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# S-fuzzy prime ideal theorem

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Abstract. The notions of a S-fuzzy $\wedge$-sub semi lattice, a S-fuzzy ideal and a S-fuzzy prime ideal of a bounded lattice with truth values in a bounded $\wedge$-sub semi lattice S are introduced which generalize the existing notions with truth values in a unit interval of real numbers. Finally, S-fuzzy prime ideal theorem is proved.

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## 1. Introduction

After an introduction and development of fuzzy sets by Zadeh[16], the fuzzy set theory is developed by others in many directions and found applications in various areas of sciences. The study of fuzzy algebraic structures started with introduction of the concept of the fuzzy subgroup of a group in the pioneering paper of Rosenfeld [12]. Since then many researchers have been engaged in extending the concepts and results of abstract algebra to broader framework of the fuzzy sets. Liu [6] introduced and examined the notion of a fuzzy ideal of a ring. Several authors have obtained interesting results on fuzzy ideals of different algebraic structures. Mukherjee and Sen [11] introduced fuzzy prime ideal of a ring. A comprehensive survey of the literature on these developments is given in [7]. Mukherjee and Sen [10], Malik and Mordeson [8], Mashinchi and Zahedi [9], Zahedi [17], studied fuzzy prime ideals of a ring. As in ring theory, work on fuzzy ideals and fuzzy prime ideals of a lattice may be found in [13], [15], [5]. In particular, Swami and Raju [13] introduced the notion of L - fuzzy ideals and L - fuzzy prime ideals of a distributive lattice where L stands for a complete Brouwerian lattice. A study of L- fuzzy prime ideals and L- fuzzy maximal ideals can also be seen in [14] by Swamy and Swamy.

Analogous to prime ideal theorem in a distributive lattice [3], prime fuzzy ideal theorem is proved in [13]. Koguep et. al.[5], introduce fuzzy prime ideal of lattice and proved fuzzy prime ideal theorem. In this topic, we introduce the concept of a S-fuzzy $\wedge$-sub semi lattice of a bounded lattice and proved the important result of generalizing the Fuzzy Stone's theorem i.e., S-fuzzy prime ideal theorem (where S denotes a bounded $\wedge$-sub semi lattice

## 2. Preliminaries

Now onwards $X$ denotes a bounded lattice with the least element 0 and the greatest element 1. S denotes a bounded $\wedge$ - sub semi lattice consisting of at least two elements unless otherwise stated. For the basic definitions in lattice theory and fuzzy set theory the reader is referred to [3] and [4] respectively.

We begin with the definition of an ideal, a dual ideal and a prime ideal of a lattice.

**Definition 2.1** ([3])**.** A non-empty subset $I$ of $X$ is called an ideal of $X$ if, for any $a, b \in I$ and $x \in X$, $a \vee b \in I$ and $a \wedge x \in I$ .

**Definition 2.2** ([3])**.** A non-empty subset $D$ of $X$ is called a dual ideal of $X$ if, for any $a, b \in D$ and $x \in X$, $a \wedge b \in D$ and $a \vee x \in D$.

**Definition 2.3** ([3])**.** A proper ideal $P$ of $X$ is called a prime ideal of $X$ if, for any $a, b \in X$, $a \wedge b \in P$ implies $a \in P$ or $b \in P$.

**Definition 2.4** ([3])**.** A proper dual ideal $P$ of $X$ is called a prime dual ideal of $X$ if, for any $a, b \in X$, $a \vee b \in P$ implies $a \in P$ or $b \in P$.

We also define a S-fuzzy set of a lattice as,

**Definition 2.5.** By a S-fuzzy set of $X$, we mean a mapping from $X$ into $S$. The set of all S-fuzzy sets of $X$ is called the S - power fuzzy set of $X$ and is denoted by $S^X$

Some basic terminology of a S-fuzzy set of a bounded lattice, which will be needed in sequel, are defined as follows.

**Definition 2.6.** For $\mu \in S^X$, the set $Supp(\mu) = \{x \in X/\mu(x) > 0\}$.

**Definition 2.7.** For any $\alpha \in S$, the set $\mu_\alpha = \{x \in X/\mu(x) \geq \alpha\}$ is called the $\alpha$ - cut of $\mu$ and the set $\mu_{[\alpha]} = \{x \in X/\mu(x) > \alpha\}$ is called the strong $\alpha$ - cut of $\mu$.

We need the following results in sequel.

**Results 2.8:**

**I)** Let $X$ be a bounded distributive lattice. Let $I$ be an ideal of $X$ and D be a dual ideal of $X$ such that $I \cap D = \varnothing$. Then there exists a prime ideal $P$ of $X$ containing $I$ and disjoint with $D$.

**II)** Let $X$ be a bounded distributive lattice. Let $S$ be a $\wedge$-sub semi lattice and $I$ be an ideal of $X$ such that $S \cap I = \varnothing$. Then there exists a prime ideal $P$ of $X$ containing $I$ and disjoint with $S$.

## 3. Fuzzy $\wedge$-sub semi lattice

We begin with defining a S-fuzzy $\wedge$-sub semi lattice of $X$ as follows.

**Definition 3.1.** Let $\mu \in S^X$. $\mu$ is a S-fuzzy $\wedge$-sub semi lattice of $X$ if, for all $x, y \in X$, $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$.

If $S = [0, 1]$ then a S-fuzzy $\wedge$ - sub semi lattice of $X$ is called a fuzzy $\wedge$ - sub semi lattice of $X$.

**Example 3.2. I)** Consider the lattice $X = \{0, a, b, 1\}$ as shown in Hasse diagram of Figure 1

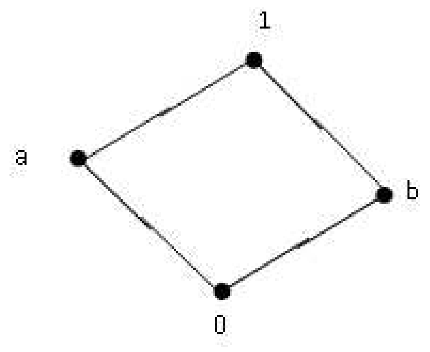


Figure 1.

Define $\mu : X \to [0, 1]$ as, $\mu(0) = 0, \mu(a) = 0,$ $\mu(b) = 0.8,$ $\mu(1) = 1.$ Then $\mu$ is a fuzzy $\wedge$ - sub semi lattice of $X$.

**II)** Consider lattice $X = \{0, a, b, c, 1\}$ and $\wedge$-sub semi lattice $S = \{0, x, y, z, t\}$ shown in the Hasse diagrams of Figure 2.

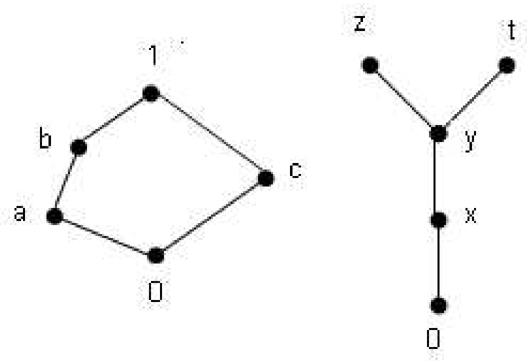


Figure 2.

Let $\mu \in S^X$. $\mu(0) = 0,$ $\mu(a) = x,$ $\mu(b) = y,$ $\mu(c) = 0,$ $\mu(1) = t.$ Then $\mu$ is a S-fuzzy $\wedge$ - sub semi lattice of $X$.

**Remark 3.3.** Every S-fuzzy set need not be a S-fuzzy $\wedge$-sub semi lattice of $X$. For this consider lattice $X = \{0, a, b, c, 1\}$ and $\wedge$-sub semi lattice $S = \{0, x, y, z, t\}$ as shown in the Hasse diagrams of Figure 2. Let $\mu \in S^X$. $\mu(0) = 0,$ $\mu(a) = x,$ $\mu(b) = y,$ $\mu(c) = y,$ $\mu(1) = t.$ Then $\mu$ is not a S-fuzzy $\wedge$-sub semi lattice of $X$ as

$\mu(a \wedge c) = \mu(0) = 0$ and $\mu(a) = x$, $\mu(c) = y$. Thus $\mu(a) \wedge \mu(c) = x \wedge y = x$. Hence $\mu(a \wedge c) \ngeqslant \mu(a) \wedge \mu(c)$.

The characterization of S-fuzzy $\wedge$-sub semi lattice is given by its $\alpha$ - cut as,

**Theorem 3.4.** *Let $\mu \in S^X$. $\mu$ is a S-fuzzy $\wedge$-sub semi lattice of $X$ if and only if $\mu_t$ is a $\wedge$-sub semi lattice of $X$ for all $t \in S$ and $\mu_t \neq \varnothing$.*

*Proof.* Let $\mu$ be a S-fuzzy $\wedge$-sub semi lattice of $X$. Let $\mu_t \neq \varnothing$ for $t \in S$. Let $x, y \in \mu_t$. Hence $\mu(x) \geq t$ and $\mu(y) \geq t$. Also $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$ (As $\mu$ is a S-fuzzy $\wedge$ - sub semi lattice of $X$). Thus $\mu(x \wedge y) \geq t \wedge t = t$. Hence $x \wedge y \in \mu_t$. Therefore $\mu_t$ is a $\wedge$-sub semi lattice of $X$.

Conversely, let $\mu_t$ be a $\wedge$-sub semi lattice of $X$ for all $t \in S$ and $\mu_t \neq \varnothing$. Let $x, y \in X$. Let $\mu(x) \wedge \mu(y) = t$. Hence $\mu(x) \geq t$ and $\mu(y) \geq t$. Therefore $x, y \in \mu_t$. Thus $x \wedge y \in \mu_t$ (As $\mu_t$ is a $\wedge$-sub semi lattice of $X$). Hence $\mu(x \wedge y) \geq t = \mu(x) \wedge \mu(y)$. This shows that $\mu$ is a S-fuzzy $\wedge$-sub semi lattice of $X$. □

Generalizing the concepts defined by Attallah [1], we define,

**Definition 3.5.** Let $\mu \in S^X$. $\mu$ is said to be a S-fuzzy sub lattice of $X$ if for all $x, y \in X$, $\mu(x \wedge y) \geqslant \mu(x) \wedge \mu(y)$ and $\mu(x \vee y) \geqslant \mu(x) \wedge \mu(y)$. Equivalently, $\mu$ is said to be a S- fuzzy sub lattice of $X$ if for all $x, y \in X$, $\mu(x \wedge y) \wedge \mu(x \vee y) \geqslant \mu(x) \wedge \mu(y)$.

If $S = [0, 1]$, then a S-fuzzy sub lattice of $X$ is called a fuzzy sub lattice of $X$.

**Remark 3.6.** Every S-fuzzy sublattice of $X$ is a S-fuzzy $\wedge$-sub semi lattice of $X$ but converse need not be true. For this consider lattice $X = \{0, a, b, c, 1\}$ and $\wedge$-sub semi lattice $S = \{0, x, y, z, t\}$ as shown in the Hasse diagrams of Figure 2. Let $\mu \in S^X$. $\mu(0) = y$, $\mu(a) = y$, $\mu(b) = y$, $\mu(c) = x$, $\mu(1) = 0$. Then $\mu$ is a S-fuzzy $\wedge$ - sub semi lattice of $X$ but it is not a S-fuzzy sublattice as $\mu(a \vee c) = \mu(1) = 0$ and $\mu(a) = y$, $\mu(c) = x$.
Thus $\mu(a) \wedge \mu(c) = y \wedge x = x$. Hence $\mu(a \vee c) \ngeqslant \mu(a) \wedge \mu(c)$.

We now define S-fuzzy ideal and S-fuzzy dual ideal of a bounded lattice as,

**Definition 3.7.** Let $\mu$ be a S-fuzzy sub lattice of $X$. Then
i) $\mu$ is a S-fuzzy ideal of $X$ if $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ for all $x, y \in X$.
ii) $\mu$ is said to be S-fuzzy dual ideal of $X$ if $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$ for all $x, y \in X$.

If $S = [0, 1]$ then a S-fuzzy ideal (dual ideal) of $X$ coincides with fuzzy ideal (dual ideal) of $X$.

**Example 3.8.** I) Consider the lattice $X = \{0, a, b, 1\}$ and chain $S = [0, 1]$ as shown by the Hasse diagrams of Figure 1. Define $\mu : X \to [0, 1]$ as, $\mu(0) = 1$, $\mu(a) = 0$, $\mu(b) = 1$, $\mu(1) = 0$. Then $\mu$ is a fuzzy ideal of $X$.

II) Consider lattice $X = \{0, a, b, c, 1\}$ and $\wedge$-sub semi lattice $S = \{0, x, y, z, t\}$ as shown in the Hasse diagrams of Figure 2. Let $\mu \in S^X$. $\mu(0) = y$, $(a) = x$, $\mu(b) = x$, $\mu(c) = 0$, $\mu(1) = 0$. Then $\mu$ is a S-fuzzy ideal of $X$.

**Remark 3.9.** Every S-fuzzy $\wedge$-sub semi lattice ( S-fuzzy sublattice ) of $X$ is not necessarily a S-fuzzy ideal or a S-fuzzy dual ideal of $X$. For this consider lattice $X = \{0, a, b, c, 1\}$ and $\wedge$-sub semi lattice $S = \{0, x, y, z, t\}$ as shown in the Hasse diagrams of Figure 2. Let $\mu \in S^X$. $\mu(0) = y$, $\mu(a) = 0$, $\mu(b) = x$, $\mu(c) = 0$, $\mu(1) = 0$. Then $\mu$ is a S-fuzzy $\wedge$-sub semi lattice of $X$. Also it is a S-fuzzy sublattice but it is neither S-fuzzy ideal nor it is S-fuzzy dual ideal as $\mu(a \vee b) = \mu(b) = x$ and $\mu(a) = 0$, $\mu(b) = x$. Thus $\mu(a) \wedge \mu(b) = 0 \wedge x = 0$. Hence $\mu(a \vee b) \neq \mu(a) \wedge \mu(b)$. Also $\mu(0 \wedge a) = \mu(0) = y$. As $\mu(0) = y$ and $\mu(a) = 0$ we get $\mu(0) \wedge \mu(a) = y \wedge 0 = 0$. Hence $\mu(0 \wedge a) \neq \mu(0) \wedge \mu(a)$.

The characterization of S-fuzzy ideal of a bounded lattice are given below.

**Theorem 3.10.** I) *Let $\mu \in S^X$. Then $\mu$ is a S-fuzzy ideal ( dual ideal ) of $X$ if and only if $\mu_t$ is an ideal ( dual ideal ) of $X$ for all $t \in S$ and $\mu_t \neq \varnothing$.*

*Proof.* Let $\mu$ be a S-fuzzy ideal of $X$. Then $\mu$ is a S-fuzzy $\wedge$-sub semi lattice of $X$. Thus by Theorem 3.4, $\mu_t$ is a $\wedge$-sub semi lattice of $X$ for all $t \in S$ and $\mu_t \neq \varnothing$. Let $\mu_t \neq \varnothing$ for $t \in S$. Let $x, y \in \mu_t$. Hence $\mu(x) \geq t$ and $\mu(y) \geq t$. Also $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$ (As $\mu$ is a S-fuzzy - sublattice of $X$). Thus $\mu(x \vee y) \geq t \wedge t = t$. Hence $x \vee y \in \mu_t$. Therefore $\mu_t$ is a sublattice of $X$. Let $x \leq a, x \in X$ and $a \in \mu_t$. Then $\mu(a) = \mu(x \vee a)$ imply $\mu(a) = \mu(x) \wedge \mu(a)$ (As $\mu$ is a S-fuzzy ideal of $X$). Therefore $\mu(x) \geq \mu(a) \geq t$. Hence $x \in \mu_t$. This shows that $\mu_t$ is an ideal of $X$.

Conversely, let $\mu_t$ be an ideal of $X$ for all $t \in S$ and $\mu_t \neq \varnothing$. Then $\mu_t$ is a $\wedge$-sub semi lattice of $X$ for all $t \in S$ and $\mu_t \neq \varnothing$. Thus by Theorem 3.4, $\mu$ is a S-fuzzy $\wedge$-sub semi lattice of $X$. Let $x, y \in X$. Let $\mu(x) \wedge \mu(y) = t$. Hence $\mu(x) \geq t$ and $\mu(y) \geq t$. Therefore $x, y \in \mu_t$. Thus $x \vee y \in \mu_t$ (As $\mu_t$ is a sublattice of $X$). Hence $\mu(x \vee y) \geq t = \mu(x) \wedge \mu(y)$. This shows that $\mu$ is a S-fuzzy sublattice of $X$.

Let $x, y \in X$. Let $\mu(x \vee y) = t$. Then $\mu_t \neq \varnothing$ as $\mu(x \vee y) \in \mu_t$. As $\mu_t$ is an ideal of $X$, $x, y \in \mu_t$. Hence $\mu(x) \geq t$ and $\mu(y) \geq t$. Therefore $\mu(x) \geq \mu(x \vee y)$ and $\mu(y) \geq \mu(x \vee y)$. Thus $\mu(x) \wedge \mu(y) \geq \mu(x \vee y)$. As $\mu$ is a S-fuzzy sublattice of $X$, $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$. Hence $\mu(x \vee y) = \mu(x) \wedge \mu(y)$. This shows that $\mu$ is a S-fuzzy ideal of $X$.

Similarly we can prove that $\mu$ is a S-fuzzy dual ideal of $X$ if and only if $\mu_t$ is a dual ideal of $X$ for all $t \in S$ and $\mu_t \neq \varnothing$. ☐

II) Let $\mu$ be a S- fuzzy sub lattice of $X$. Then

(i) $\mu$ is a S- fuzzy ideal of $X$ if and only if $x \leq y \Rightarrow \mu(x) \geq \mu(y)$, for all $x, y \in X$.

(ii) $\mu$ is a S- fuzzy dual ideal of $X$, if and only if, $x \leq y \Rightarrow \mu(x) \leq \mu(y)$, for all $x, y \in X$.

*Proof.* i ) Let $\mu$ be a S-fuzzy ideal of $X$. Let $x, y \in X$ such that $x \leq y$. Then $x \vee y = y$. As $\mu$ is a S-fuzzy ideal of $X$, $\mu(x) \wedge \mu(y) = \mu(x \vee y) = \mu(y)$. Therefore $\mu(y) \leq \mu(x)$. This shows that $x \leq y \Rightarrow \mu(x) \geq \mu(y)$, for all $x, y \in X$.

Conversly let $x \leq y \Rightarrow \mu(x) \geq \mu(y)$, for all $x, y \in X$. Let $x, y \in X$. As $\mu$ is a S-fuzzy sublattice of $X$, $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$. $x \leq x \vee y$ and $y \leq x \vee y$ imply $\mu(x) \geq \mu(x \vee y)$ and $\mu(y) \geq \mu(x \vee y)$. Then $\mu(x) \wedge \mu(y) \geq \mu(x \vee y)$ imply $\mu(x) \wedge \mu(y) = \mu(x \vee y)$. This shows that $\mu$ is a S-fuzzy ideal of $X$.

ii ) Same as (i). ☐

By Theorem 3.10(II)(i), we get following remarks as,

**Remark 3.11.** If $\mu$ is a S-fuzzy ideal of $X$ then

i) $\mu(0) \geq \mu(x) \geq \mu(1)$ for all $x \in X$.

ii) $x \in (y) \Rightarrow \mu(x) \geq \mu(y)$ for all $x, y \in X$.

In following theorem we prove that the set of all elements which maps to image of 0 under S-fuzzy ideal of a bounded lattice is an ideal of a lattice.

**Theorem 3.12.** *Let $\mu$ be a S-fuzzy ideal of $X$. Define $X_\mu = \{x \in X/\mu(x) = \mu(0)\}$. Then $X_\mu$ is an ideal of $X$.*

*Proof.* $X_\mu \neq \varnothing$ as $0 \in X_\mu$. Let $x, y \in X_\mu$. Then $\mu(x) = \mu(0)$ and $\mu(y) = \mu(0)$. As $x \wedge y \leq x$, by Theorem 3.10(II)(i), $\mu(x \wedge y) \geq \mu(x)$. By Remark 3.11 (i), $\mu(x \wedge y) \leq \mu(0)$. Therefore $\mu(x \wedge y) = \mu(0)$. Hence $x \wedge y \in X_\mu$. As $\mu$ is a S-fuzzy ideal of $X$, $\mu(x \vee y) = \mu(x) \wedge \mu(y) = \mu(0) \wedge \mu(0) = \mu(0)$. Hence $x \vee y \in X_\mu$.

Let $x \leq a, x \in X$ and $a \in X_\mu$. Then $\mu(a) = \mu(0)$. As $\mu$ is a S-fuzzy ideal of $X$, by Remark 3.11 (i), $\mu(x) \leq \mu(0)$. As $x \leq a$, by Theorem 3.10(II)(i), $\mu(x) \geq \mu(a) = \mu(0)$. Therefore $\mu(x) = \mu(0)$. Hence $x \in X_\mu$. This shows that $X_\mu$ is an ideal of $X$. □

Using the Theorem 3.12 we prove,

**Theorem 3.13.** *Let $A$ be a non-empty subset of $X$. Let $r, t \in S$ such that $r \leq t$. Define $\mu_A : X \to S$ as,*

$$\begin{aligned} \mu_A(x) &= t \quad if x \in A \\ &= r \quad if x \notin A \end{aligned}$$

*Then $A$ is an ideal of $X$ if and only if $\mu_A$ is a S-fuzzy ideal of $X$. Moreover $X_\mu = A$.*

*Proof.* Let $A$ be an ideal of $X$. Let $x, y \in X$. If $x \wedge y \in A$ then $\mu_A(x \wedge y) = t \geq \mu_A(x) \wedge \mu_A(y)$. If $x \wedge y \notin A$ then $x \notin A$ and $y \notin A$ (As $A$ is an ideal of $X$) and therefore $\mu_A(x \wedge y) = \mu_A(x) = \mu_A(y) = r$. Hence $\mu_A(x \wedge y) \geq \mu_A(x) \wedge \mu_A(y)$. Similarly we can prove that $\mu_A(x \vee y) \geq \mu_A(x) \wedge \mu_A(y)$. This shows that $\mu_A$ is a S-fuzzy sublattice of $X$.

Let $x, y \in X$ such that $x \leq y$. If $x \in A$ then $\mu_A(x) = t \geq \mu_A(y)$. If $x \notin A$ then $y \notin A$ (As $A$ is an ideal of $X$) and therefore $\mu_A(x) = \mu_A(y) = r$. Hence $\mu_A(x) \geq \mu_A(y)$. By Theorem 3.10 (II) (i), $\mu_A$ is a S-fuzzy ideal of $X$.

Conversly, let $\mu_A$ be a S-fuzzy ideal of $X$. Let $x \in A$. Then $\mu_A(x) = t$. By Remark 3.11 (i), $\mu_A(0) \geq \mu_A(x) = t$. Therefore $\mu_A(0) = t$. Hence $0 \in A$. Thus $X_\mu = \{x \in X/\mu(x) = \mu(0) = t\} = A$. By Theorem 3.12, $X_\mu = A$ is an ideal of $X$. □

Thus using Theorem 3.13, we can prove that a non-empty subset of a bounded lattice is an ideal if and only if its characteristic function is a S-fuzzy ideal of a lattice which is stated in following corollary as,

**Corollary 3.14.** *Let $A$ be a non-empty subset of $X$. Then $A$ is an ideal of $X$ if and only if characteristic function of $A$, $\chi_A$ is a S-fuzzy ideal of $X$.*

We now define S-fuzzy prime ideal of a bounded lattice as,

**Definition 3.15.** Let $\mu$ be a S- fuzzy ideal (dual ideal) of $X$. $\mu$ a S-fuzzy prime ideal (S-fuzzy prime dual ideal) of $X$ if $\mu(x \wedge y) = \mu(x)$ or $\mu(x \wedge y) = \mu(y)$ ($\mu(x \vee y) = \mu(x)$ or $\mu(x \wedge y) = \mu(y)$) for all $x, y \in X$.

**Example 3.16.** $\mu \in S^X$ defined in Example 3.8 (I) is a S-fuzzy prime ideal of $X$.

**Remark 3.17.** Every S-fuzzy ideal of $X$ is not necessarily a S-fuzzy prime ideal of $X$. For this consider $\mu \in S^X$ defined in Example 3.8 (II). $\mu$ is a S-fuzzy ideal of $X$ but $\mu$ is not a S-fuzzy prime ideal of $X$ as $\mu(a \wedge c) = \mu(0) = y$ and $\mu(a) = x$ and $\mu(c) = 0$. Hence $\mu(a \wedge c) \neq \mu(a)$ and $\mu(a \wedge c) \neq \mu(c)$.

In the following Theorem, we prove that every prime ideal of a bounded lattice induces a S-fuzzy prime ideal of a bounded lattice.

**Theorem 3.18.** *Let $P$ be a non-empty subset of $X$. Let $r, t \in S$ such that $r \leq t$. Define $\mu_P : X \to S$ as,*

$$\begin{aligned} \mu_P(x) &= t \quad if x \in P \\ &= r \quad if x \notin P \end{aligned}$$

*Then $P$ is a prime ideal of $X$ if and only if $\mu_P$ is a S-fuzzy prime ideal of $X$.*

*Proof.* Let $P$ be a prime ideal of $X$. Then by Theorem 3.13, $\mu_P$ is a S-fuzzy ideal of $X$. Let $x, y \in X$. If $x \wedge y \in P$ then $x \in P$ or $y \in P$ (As $P$ is a prime ideal of $X$). Then $\mu_P(x \wedge y) = t$ and $\mu_P(x) = t$ or $\mu_P(y) = t$. Therefore $\mu_P(x \wedge y) = \mu_P(x)$ or $\mu_P(x \wedge y) = \mu_P(y)$. If $x \wedge y \notin P$ then $x \notin P$ and $y \notin P$ (As $P$ is an ideal of $X$) and therefore $\mu_P(x \wedge y) = \mu_P(x) = \mu_P(y) = r$. Thus for $x, y \in X$, $\mu_P(x \wedge y) = \mu_P(x)$ or $\mu_P(x \wedge y) = \mu_P(y)$. This shows that $\mu_P$ is a S-fuzzy prime ideal of $X$.

Conversely, let $\mu_P$ be a S-fuzzy prime ideal of $X$. Then by Theorem 3.13, P is an ideal of $X$. Let $x \wedge y \in P$. Then $\mu_P(x \wedge y) = t$. As $\mu_P$ is a S-fuzzy prime ideal of $X$, $\mu_P(x \wedge y) = \mu_P(x)$ or $\mu_P(x \wedge y) = \mu_P(y)$. Therefore $\mu_P(x) = t$ or $\mu_P(y) = t$. Hence $x \in P$ or $y \in P$. This shows that $P$ is a prime ideal of $X$. ☐

Using Theorem 3.18 we can prove that a non-empty subset of a bounded lattice is a prime ideal if and only if its characteristic function is a S-fuzzy prime ideal of a lattice which is stated in following corollary as,

**Corollary 3.19.** *Let $P$ be a non-empty subset of $X$. Then $P$ is a prime ideal of $X$ if and only if characteristic function of $P$, $\chi_P$ is a S-fuzzy prime ideal of $X$.*

## 4. S-fuzzy prime ideal theorem

Generalizing the concept defined by Dheena and Mohanraaj [2] we define,

**Definition 4.1.** Let $\lambda$ and $\mu \in S^X$. We write $\lambda \cap \mu = 0$ if there exist $t$ ($\neq 1$, if 1 exist) $\in S$ such that $\lambda_t \neq \varnothing$, $\mu_t \neq \varnothing$ and $\lambda_t \cap \mu_t = \varnothing$.

**Example 4.2.** Consider the lattice $X = \{0, a, b, c, 1\}$ and $\wedge$-sub semi lattice $S = \{0, x, y, z, t\}$ as shown in the Hasse diagrams of Figure 2. Let $\mu$, $\lambda \in S^X$. $\mu(0) = 0$, $\mu(a) = x$, $\mu(b) = y$, $\mu(c) = 0$, $\mu(1) = t$ and $\lambda(0) = t$, $\lambda(a) = y$, $\lambda(b) = t$, $\lambda(c) = t$, $\lambda(1) = x$. Also $\lambda_t = \{0, b, c\}$ and $\mu_t = \{1\}$. We have $\lambda_t \neq \varnothing$, $\mu_t \neq \varnothing$ and $\lambda_t \cap \mu_t = \varnothing$. Thus $\lambda \cap \mu = 0$.

Now we prove our main result.

**Theorem 4.3.** (*Generalization Of Fuzzy Stone's Theorem*) *Let $S$ be a bounded $\wedge$-sub semi lattice. Let $\lambda$ be a S-fuzzy ideal and $\gamma$ be a S-fuzzy $\wedge$-sub semi lattice of bounded distributive lattice $X$ such that $\lambda \cap \gamma = 0$. Then there exist a S-fuzzy prime ideal $\mu$ of $X$ such that $\lambda \leq \mu$ and $\mu \cap \gamma = 0$.*

*Proof.* As $\lambda \cap \gamma = 0$, there exist $t(\neq 1) \in S$ such that $\lambda_t \neq \varnothing$, $\gamma_t \neq \varnothing$ and $\lambda_t \cap \gamma_t = \varnothing$. As $\lambda$ is a S-fuzzy ideal of $X$ and $\lambda_t \neq \varnothing$, $\lambda_t$ is an ideal in $X$ (See Theorem 3.10 (I)). As $\gamma$ is S - fuzzy meet sub semi lattice and $\gamma_t \neq \varnothing$, $\gamma_t$ is a $\wedge$-sub semi lattice of $X$ (See Theorem 3.4). But then $\lambda_t \cap \gamma_t = \varnothing$ will imply the existence of a prime ideal $P$ of $X$ containing $\lambda_t$ and disjoint with $\gamma_t$ (By Result 2.8(II)).
Define $\mu : X \to S$ by,

$$\begin{aligned} \mu(x) &= 1 && \text{if } x \in P \\ &= \lambda(x) && \text{if } x \notin P. \end{aligned}$$

For $x \notin P$ and $\lambda_t \subseteq P$, $x \notin \lambda_t$ and hence $\lambda(x) \ngeq t$ and $t < 1$. Then by Theorem 3.18, $\mu$ is S-fuzzy prime ideal of $X$. Select $x \in X$. If $x \in P$, then obviously, $\lambda(x) \leq \mu(x) = 1$. If $x \notin P$ then $\mu(x) = \lambda(x)$. Thus in any case $\lambda(x) \leq \mu(x)$. Hence $\lambda \leq \mu$. Again for $x \notin P$ and $\lambda_t \subseteq P$, $x \notin \lambda_t$ and hence $\mu(x) = \lambda(x) \ngeq t$ and hence $\mu_t = P$. As $\lambda_t(\neq \varnothing) \subseteq P$, $\mu_t \neq \varnothing$. Therefore $P \cap \gamma t = \varnothing$ implies $\mu_t \cap \gamma_t = \varnothing$. This shows that $\mu \cap \gamma = 0$. □

As every S-fuzzy dual ideal $D$ of $X$ is a S-fuzzy $\wedge$-sub semi lattice of $X$, we obtain the fuzzification of Stone's Theorem [3] as follows.

**Corollary 4.4.** (*S-fuzzy prime ideal theorem*) *Let $S$ be a bounded $\wedge$ - sub semi lattice. Let $\lambda$ be a S-fuzzy ideal and $\gamma$ be a S-fuzzy dual ideal of bounded distributive lattice $X$ such that $\lambda \cap \gamma = 0$. Then there exist a S - fuzzy prime ideal $\mu$ of $X$ such that $\lambda \leq \mu$ and $\mu \cap \gamma = 0$.*

By taking $S = [0, 1]$ in particular, we get

**Corollary 4.5.** (*Fuzzy prime ideal theorem*) *Let $\lambda$ be a S-fuzzy ideal and $\gamma$ be a S-fuzzy dual ideal of bounded distributive lattice $X$ such that $\lambda \cap \gamma = 0$. Then there exist a S-fuzzy prime ideal $\mu$ of $X$ such that $\lambda \leq \mu$ and $\mu \cap \gamma = 0$.*

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