Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 6, No. 3, (November 2013), pp. 643–656
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

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On multisets and multigroups

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Received 8 February, 2013; Revised 13 March, 2013; Accepted 12 April, 2013

Abstract. In this paper we have introduced the notion of multigroups (in short mgroups) and study its important properties. We have shown that the intersection of two mgroups is again a mgroup but their union may not be a mgroup. It has been also shown that the homomorphic image and pre image of a mgroup is again a mgroup. Next we have defined the notions of submgroups, abelian mgroups, normal mgroups and factor mgroups and study some of their properties. In this paper we have also studied some basic results regarding msets, like functional image and pre image of a mset under a mapping, decomposition theorems of msets etc.

2010 AMS Classification: 54A40, 03E72, 20N25, 06D72

Keywords: Multisets, Multigroups, Submultigroups, Abelian multigroups, Normal multigroups, Factor multigroups.

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1. Introduction

Theory of Multisets is an important generalization of classical set theory which has emerged by violating a basic property of classical sets that an element can belong to a set only once. The term multiset (mset in short) as Knuth [21] notes, was first suggested by N. G. de Bruijn [5] in a private communication to him. Owing to its aptness, it has replaced a variety of terms viz. list, heap, bunch, bag, sample, weighted set, occurrence set and fireset (finitely repeated element set) used in different contexts but conveying synonimity with mset. It is a "set" where an element can occur more than once. Multisets are very useful structures arising in many areas of mathematics and computer science such as database queries. Many authors like Yager [30], Miyamoto [25], Hickman [19], Blizard [3], Girish and John [16, 17] etc. have studied the properties of multisets. Some authors have also generalized the notion of multisets in cases of fuzzy sets and soft sets to form fuzzy multisets [23], soft multisets [24, 2] etc. One of the important applications of fuzzy multisets is in

information retrieval on the web, since an information item may appear more than once with possibly different degrees of relevance to a query. Except this, multisets and fuzzy multisets have been applied in multiple type of scenario's such as in statistics, multicriteria decision making, knowledge representation in data based systems, biological systems and membrane computing [30, 26, 20, 27, 28, 22]. More works on multisets can be found in [4, 18, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Again the theory of groups is one of the most important algebraic structures in modern mathematics. Several authors have introduced the notion of group in fuzzy sets [29], intuitionistic L-fuzzy sets [15], soft sets [1] etc. Therefore the study of group structure in multisets is very natural. Here we have introduced a notion of group in multiset setting. The organization of the rest of this paper is as follows:
In section 2, some preliminary definitions and results regarding multisets have been introduced. Also we have studied the image and preimage of a mset under a mapping. A decomposition of a multiset into a family of ordinary set and vice-versa has also been discussed here. In section 3, the notion of multigroup has been introduced. Several properties regarding multigroups are studied and notions like normal multigroup, factor multigroup etc are also defined in this section.

2. Preliminaries

In this section definition of a multiset (mset in short) is given and studied its properties. Unless otherwise stated, X will be assumed to be an initial universal set and $MS(X)$ denote the set of all mset over X.

Definition 2.1 ([17])**.** An mset M drawn from the set X is represented by a Count function C_M defined as $C_M : X \to N$ where N represents the set of non negative integers.

Here $C_M(x)$ is the number of occurrence of the element x in the mset M. The presentation of the mset M drawn from $X = \{x_1, x_2, \dots, x_n\}$ will be as $M = \{x_1/m_1, x_2/m_2, \dots, x_n/m_n\}$ where m_i is the number of occurances of the element $x_i,\ i = 1, 2, \dots, n$ in the mset M.

Also here for any positive integer w, $[X]^w$ is the set of all msets whose elements are in X such that no element in the mset occurs more than w times and $[X]^\infty$ is the set of all msets whose elements are in X such that there is no limit on the number of occurances of an element in a mset. As in [17], $[X]^w$ and $[X]^\infty$ will be referred to as mset spaces.

Definition 2.2 ([17])**.** Let M_1 and M_2 be two msets dwawn from a set X. Then M_1 is said to be *submset* of M_2 if $C_{M_1}(x) \leq C_{M_2}(x),\ \forall\ x \in X$. This relation is denoted by $M_1 \subseteq M_2$. M_1 is said to be *equal* to M_2 if $C_{M_1}(x) = C_{M_2}(x),\ \forall\ x \in X$. It is denoted by $M_1 = M_2$.

Definition 2.3 ([17])**.** Let $\{M_i; i \in I\}$ be a nonempty family of msets drawn from the set X. Then

(a) Their *Intersection*, denoted by $\bigcap_{i\in I} M_i$ where
$C_{\bigcap_{i\in I} M_i}(x) = \bigwedge_{i\in I} C_{M_i}(x),\ \forall\ x \in X$.

(b) Their *Union*, denoted by $\bigcup_{i\in I} M_i$ where
$C_{\bigcup_{i\in I} M_i}(x) = \bigvee_{i\in I} C_{M_i}(x),\ \forall\ x \in X$.

(c) The *Complement* of any mset M_i in $[X]^w$, denoted by M_i^c where $C_{M_i^c}(x) = w - C_{M_i}(x),\ \forall\ x \in X$.

Definition 2.4. Let M be a mset over a set X. Then we define a set $M_n = \{x \in X :\ C_M(x) \geq n\}$, where n is a natural number, is called n-level set of M.

Example 2.5. Let $X = \{a,\ b,\ c\}$ and $M = \{a,\ a,\ b,\ b,\ b\}$. Then $M \in MS(X)$ and the level sets of M are $M_1 = \{a,\ b\},\ M_2 = \{a,\ b\},\ M_3 = \{b\}$ and $M_n = \{\phi\},\ n \geq 4$.

Proposition 2.6. *Let $A,\ B$ be msets over X and $m, n \in \mathbb{N}$.*

(i) *If $A \subseteq B$, then $A_n \subseteq B_n$:*
(ii) *If $m \leq n$, then $A_m \supseteq A_n$;*
(iii) $(A \cap B)_n = A_n \cap B_n$;
(iv) $(A \cup B)_n = A_n \cup B_n$;
(v) *$A = B$ iff $A_n = B_n,\ \forall\ n \in \mathbb{N}$.*

Proof. The proofs are straightforward. $\square$

Definition 2.7. Let $P \subseteq X$. Then for each $n \in \mathbb{N}$, we define a mset nP over X, where $C_{nP}(x) = n,\ \forall\ x \in X$.

Example 2.8. Let $X = \{a,\ b,\ c\}$ and $P = \{a,\ b\}$. Then $1P = \{a,\ b\},\ 2P = \{a,\ a,\ b,\ b\},\ 3P = \{a,\ a,\ a,\ b,\ b,\ b\}, \ldots, nP = \{a,\ a,\ \ldots\ n \text{ times},\ b,\ b,\ \ldots\ n \text{ times}\}$ and $nP \in MS(X),\ \forall\ n \in \mathbb{N}$.

Theorem 2.9. (First Decomposition Theorem) *If $A_n,\ n \in \mathbb{N}$ be the level sets of a mset A over X, then $C_A(x) = \sum_{n\in\mathbb{N}} \chi_{A_n}(x)$, where χ_{A_n} is the characteristic function of A_n.*

Proof. Let $x \in X$ and $x \in A_p,\ p = 1, 2, \ldots, p$ but $x \notin A_{p+n},\ n \in \mathbb{N}$.
Then $C_A(x) = p$ and
$\sum_{n\in\mathbb{N}} \chi_{A_n}(x) = \sum_{n=1}^{p} \chi_{A_n}(x) + \sum_{n\in\mathbb{N}} \chi_{A_{p+n}}(x) = [1+1+\ldots p \text{ times}] + [0+\ldots] = p$
Therefore, $C_A(x) = \sum_{n\in\mathbb{N}} \chi_{A_n}(x)$. $\square$

Theorem 2.10. (Second Decomposition Theorem) *If $A_n,\ n \in \mathbb{N}$ be the level sets of a mset A over X, then $A = \bigcup_{n\in\mathbb{N}} nA_n$, where $\bigcup$ denotes the standrad mset union.*

Proof. Let $x \in X$ and $C_A(x) = p$.
Then $x \in A_n,\ n = 1, 2, \ldots, p$ and $x \notin A_n,\ \forall\ n \geq (p+1)$.

$$\begin{aligned} \text{Now } C_{\bigcup_{n\in\mathbb{N}} nA_n}(x) &= \vee_{n\in\mathbb{N}}\{[nA_n](x)\} \\ &= [1A_1](x) \vee [2A_2](x) \vee \ldots \vee [pA_p](x) \vee [(p+1)A_{p+1}](x) \vee \ldots \\ &= \vee\{1, 2, \ldots, p, 0, 0, \ldots\} \\ &= p = C_A(x),\ \forall\ x \in X. \end{aligned}$$

Therefore $A = \bigcup_{n\in\mathbb{N}} nA_n$. $\square$

Definition 2.11. Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. Then

(i) the *image* of a mset $M \in [X]^w$ under the mapping f is denoted by $f(M)$ or $f[M]$, where

$$C_{f(M)}(y) = \begin{cases} \bigvee_{f(x)=y} C_M(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

(ii) the *inverse image* of a mset $N \in [Y]^w$ under the mapping f is denoted by $f^{-1}(N)$ or $f^{-1}[N]$ where $C_{f^{-1}(N)}(x) = C_N[f(x)]$.

Proposition 2.12. *Let X, Y and Z be three nonempty sets and $f : X \to Y$, $g : Y \to Z$ be two mappings. If $M_i \in [X]^w$, $N_i \in [Y]^w$, $i \in I$ then*

(i) $M_1 \subseteq M_2 \Rightarrow f(M_1) \subseteq f(M_2)$.
(ii) $f[\bigcup_{i\in I} M_i] = \bigcup_{i\in I} f[M_i]$.
(iii) $N_1 \subseteq N_2 \Rightarrow f^{-1}(N_1) \subseteq f^{-1}(N_2)$.
(iv) $f^{-1}[\bigcup_{i\in I} M_i] = \bigcup_{i\in I} f^{-1}[M_i]$.
(v) $f^{-1}[\bigcap_{i\in I} M_i] = \bigcap_{i\in I} f^{-1}[M_i]$.
(vi) $f(M_i) \subseteq N_j \Rightarrow M_i \subseteq f^{-1}[N_j]$.
(vii) $g[f(M_i)] = [gf](M_i)$ *and* $f^{-1}[g^{-1}(N_j)] = [gf]^{-1}(N_j)$.

Proof. We give proof of (vii) and others are straightforward. Let $M_i \in [X]^w$ and $z \in Z$. Then

$$\begin{aligned} C_{g[f(M_i)]}(z) &= \vee\{[C_{f(M_i)]}(y);\ y \in Y,\ g(y) = z\} \\ &= \vee\{\vee\{C_{M_i}(x);\ x \in X,\ f(x) = y,\};\ y \in Y,\ g(y) = z\} \\ &= \vee\{C_{M_i}(x);\ x \in X,\ [gf](x) = z\} \\ &= C_{[gf](M_i)}(z). \end{aligned}$$

Therefore $g[f(M_i)] = [gf](M_i)$. Again let $N_j \in [Z]^w$ and $x \in X$. Then

$$C_{[(gf)^{-1}](N_j)}(x) = C_{N_j}[gf](x) = C_{N_j}g[f(x)] = C_{g^{-1}(N_j)}f(x) = C_{f^{-1}[g^{-1}(N_j)]}(x).$$

Therefore $f^{-1}[g^{-1}(N_j)] = [gf]^{-1}(N_j)$. □

Proposition 2.13. *Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. If $M \in [X]^w$ then*

(i) $M \widetilde{\subset} f^{-1}[f(M)]$.
(ii) $f^{-1}[f(M)] = M$, *if f is injective.*

Proof. (i) Since

$$C_{f^{-1}[f(M)]}(x) = C_{f(M)}[f(x)] = \vee\{C_M(x');\ x' \in X,\ f(x') = f(x)\} \geq C_M(x),$$

it follows that $M \widetilde{\subset} f^{-1}[f(M)]$.
(ii) If f is injective, then

$$C_{f^{-1}[f(M)]}(x) = C_{f(M)}[f(x)] = \vee\{C_M(x');\ x' \in X,\ f(x') = f(x)\} = C_M(x),$$

and hence $f^{-1}[f(M)] = M$, if f is injective. □

Proposition 2.14. *Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. If $N \in [Y]^w$ then*

(i) $f[f^{-1}(N)] \widetilde{\subset} N$.
(ii) $f[f^{-1}(N)] = N$, *if f is surjective.*

Proof. The proof is similar to that of Proposition 2.13. □

Definition 2.15. An mset containing only one element x of n times is called a singleton mset and it is denoted by n_x.

3. Multigroups

Throughout this section, let X be a group and e be the identity element of X. Also throughtout the rest of the paper we assume that msets are taken from $[X]^w$ and $MG(X)$ denote the set of all mgroups over a group X.

Definition 3.1. Let $A,\ B \in [X]^w$. Then we define $A \circ B$ and A^{-1} as follows:
$C_{A\circ B}(x) = \bigvee\{C_A(y) \wedge C_B(z);\ y,\ z \in X \textit{ and } yz = x\}$ and $C_{A^{-1}}(x) = C_A(x^{-1})$.

Proposition 3.2. *If $A,\ B,\ C,\ A_i \in [X]^w$, then the following assertions hold:*

(i) $C_{A\circ B}(x) = \bigvee_{y\in X}[C_A(y) \wedge C_B(y^{-1}x)] = \bigvee_{y\in X}[C_A(xy^{-1}) \wedge C_B(y)],\ \forall\ x \in X$;
(ii) $[A^{-1}]^{-1} = A$;
(iii) $A \subseteq B \Rightarrow A^{-1} \subseteq B^{-1}$;
(iv) $[\bigcup_{i\in I} A_i]^{-1} = \bigcup_{i\in I}[A_i^{-1}]$;
(v) $[\bigcap_{i\in I} A_i]^{-1} = \bigcap_{i\in I}[A_i^{-1}]$;
(vi) $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$;
(vii) $(A \circ B) \circ C = A \circ (B \circ C)$.

Proof. (i) Since X is a group, it follows that for each $x,\ y \in X$ there exists a unique $z \in X$ such that $yz = x$. Thus

$$\begin{aligned} C_{A\circ B}(x) &= \vee\{C_A(y) \wedge C_B(z);\ y,\ z \in X \text{ and } yz = x\} \\ &= \vee_{y\in X}[C_A(y) \wedge C_B(y^{-1}x)] = \vee_{y\in X}[C_A(xy^{-1}) \wedge C_B(y)],\ \forall\ x \in X. \end{aligned}$$

(ii) From Definition 3.1, we have $C_{[A^{-1}]^{-1}}(x) = C_{[A]^{-1}}(x^{-1}) = C_A(x),\ \forall\ x \in X$. Hence $[A^{-1}]^{-1} = A$.

(iii) Since $A \subseteq B$, it follows that $C_A(x) \le C_B(x),\ \forall\ x \in X$.
So, $C_{A^{-1}}(x) = C_A(x^{-1}) \le C_B(x^{-1}) = C_{B^{-1}}(x),\ \forall\ x \in X$.
Hence $A^{-1} \subseteq B^{-1}$.

(iv) Since

$$\begin{aligned} C_{[\bigcup_{i\in I} A_i]^{-1}}(x) &= C_{[\bigcup_{i\in I} A_i]}(x^{-1}) = \vee_{i\in I}[C_{A_i}(x^{-1})] \\ &= \vee_{i\in I}[C_{A_i^{-1}}(x)] = C_{[\bigcup_{i\in I} A_i^{-1}]}(x),\ \forall\ x \in X. \end{aligned}$$

Hence, $[\bigcup_{i\in I} A_i]^{-1} = \bigcup_{i\in I}[A_i^{-1}]$.

(v) Similar to that of item (iv).

(vi) Now,

$$\begin{aligned} & C_{(A\circ B)^{-1}}(x) = C_{(A\circ B)}(x^{-1}) \\ & = \vee_{y\in X}[C_A(y) \wedge C_B(z) :\ \forall\ y, z \in X \text{ such that } yz = x^{-1}] \\ & = \vee_{y\in X}[C_{A^{-1}}(y^{-1}) \wedge C_{B^{-1}}(z^{-1}) :\ \forall\ y^{-1}, z^{-1} \in X \text{ such that } (yz)^{-1} = x] \\ & = \vee_{z^{-1}\in X}[C_{B^{-1}}(z^{-1}) \wedge C_{A^{-1}}(y^{-1}) :\ \forall\ z^{-1}, y^{-1} \in X \text{ such that } z^{-1}y^{-1} = x] \\ & = C_{B^{-1}\circ A^{-1}}(x),\ \forall\ x \in X. \end{aligned}$$

Therefore, $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$.

(vii) Can be proved in a similar way. □

Remark 3.3. If $e \in B_*$, then $A \subseteq (A \circ B)$.

Definition 3.4. Let X be a group. A multiset G over X is said to be a *multigroup* over X if the Count function G or C_G satisfies the following two conditions.

(i) $C_G(xy) \geq [C_G(x) \wedge C_G(y)],\ \forall\ x, y \in X$;
(ii) $C_G(x^{-1}) \geq C_G(x),\ \forall\ x \in X$.

The set of all multigroups over X is denoted by $MG(X)$.

Example 3.5. Let $X = \{e,\ a,\ b,\ c\}$ be Klein's 4-group and

$$G = \{e,\ e,\ e, a,\ a,\ b,\ b,\ b,\ c,\ c\}$$

be a multiset over X. Now
$C_G(ea) = C_G(a) = 2 \geq [C_G(e) \wedge C_G(a)],\ C_G(eb) = C_G(b) = 3 \geq [C_G(e) \wedge C_G(b)]$,
$C_G(ec) = C_G(c) = 2 \geq [C_G(e) \wedge C_G(c)],\ C_G(ab) = C_G(c) = 2 \geq [C_G(a) \wedge C_G(b)]$,
$C_G(bc) = C_G(a) = 2 \geq [C_G(b) \wedge C_G(c)],\ C_G(ca) = C_G(b) = 3 \geq [C_G(c) \wedge C_G(a)]$,
$C_G(a^2) = C_G(e) = 3 \geq [C_G(a) \wedge C_G(a)],\ C_G(b^2) = C_G(e) = 3 \geq [C_G(b) \wedge C_G(b)]$,
$C_G(c^2) = C_G(e) = 3 \geq [C_G(c) \wedge C_G(c)],\ C_G(e^2) = C_G(e) = 3 \geq [C_G(e) \wedge C_G(e)]$ and
$C_G(a^{-1}) = C_G(a) = 2,\ C_G(b^{-1}) = C_G(b) = 3,\ C_G(c^{-1}) = C_G(c) = 2,\ C_G(e^{-1}) = C_G(e) = 3$.
Therefore G is a multigroup over X.

Proposition 3.6. *Let $A \in MG(X)$. Then*

(i) $C_A(e) \geq C_A(x),\ \forall\ x \in X$;
(ii) $C_A(x^n) \geq C_A(x),\ \forall\ x \in X$;
(iii) $C_A(x^{-1}) = C_A(x),\ \forall\ x \in X$;
(iv) $A = A^{-1}$.

Proof. Let $x,\ y \in G$. (i) $C_A(e) = C_A(xx^{-1}) \geq [C_A(x) \wedge C_A(x^{-1})] = [C_A(x) \wedge C_A(x)] = C_A(x),\ \forall\ x \in X$;

(ii) $C_A(x^n) \geq C_A(x^{n-1}) \wedge C_A(x) \geq C_A(x^{n-2}) \wedge C_A(x) \wedge C_A(x) \geq [C_A(x) \wedge C_A(x) \wedge \ldots \wedge C_A(x)] = C_A(x)$.

(iii) Since $C_A(x^{-1}) \geq C_A(x) = C_A([x^{-1}]^{-1}) \geq C_A(x^{-1})$.
Hence $C_A(x^{-1}) = C_A(x)$.

(iv) Since $C_{A^{-1}}(x) = C_A(x^{-1}) = C_A(x)$. Hence $A = A^{-1}$. □

Proposition 3.7. *Let A be a mset. Then $A \in MG(X)$ iff $C_A(xy^{-1}) \geq [C_A(x) \wedge C_A(y)],\ \forall\ x, y \in X$.*

Proof. Let $A \in MG(X)$. Then

$$C_A(xy^{-1}) \geq [C_A(x) \wedge C_A(y^{-1})] = [C_A(x) \wedge C_A(y)],\ \forall\ x,\ y \in X.$$

Therefore the given condition is satisfied.

Conversely let the given condition be satisfied. Now, $C_A(e) = C_A(xx^{-1}) \geq C_A(x) \wedge C_A(x) = C_A(x),\ \forall\ x \in X$. ......(1)
Again $C_A(x^{-1}) = C_A(ex^{-1}) \geq C_A(e) \wedge C_A(x) = C_A(x),\ \forall\ x \in X$ [From (1)]......(2)
Also $C_A(xy) = C_A[x(y^{-1})^{-1}] \geq [C_A(x) \wedge C_A(y^{-1})$
$\geq [C_A(x) \wedge C_A(y)]$ [From (2)]..............(3)
Therefore, from (2) and (3) we have $A \in MG(X)$. □

Definition 3.8. Let $A \in MG(X)$. Then define $A_n = \{x \in X;\ C_A(x) \geq n,\ n \in \mathbb{N}\}$.

Proposition 3.9. *Let $A \in MG(X)$. Then $A_n,\ n \in \mathbb{N}$ are subgroups of X.*

Proof. Let $x,\ y \in A_n$. Then $C_A(x) \geq n$ and $C_A(y) \geq n$. Since $A \in MG(X)$, it follows that $C_A(xy^{-1}) \geq [C_A(x) \wedge C_A(y)] \geq n$. Hence $xy^{-1} \in A_n$. Therefore, $A_n,\ n \in \mathbb{N}$ are subgroups of X. $\square$

Proposition 3.10. *If $A_n,\ n \in \mathbb{N}$ are subgroups of X, then the mset A, define in Theorem 2.9, is a mgroup over X.*

Proof. Let $x,\ y \in X$ and $C_A(x) = p,\ C_A(y) = q$. Then $x \in A_p,\ p = 1, 2, \ldots, p,\ x \notin A_{p+n},\ n \in \mathbb{N},\ x \in A_q,\ q = 1, 2, \ldots, q$ and $x \notin A_{q+n},\ n \in \mathbb{N}$. Let $min\{p,\ q\} = p$. Since $A_n,\ n \in \mathbb{N}$ are subgroups of X, it follows that $xy^{-1} \in A_p,\ p = 1, 2, \ldots, p$ and hence $C_A(xy^{-1}) \geq p = min\{p,\ q\} = C_A(x) \wedge C_A(y)$. Therefore, $A \in MG(X)$. $\square$

Definition 3.11. Let $A \in MG(X)$. Then define $A^* = \{x \in X;\ C_A(x) = C_A(e)\}$ and $A_* = \{x \in X;\ C_A(x) > 0\}$.

Proposition 3.12. *Let $A \in MG(X)$. Then A_* and A^* are subgroups of X.*

Proof. Let $x, y \in A^*$. Then $C_A(x) = C_A(y) = C_A(e)$. Now

$$C_A(xy^{-1}) \geq [C_A(x) \wedge C_A(y)] = [C_A(e) \wedge C_A(e)] = C_A(e) \geq C_A(xy^{-1}).$$

So, $C_A(xy^{-1}) = C_A(e),\ \forall\ x, y \in X$ and hence $x,\ y \in A^* \Rightarrow xy^{-1} \in A^*$. Therefore A^* is a subgroup of X.

Again let $x, y \in A_*$. Then $C_A(x) > 0$ and $C_A(y) > 0$. Now $C_A(xy^{-1}) \geq [C_A(x) \wedge C_A(y)] > 0$. Therefore, $x,\ y \in A_* \Rightarrow xy^{-1} \in A_*$ and hence A_* is a subgroup of X. $\square$

Proposition 3.13. *Let $A \in MS(X)$. Then $A \in MG(X)$ iff A satisfies the following conditions:*

(a) (i) $A \circ A \subseteq A$;
(ii) $A^{-1} \subseteq A$ *or* $A \subseteq A^{-1}$ *or* $A^{-1} = A$.
or
(b) $A \circ A^{-1} \subseteq A$.

Proof. Let $A \in MG(X)$. Then $C_A(yz) \geq C_A(y) \wedge C_A(z),\ \forall\ y, z \in X$. Thus, $C_A(x) \geq \{C_A(y) \wedge C_A(z) :\ yz = x\}$. Hence

$$C_A(x) \geq \vee_{y,z \in X}\{C_A(y) \wedge C_A(z) :\ yz = x\} = C_{A \circ A}(x),\ \forall\ x \in X.$$

Therefore, $A \circ A \subseteq A$. Again since $C_{A^{-1}}(x) = C_A(x^{-1}) = C_A(x)$, it follows that $A = A^{-1}$ and hence $A \subseteq A^{-1}$ and $A^{-1} \subseteq A$. Thus the given conditions are satisfied.

Conversely let the given conditions are satisfied. Let $x,\ y \in X$. Then $C_A(xy^{-1}) \geq C_{A \circ A}(xy^{-1}) = \bigvee_{z \in X}[C_A(z) \wedge C_A(z^{-1}xy^{-1})] \geq [C_A(x) \wedge C_A(y^{-1})] = [C_A(x) \wedge C_A(y)]\}$. Therefore $A \in MG(X)$.

Similarly we can proof the 'or' part. $\square$

Remark 3.14. By Proposition 3.2, the condition a(i) is equivalent to a(i') $A \circ A = A$ and condition (b) is equivalent to (b') $A \circ A^{-1} = A$.

Proposition 3.15. *Let $A,\ B \in MG(X)$. Then $A \circ B \in MG(X)$ iff $A \circ B = B \circ A$.*

Proof. Since $A,\ B \in MG(X)$, it follows that $A = A^{-1}$ and $B = B^{-1}$. Suppose $A \circ B \in MG(X)$. Then $A \circ B = (A \circ B)^{-1} = B^{-1} \circ A^{-1} = B \circ A$.

Conversely let $A \circ B = B \circ A$. Then $(A \circ B)^{-1} = (B \circ A)^{-1} = A^{-1} \circ B^{-1} = A \circ B$ and $(A \circ B) \circ (A \circ B) = A \circ (B \circ A) \circ B = A \circ (A \circ B) \circ B = (A \circ A) \circ (B \circ B) \subseteq A \circ B$. Therefore $A \circ B \in MG(X)$. □

Proposition 3.16. *Let $A,\ B \in MG(X)$. Then $A \cap B \in MG(X)$.*

Proof. Since $A,\ B \in MG(X)$, we have $C_A(xy^{-1}) \geq [C_A(x) \wedge C_A(y)]$ and $C_B(xy^{-1}) \geq [C_B(x) \wedge C_B(y)],\ \forall\ x, y \in X$. Now

$$\begin{aligned}
C_{A\cap B}(xy^{-1}) &= \wedge\{C_A(xy^{-1}),\ C_B(xy^{-1})\} \\
&\geq \wedge\{[C_A(x) \wedge C_A(y)],\ [C_B(x) \wedge C_B(y)]\} \\
&= C_A(x) \wedge C_A(y) \wedge C_B(x) \wedge C_B(y) \\
&= [C_A(x) \wedge C_B(x)]\ \wedge\ [C_A(y) \wedge C_B(y)] \\
&= C_{A\cap B}(x) \wedge C_{A\cap B}(y) \\
\text{and } C_{A\cap B}(x^{-1}) &= C_A(x^{-1}) \wedge C_B(x^{-1}) \\
&= C_A(x) \wedge C_B(x) = C_{A\cap B}(x).
\end{aligned}$$

Therefore $A \cap B \in MG(X)$. □

Remark 3.17. If $\{A_i,\ i \in I\}$ be a family of multigroups over a group X, then their intersection $\cap_{i\in I} A_i$ is a multigroup over X.

Remark 3.18. If $A,\ B \in MG(X)$, then the following example shows that their union $A \cup B$ is not a multigroup over X in general.

Example 3.19. Let $X = K_4 = \{e, a, b, c\}$ be the Klein's 4-group,

$$A = \{e, e, a\} \text{ and } B = \{e, e, b\}. \text{ Then } A,\ B \in MG(X).$$

Clearly $A \cup B = \{e, e, a, b\}$ and $C_{A\cup B}(c) = C_{A\cup B}(ab) = 0 \ngeq \bigwedge[C_{A\cup B}(a),\ C_{A\cup B}(b)] = 1$. Therefore $A \cup B$ is not a multigroup over X.

Definition 3.20. Let A and B be two multigroups over a group X. Then A is said to be a submultigroup of B if $A \subseteq B$.

Example 3.21. Let $X = \{e,\ a,\ b,\ c\}$ be the Klein's 4-group,

$$A = \{e,\ e,\ a,\ a,\ b,\ b,\ c,\ c\} \text{ and } B = \{e,\ e,\ e,\ a,\ a.\ b,\ b,\ b,\ c,\ c\}.$$

Then clearly $A,\ B \in MG(X)$ and $A \subseteq B$. Therefore, A is a submultigroup of B.

Proposition 3.22. *Let $X,\ Y$ be two groups and $f : X \to Y$ be a homomorphism. If $A \in MG(X)$, then $f(A) \in MG(Y)$.*

Proof. Let $u,\ v \in Y$.

Case-I: Let $u,\ v \notin f(X)$. Then $C_{f(A)}(u) \wedge C_{f(A)}(v) = 0 \wedge 0 \leq C_{f(A)}(uv)$.

Case-II: Let $u \notin f(X)$. Then $u^{-1} \notin f(X)$ and $C_{f(A)}(u) \wedge C_{f(A)}(v) = 0 \wedge C_{f(A)}(v) = 0 \leq C_{f(A)}(uv)$.

Case-III: Let $u,\ v \in f(X)$. Then $\exists\ x,\ y \in X$ such that $f(x) = u$ and $f(y) = v$. Now

$$\begin{aligned} C_{f(A)}(uv) &= \vee\{C_A(w);\ w \in X,\ f(w) = uv\} \\ &\geq \{C_A(xy);\ x, y \in X,\ f(x) = u,\ f(y) = v\} \\ &\geq \{C_A(x) \wedge C_A(y);\ x, y \in X,\ f(x) = u,\ f(y) = v\} \\ &= [\vee\{C_A(x);\ x \in X,\ f(x) = u\}] \wedge [\vee\{C_A(y);\ y \in X,\ f(y) = v\}] \\ &= [C_{f(A)}(u)] \wedge [C_{f(A)}(v)]. \end{aligned}$$

Also

$$\begin{aligned} C_{f(A)}(u^{-1}) &= \vee\{C_A(z);\ z \in Y,\ f(z) = u^{-1}\} \\ &= \vee\{C_A(z^{-1}),\ z \in Y,\ f(z^{-1}) = u\} = C_{f(A)}(u). \end{aligned}$$

Therefore, $f(A) \in MG(Y)$. $\square$

Proposition 3.23. *Let X, Y be two groups and $f : X \to Y$ be a homomorphism. If $B \in MG(Y)$, then $f^{-1}(B) \in MG(X)$.*

Proof. Proof is similar to that of Proposition 3.22. $\square$

Proposition 3.24. *Let $A \in MS(X)$. Then the following assertions are equivalent:*

(i) $C_A(xy) = C_A(yx),\ \forall x,\ y \in X$;
(ii) $C_A(xyx^{-1}) = C_A(y)\ \forall x,\ y \in X$;
(iii) $C_A(xyx^{-1}) \geq C_A(y)\ \forall x,\ y \in X$;
(iv) $C_A(xyx^{-1}) \leq C_A(y)\ \forall x,\ y \in X$.

Proof. $(i) \Rightarrow (ii)$: Let $x,\ y \in X$. Then $C_A(xyx^{-1}) = C_A(x^{-1}xy) = C_A(y)$.
$(ii) \Rightarrow (iii)$: straightforward.
$(iii) \Rightarrow (iv)$: $C_A(xyx^{-1}) \leq C_A(x^{-1}[xyx^{-1}](x^{-1})^{-1}) = C_A(y)$.
$(iv) \Rightarrow (i)$: Let $x,\ y \in X$. Then

$$C_A(xy) = C_A(x[yx]x^{-1}) \leq C_A(yx) = C_A(y[xy]y^{-1}) \leq C_A(xy).$$

Hence $C_A(xy) = C_A(yx)$. Thus the above assertions are equivalent. $\square$

Proposition 3.25. *Let $A \in MS(X)$. Then the following assertions are equivalent.*

(i) $C_A(xy) = C_A(yx),\ \forall x,\ y \in X$;
(ii) $A \circ B = B \circ A,\ \forall\ B \in MS(X)$.

Proof. $(i) \Rightarrow (ii)$: Let $x \in X$. Then

$$\begin{aligned} C_{A\circ B}(x) &= \vee_{y\in X}[C_A(xy^{-1}) \wedge C_B(y)] \\ &= \vee_{y\in X}[C_B(y) \wedge C_A(y^{-1}x)] \\ &= C_{B\circ A}(x). \end{aligned}$$

Hence $A \circ B = B \circ A$.

$(ii) \Rightarrow (i)$: Let $x, y \in X$, $[C_A(xy) \vee C_A(yx)] = n$ and $T = n_{y^{-1}}$. Then $T \in MS(X)$. Since $A \circ B = B \circ A$, $\forall\ B \in MS(X)$, it follows that $A \circ T = T \circ A$ and hence $C_{A\circ T}(x) = C_{T\circ A}(x)$, $\forall\ x \in X$. Now $C_{A\circ T}(x) = \vee_{uv=x}\{C_A(u) \wedge C_T(v)\} = C_A(xy) \wedge C_T(y^{-1}) = C_A(xy)$.

Similarly we have $C_{T\circ A}(x) = C_A(yx)$. Hence $C_A(xy) = C_A(yx)$, $\forall\ x, y \in X$. $\square$

Definition 3.26. A mgroup G over X is called *abelian* over X if $C_G(xy) = C_G(yx),\ \forall\ x, y \in X$. Let $AMG(X)$ denote the set of all abelian mgroups over X.

Example 3.27. Let X be an abelian group and G be a mgroup over X. Then G is an abelian mgroup over X.

Proposition 3.28. *Let $A \in AMG(X)$. The $A^*,\ A_*$ and $A_n,\ n \in \mathbb{N}$ are normal subgroups of X.*

Proof. By Proposition 3.12 and 3.9, $A^*,\ A_*$ and $A_n,\ n \in \mathbb{N}$ are subgroups of X. Let $x \in X$ and $y \in A_*$. Then $C_A(y) > 0$ and by Proposition 3.24, $C_A(xyx^{-1}) = C_A(y) > 0$. Thus $xyx^{-1} \in A_*$ and hence A_* is a normal subgroup of X.

Similarly A^* is also a normal subgroup of X.

Let $x \in X$ and $y \in A_n$. Then $C_A(y) \geq n$ and by Proposition 3.24, $C_A(xyx^{-1}) = C_A(y) \geq n$. Thus $xyx^{-1} \in A_n$ and hence $A_n,\ n \in \mathbb{N}$ are a normal subgroups of X.

Similarly A_* is also a normal subgroup of X. □

Definition 3.29. Let $H \in MG(X)$ and $x \in X$. Also let e be the identity element of X and $[C_H(e)]_x$ be as in Definition 2.15. Then

(i) the mset $[C_H(e)]_x \circ H$ is called a left mcoset of H in X and is denoted by xH, where
$C_{xH}(z) = \vee\{C_{[C_H(e)]_x}(u) \wedge C_H(v);\ u, v \in X,\ uv = z\}$
$= C_{[C_H(e)]_x}(x) \wedge C_H(x^{-1}z) = C_H(e) \wedge C_H(x^{-1}z) = C_H(x^{-1}z)$.

(ii) the mset $H \circ [C_H(e)]_x$ is called a right mcoset of of H in X and is denoted by Hx, where
$C_{Hx}(z) = \vee\{C_H(u) \wedge C_{[C_H(e)]_x}(v);\ u, v \in X,\ uv = z\}$
$= C_H(zx^{-1}) \wedge C_{[C_H(e)]_x}(x) = C_H(zx^{-1}) \wedge C_H(e) = C_H(zx^{-1})$.

Remark 3.30. If $H \in AMG(X)$, then $xH = Hx,\ \forall\ x \in X$.

Proposition 3.31. *Let $H \in MG(X)$. Then $\forall\ x, y \in X$,*

(i) $xH = yH \Longleftrightarrow xH_* = yH_*$;
(ii) $Hx = Hy \Longleftrightarrow H_*x = H_*y$

Proof. (i) Let $xH = yH$. Then $[C_H(e)]_x \circ H = [C_H(e)]_y \circ H$ and hence $C_H(x^{-1}z) = C_H(y^{-1}z),\ \forall\ z \in X$.In particular $C_H(x^{-1}y) = C_H(y^{-1}y) = C_H(e)$. Thus $x^{-1}y \in H_*$ and hence $xH_* = yH_*$.

Conversely let $xH_* = yH_*$. Then $x^{-1}y,\ y^{-1}x \in H_*$. Now

$$\begin{aligned} C_H(x^{-1}z) &= C_H([x^{-1}y][y^{-1}z]) \geq C_H(x^{-1}y) \wedge C_H(y^{-1}z) \\ &= C_H(e) \wedge C_H(y^{-1}z) = C_H(y^{-1}z),\ \forall\ z \in X. \end{aligned}$$

Similarly, $C_H(y^{-1}z) \geq C_H(x^{-1}z),\ \forall\ z \in X$. Therefore, $C_H(x^{-1}z) = C_H(y^{-1}z),\ \forall\ z \in X$ which shows that $xH = yH$.

(ii) Proof is similar to that of part (i). □

Proposition 3.32. *Let $H \in AMG(X)$. If $xH = yH$, then $C_H(x) = C_H(y),\ \forall\ x, y \in X$.*

Proof: Let $xH = yH$. Then by Proposition 3.31, $x^{-1}y,\ y^{-1}x \in H_*$.
Since $H \in AMG(X)$, it follows that
$C_H(x) = C_H(y^{-1}xy) \geq [C_H(y^{-1}x) \wedge C_H(y) = C_H(e) \wedge C_H(y) = C_H(y)$.
Similarly $C_H(y) \geq C_H(x)$ and therefore $C_H(x) = C_H(y)$.

Proposition 3.33. *Let $H \in AMG(X)$ and define $X/H = \{xH;\ x \in X\}$. Then the following assertions hold:*

(i) $(xH) \circ (yH) = (xy)H,\ \forall\ x, y \in X$;
(ii) *If $xH = x_1H$ and $yH = y_1H$ then $(xy)H = (x_1y_1)H$;*
(iii) *$(X/H, \circ)$ is a group;*
(iv) $X/H \simeq X/H^*$;

Proof. (i) For all $x, y \in X$, we have

$$\begin{aligned}
(xH) \circ (yH) &= ([C_H(e)]_x \circ H) \circ ([C_H(e)]_y \circ H) \\
&= ([C_H(e)]_x \circ H) \circ (H \circ [C_H(e)]_y)(\text{by Proposition } 3.25) \\
&= [C_H(e)]_x \circ (H \circ H) \circ [C_H(e)]_y(\text{by associativity of } \circ) \\
&= [C_H(e)]_x \circ H \circ [C_H(e)]_y(\text{by Remark } 3.14) \\
&= [C_H(e)]_x \circ [C_H(e)]_y \circ H(\text{since H is abelian}) \\
&= ([C_H(e)]_{xy} \circ H = (xy)H.
\end{aligned}$$

(ii) Let $xH = x_1H$ and $yH = y_1H$. Then $C_H(x^{-1}z) = C_H(x_1^{-1}z)$ and $C_H(y^{-1}z) = C_H(y_1^{-1}z),\ \forall\ z \in X$. Now

$$\begin{aligned}
C_H[(xy)^{-1}z] &= C_H[y^{-1}x^{-1}z] = C_H[y_1^{-1}x^{-1}z] \\
&= C_H[x^{-1}zy_1^{-1}] = C_H[x_1^{-1}zy_1^{-1}] \\
&= C_H[y_1^{-1}x_1^{-1}z] = C_H[(x_1y_1)^{-1}z],\ \forall\ z \in X.
\end{aligned}$$

Therefore $(xy)H = (x_1y_1)H$.

(iii) From (ii), the composition is well defined. From (i), X/H is closed under the operation $\circ$. Also by Proposition 3.2, $\circ$ is associative. Now $H \circ xH = eH \circ xH = (ex)H = xH,\ \forall\ x \in X$ and $x^{-1}H \circ xH = (xx^{-1})H = eH = H,\ \forall\ x \in X$. Therefore, $(X/H, \circ)$ is a group.

(iv) Since $H \in AMG(X)$, it follows that H^* is a normal subgroup of X and hence X/H^* is a group. Now let $f : X/H \to X/H^*$, defined by $f(xH) = xH^*$. Then f is an isomorphism and hence $X/H \simeq X/H^*$. □

The group X/H is called the factor mgroup (or quotient mgroup) of X relative to the normal mgroup H.

Proposition 3.34. *Let $A \in MG(X)$ and N be a normal subgroup of X. Define $\hat{A} \in MS(X/N)$ such that $C_{\hat{A}}(xN) = \bigvee\{C_A(z);\ z \in xN\}, \forall\ x \in X$. Then $\hat{A} \in MG(X/N)$.*

Proof. Now

$$\begin{aligned} C_{\hat{A}}[(xN)^{-1}] &= C_{\hat{A}}(x^{-1}N) = \vee\{C_A(z);\ z \in x^{-1}N\} \\ &= \vee\{C_A(w^{-1});\ w^{-1} \in x^{-1}N\} \\ &= \vee\{C_A(w);\ w \in xN\}(w \in xN if and only if w^{-1} \in x^{-1}N, \\ &\qquad \text{as } N \text{ is a normal subgroup of } X) \\ &= C_{\hat{A}}(xN),\ \forall\ x \in X. \end{aligned}$$

Again

$$\begin{aligned} C_{\hat{A}}(xNyN) &= \vee\{C_A(z);\ z \in xyN\} = \vee\{C_A(uv);\ u \in xN,\ v \in yN\} \\ &\geq \vee\{C_A(u) \wedge C_A(v);\ u \in xN,\ v \in yN\} \\ &= [\vee\{C_A(u);\ u \in xN\}] \wedge [\vee\{C_A(v);\ v \in yN\}] \\ &= C_{\hat{A}}(xN) \wedge C_{\hat{A}}(yN),\ \forall\ x, y \in X. \end{aligned}$$

Hence $\hat{A} \in MG(X/N)$. □

Definition 3.35. The mgroup $\hat{A}$, as in Proposition 3.34 is called the factor mgroup of the mgroup A over X relative to the normal subgroup N of X and is denoted by A/N.

Proposition 3.36. *Let $H \in AMG(X)$ and Y be a group. Suppose that $f : X \to Y$ be an onto homomorphism. Then $f(H) \in AMG(Y)$.*

Proof. By Proposition 3.22, $f(H) \in MG(Y)$. Now let $y, z \in Y$. Since f is onto, $\exists\ u \in X$ such that $f(u) = z$. Thus

$$\begin{aligned} C_{f(H)}(zyz^{-1}) &= \vee\{C_H(w);\ w \in X,\ f(w) = zyz^{-1}\} \\ &= \vee\{C_H(u^{-1}wu);\ w \in X,\ f(u^{-1}wu) = y\} \\ &= \vee\{C_H(v);\ v \in X,\ f(v) = y\} = C_{f(H)}(y). \end{aligned}$$

Therefore, $f(H) \in AMG(Y)$. □

Proposition 3.37. *Let $H \in AMG(Y)$ and X be a group. Suppose that $f : X \to Y$ be an into homomorphism. Then $f^{-1}(H) \in AMG(X)$.*

Proof. By Proposition 3.23, $f^{-1}(H) \in MG(X)$. Let $x, z \in X$. Thus

$$\begin{aligned} C_{f^{-1}(H)}(xz) &= C_H[f(xz)] = C_H[f(x)f(z)] = C_H[f(z)f(x)] \\ &= C_H[f(zx)] = C_{f^{-1}(H)}(zx). \end{aligned}$$

Therefore, $f^{-1}(H) \in NMG(X)$. □

4. CONCLUSION

In this paper, we have introduced the notion of mgroups for the first time, and studied its important properties. We have studied some basic properties of msets also. The theory of msets and mgroups can be very useful in many areas like information retrival on the web, data encription, data mining, coding theory, decision making etc. One can further study the deeper properties of mgroups viz. the isomorphism theorems etc. In future we will study mgroup structures on several hybrid sets like fuzzy msets, soft msets etc.

Acknowledgements. The authors express their sincere thanks to the reviewers for their valuable suggestions. The authors are also thankful to the editors-in-chief and managing editors for their important comments which hepled improve the presentation of the paper. The present work is partially supported by the Special Assistance Programme (SAP) of UGC, New Delhi, India [Grant No. F 510/8/DRS/2009 (SAP -II)].

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