_\alpha(Tx_{n_i}, Tx^*) \leq Kd^1_\alpha(x_{n_i}, x^*)$ and
$d^2_\alpha(Tx_{n_i}, Tx^*) \leq Kd^2_\alpha(x_{n_i}, x^*)\ \forall \alpha \in (0, 1]$. (4. 4. 1)
Since $\{x_{n_i}\} \to x^*$, thus $d(x_{n_i}, x^*) \to \bar{0}$ as $n \to \infty$.
i. e. $d^1_\alpha(x_{n_i}, x^*) \to 0$ and $d^2_\alpha(x_{n_i}, x^*) \to 0\ \forall \alpha \in (0, 1]$ as $n \to \infty$.
Fom (4. 4. 1) we have, $d^1_\alpha(Tx_{n_i}, Tx^*) \to 0$ and $d^2_\alpha(Tx_{n_i}, Tx^*) \to 0\ \forall \alpha \in (0, 1]$ as $n \to \infty$.
i. e. $d(Tx_{n_i}, Tx^*) \to \bar{0}$ as $i \to \infty$.
Hence $Tx_{n_i} \to Tx^*$ as $i \to \infty$.
Similarly $T^2 x_{n_i} \to T^2 x^*$ as $i \to \infty$.
By using Lemma 3. 15, we have
$d(Tx_{n_i}, x_{n_i}) \to Kd(Tx^*, x^*)$ and $d(T^2 x_{n_i}, Tx_{n_i}) \to Kd(T^2 x^*, Tx^*)$ as $i \to \infty$.
Also we have, $d(Tx_{n_i}, x_{n_i}) = ||y_{n_i}|| \to K||y|| = Kd(Tx^*, x^*)$ as $i \to \infty$.
Now we shall show that $Tx^* = x^*$.
If $Tx^* \neq x^*$, then $||y|| \neq \bar{0}$ and then we get
$K||y|| = Kd(Tx^*, x^*) > Kd(T^2 x^*, Tx^*) = \lim_{i \to \infty} d(T^2 x_{n_i}, Tx_{n_i}) = \lim_{i \to \infty} ||y_{n_i}|| = K||y||$.
Which is a contradiction and hence $Tx^* = x^*$.
So x^* is a fixed point of T. The uniqueness of x^* follows easily. □

Theorem 4.5. *Let (X, d) be a complete fuzzy cone metric space and P be a normal fuzzy cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the the contractive condition $d(Tx, Ty) < k(d(Tx, x) \oplus d(Ty, y))\ \forall x, y \in X$, where $k \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X. Also for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.*

Proof. Choose $x_0 \in X$.
Set $x_1 = Tx_0,\ x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0, ...$.
We have,
$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq k(d(Tx_n, x_n) \oplus d(Tx_{n-1}, x_{n-1})) = k(d(x_{n+1}, x_n) \oplus d(x_n, x_{n-1}))$.
So, $d(x_{n+1}, x_n) \leq \frac{k}{1-k} d(x_n, x_{n-1}) = hd(x_n, x_{n-1})$ where $h = \frac{k}{1-k}$.
For $n > m$,
$d(x_n, x_m) \leq d(x_n, x_{n-1}) \oplus d(x_{n-1}, x_{n-2}) \oplus ...... \oplus d(x_{m+1}, x_m)$.
i. e. $d(x_n, x_m) \leq (h^{n-1} + h^{n-2} + ... + h^m)d(x_1, x_0) = \frac{h^m}{1-h} d(x_1, x_0)$.
We have $d(x_n, x_m) \preceq K \frac{h^m}{1-h} d(x_1, x_0)$ (since P is normal).
This implies that $d(x_n, x_m) \to \bar{0}$ as $m, n \to \infty$.
Hence $\{x_n\}$ is a Cauchy sequence. By completeness of X, there is $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.
Now $d(Tx^*, x^*) \leq d(Tx_n, Tx^*) \oplus d(Tx_n, x^*) \leq k(d(Tx_n, x_n) \oplus d(Tx^*, x^*)) \oplus d(x_{n+1}, x^*)$.
$\Rightarrow d(Tx^*, x^*) \leq \frac{1}{1-k}(kd(Tx_n, x_n) \oplus d(x_{n+1}, x^*))$.
We have,
$d(Tx^*, x^*) \preceq K \frac{1}{1-k}(kd(x_{n+1}, x_n) \oplus d(x_{n+1}, x^*)) \to \bar{0}$ as $n \to \infty$.
Hence $d(Tx^*, x^*) = \bar{0}$. i. e. $Tx^* = x^*$. So x^* is a fixed point of T.
Now if y^* is another fixed point of T, then,
$d(x^*, y^*) = d(Tx^*, Ty^*) \leq k(d(Tx^*, x^*) \oplus d(Ty^*, y^*)) = \bar{0}$.
i. e. $d(x^*, y^*) = \bar{0}$.
i. e. $x^* = y^*$.
Thus fixed point of T is unique. □

Note 4.6. *The Lemma 3. 15 is not valid in classical sense when $k \neq 1$.
For, in Example 3. 7, we see that over the same linear space (real), $(E\ ,\ ||\ ||')$ is a Banach space and $(E\ ,\ ||\ ||)$ is a fuzzy Banach space (where L=min and U=max).*

*Consider this real Banach space $(E\ ,\ ||\ ||')$ and fuzzy Banach space $(E\ ,\ ||\ ||)$. According to Lemma 3. 15, if (X , d) is a fuzzy cone metric space, P is a normal cone with normal constant k and if $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \to x,\ y_n \to y$ as $n \to \infty$ then $d(x_n\ ,\ y_n) \to kd(x\ ,\ y)$ as $n \to \infty$.
This Lemma (Lemma 5) is already established in [9] in classical sense.
The statement of the Lemma is as follows:
Let $(X\ ,\ d)$ be a cone metric space, P be a normal cone with normal constant k. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \to x,\ y_n \to y$ as $n \to \infty$. Then $d(x_n\ ,\ y_n) \to d(x\ ,\ y)$ as $n \to \infty$.*

5. Conclusion

In this paper, an idea of fuzzy cone metric space is introduced which is a generalization of fuzzy metric space. In fuzzy cone metric space, range of fuzzy metric is considered as ordering fuzzy real numbers defined on a real fuzzy Banach space. It is seen that Kaleva et al. type (max, min) fuzzy metric space is a particular case of fuzzy cone metric space. I think that there is a large scope of developing more results of fuzzy functional analysis in this context.

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