Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 6, No. 3, (November 2013), pp. 575–588
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Isomorphism on vague graphs

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Received 20 September 2012; Revised 20 November 2012; Accepted 18 March 2013

Abstract. The concept of vague sets is due to Gau and Buehrer who studied the concept with the aim of interpreting the real life problems in better way than the existing mechanisms such as Fuzzy sets. In this paper, we define weak isomorphism, co-weak isomorphism, and self-weak complementary on vague graphs and investigate some of their properties.

2010 AMS Classification: 03E72

Keywords: Vague graph, Self-weak complementary of vague graph, Co-weak isomorphism.

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1. Introduction

The theory of fuzzy sets proposed by Zadeh [26] has achieved a great success in various fields. Out of several higher order fuzzy sets, intuitionistic fuzzy sets introduced by atanassov [5, 6, 7] have been found to be highly useful to deal with vagueness. Rosenfeld [19] introduced the notion of fuzzy graphs in 1975 and proposed another definitions including paths, cycles, connectedness and etc. Mordeson and Peng [13] studied operations on fuzzy graphs in 1994. The complement of a fuzzy graph was defined by Mordeson and Nair [12] and further studied by Sunitha and Vijayakumar [23]. The concept of weak isomorphism, co-weak isomorphism and isomorphism between fuzzy graphs was introduced by Bhutani in [8]. Nagoorgani and Chandrasekaran [14], defined μ-Complement of a fuzzy graph. The concept of intuitionistic fuzzy (IF) relations and intuitionistic fuzzy graphs(IFG) was introduced in [20, 21]. Nagoor gani and Shajitha Begum discussed the various types of degrees and some properties of IFG in [15]. Sunitha analyzed the properties of self-centered fuzzy graph in [22], [24]. Parvathi and Karunambigai [16, 17] introduced the concept of min-max IFG and analyzed its properties, also they analyzed the concept of operations, complements of intuitionistic fuzzy graphs. Talebi and Rashmanlou [25] studied properties of isomorphism and complement on interval- valued fuzzy graphs. Akram and Davvaz discussed the properties of strong intuitionistic fuzzy graphs and also they introduced the concept of intuitionistic fuzzy line graphs in [1]. In [2], Akram and Dudek introduced the concept of bipolar fuzzy graphs and studied some properties of interval-valued fuzzy graphs in [3]. Recently Akram, Feng, Sarwar and Jun studied certain types of vague graphs in [4].

A vague set A on a set X is a pair (t_A, f_A) where t_A and f_A are real valued functions defined on $X \to [0,1]$, such that $t_A(x) + f_A(x) \leq 1$ for all $x \in X$. The interval $[t_A(x), 1 - f_A(x)]$ is called the vague value of x in A and it is denoted by $V_A(x)$. Vague relation μ from X to Y is a vague set of $X \times Y$, where X, Y are non-empty sets, defined as $\mu = (t_\mu, f_\mu)$ where $t_\mu : X \times Y \to [0,1]$, $f_\mu : X \times Y \to [0,1]$ which satisfies the condition $t_\mu(x,y) + f_\mu(x,y) \leq 1$ for all $(x,y) \in X \times Y$. A vague relation μ on the set N is a vague relation from N to N. If σ is a vague set on a set N, then a vague relation μ on σ is a vague relation on N which satisfies $t_\mu(x,y) \leq min\{t_\sigma(x),\ t_\sigma(y)\}$, $f_\mu(x,y) \geq max\{f_\sigma(x), f_\sigma(y)\}$, for all $x, y \in N$. Atanassov [7] studied Intuitionistic fuzzy sets which are mathematically equivalent to Vague sets. We prefer the terminology of vague sets in line with Gau and Buehrer [10]. The objective of this paper is to initiate the study of vague graphs with the hope that the study promises that these vague graphs serve as a better finer tool in understanding and interpreting several real life situations. A vague graph is a pair of functions $G = (\sigma, \mu)$ where σ is a vague set of a set of nodes N and μ is a vague relation on σ. In this paper, we define weak isomorphism, co-weak isomorphism, self-weak complementary on vague graphs and investigate some of their properties.

2. Preliminaries

A fuzzy graph with S, a non empty finite set as the underlying set is a pair $G = (\sigma, \mu)$ where $\sigma : S \to [0,1]$ is a fuzzy subset of S, $\mu : S \times S \to [0,1]$ is a symmetric fuzzy relation on the fuzzy subset σ such that $\mu(x,y) \leq \min(\sigma(x), \sigma(y)),\ \ \forall\, x, y \in S$. A fuzzy relation μ is symmetric if $\mu(x,y) = \mu(y,x)$ for all $x, y \in S$. The underlying crisp graph of the fuzzy graph $G = (\sigma, \mu)$ is denoted as $G^* = (\sigma^*, \mu^*)$ where $\sigma^* = \{x \in S \mid \sigma(x) > 0\}$ and $\mu^* = \{(x,y) \in S \times S \mid \mu(x,y) > 0\}$. If $\mu(x,y) > 0$, then x and y are called neighbors. For simplicity, an edge (x,y) will be denoted by xy.
We give here a review of some definitions which are in [8], [9], [10], [11], [18] and [23].

Definition 2.1. A fuzzy graph G is said to be a complete fuzzy graph if $\mu(x,y) = \sigma(x) \wedge \sigma(y),\ \ \forall\, x, y \in \sigma^*$, it is denoted as $K_\sigma = (\sigma, \mu)$.

Definition 2.2. A homomorphism of fuzzy graphs $h : G_1 \to G_2$ is a map $h : V_1 \to V_2$ which satisfies
$\sigma_1(x) \leq \sigma_2(h(x)), \forall\, x \in V_1$ and $\mu_1(xy) \leq \mu_2((h(x)h(y)), \forall\, x, y \in V_1$.

Definition 2.3. A weak isomorphism $h : G_1 \to G_2$ is a map $h : V_1 \to V_2$ which is a bijective homomorphism that satisfies $\sigma_1(x) = \sigma_2(h(x)), \forall\, x \in V_1$.

Definition 2.4. A co-weak isomorphism $h : G_1 \to G_2$ is a map $h : V_1 \to V_2$ which is a bijective homomorphism that satisfies $\mu_1(xy) = \mu_2((h(x)h(y))\forall\, x, y \in V_1$.

Definition 2.5. An isomorphism $h : G_1 \to G_2$ is a map $h : V_1 \to V_2$ which is bijective and satisfies $\sigma_1(x) = \sigma_2(h(x)) \forall\, x \in V_1$ and $\mu_1(xy) = \mu_2((h(x)h(y)), \forall\, x, y \in V_1$, and we denote $G_1 \cong G_2$.

Definition 2.6. A fuzzy graph G is said to be a self-complementary fuzzy graph if $G \cong \overline{G}$.

Definition 2.7. A vague set A on a set X is a pair (t_A, f_A) where $t_A : X \to [0, 1]$, $f_A : X \to [0, 1]$, with $t_A(x) + f_A(x) \leq 1$ for all $x \in X$. The mapping $t_A : X \to [0, 1]$, defines the degree of membership function and the mapping $f_A : X \to [0, 1]$ defines the degree of non-membership function of the element $x \in X$, the functions t_A and f_A should satisfy the condition $t_A \leq 1 - f_A$.

Definition 2.8. The interval $[t_A(x), 1 - f_A(x)]$ is called the vague value of $x \in X$, and it is denoted by $V_A(x)$. i.e., $V_A(x) = [t_A(x), 1 - f_A(x)]$.

Definition 2.9. A vague set A of set X with $t_A(x) = 0$ and $f_A(x) = 1$, $\forall x \in X$ is called zero vague set of X.

Definition 2.10. A vague set A of a set X with $t_A(x) = 1$ and $f_A(x) = 0$, $\forall x \in X$ is called the unit vague set of X.

Definition 2.11. A vague set A of a set X with $t_A(x) = \alpha$ and $f_A(x) = 1 - \alpha$, $\forall x \in X$ is called the $\alpha-$vague set of X, where $\alpha \in [0, 1]$.

Definition 2.12. A vague set A is contained in a vague set B, $A \subseteq B$ if and only if $t_A \leq t_B$ and $f_A \geq f_B$.

Definition 2.13. Let X and Y be any non-empty sets, a vague relation μ from X to Y is defined as a vague set of $X \times Y$, it is denoted by $\mu = (t_\mu, f_\mu)$ where $t_\mu : X \times Y \to [0, 1]$, $f_\mu : X \times Y \to [0, 1]$ which satisfies the condition $t_\mu(x, y) + f_\mu(x, y) \leq 1$, $\forall (x, y) \in X \times Y$. A vague relation μ on a set N is a vague relation from N to N. If σ is a vague set on a set N, then a vague relation μ on σ is a vague relation on N, which satisfies

$$t_\mu(x, y) \leq min\{t_\sigma(x), t_\sigma(y)\},$$

and $\forall x, y \in N$

$$f_\mu(x, y) \geq max\{f_\sigma(x), f_\sigma(y)\}.$$

Now, Ramakrishna introduced the following definition [18].

Definition 2.14. Let N be a non-empty set, members of N are called nodes. A vague graph $G = (\sigma, \mu)$ with N as the set of nodes, is a pair functions (σ, μ), where σ is a vague set of N and μ is a vague relation on σ.

Definition 2.15. The complement of a vague graph $G = (\sigma, \mu)$ is a vague graph $\overline{G} = (\overline{\sigma}, \overline{\mu})$, where $V_{\overline{\sigma}}(u) = V_\sigma(u)$, for all $u \in N$, and

$$V_{\overline{\mu}}(uv) = min\{V_\sigma(u), V_\sigma(v)\} - V_\mu(uv),\ \forall u, v \in N, \text{ i.e.,}$$

$$t_{\overline{\mu}}(uv) = min\{t_\sigma(u), t_\sigma(v)\} - t_\mu(uv),$$

$\forall u, v \in N$

$$f_{\overline{\mu}}(uv) = f_\mu(uv) - max\{f_\sigma(u), f_\sigma(v)\}.$$

Definition 2.16. A vague graph G is said to be self-complementary if $G \cong \overline{G}$.

3. Some operations on vague graphs

In this section we introduce self-weak complementary on vague graphs and investigate some of their properties.

Example 3.1. Suppose that $N = \{v_1, v_2, v_3, v_4\}$ be a set of nodes $G = (\sigma, \mu)$ is a vague graph, where $\sigma = (t_\sigma, f_\sigma)$ is a vague set on N given by

$$\sigma = \{(v_1, 0.1, 0.2), (v_2, 0.4, 0.5), (v_3, 0.25, 0.5), (v_4, 0.3, 0.6)\},$$

and

$$\mu = \{(v_1v_2, 0.1, 0.5), (v_2v_3, 0.2, 0.6), (v_1v_3, 0.1, 0.5), (v_3v_4, 0.25, 0.6),$$

$$(v_1v_4, 0.1, 0.6), (v_2v_4, 0.25, 0.6)\}.$$

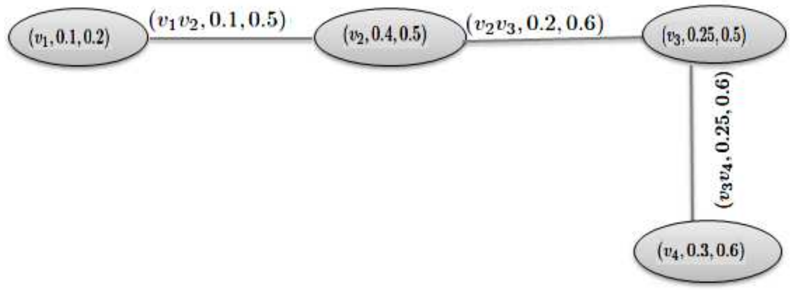

Figure 1. Vague graph G

Example 3.2. Suppose that $N = \{u_1, u_2, u_3, u_4, u_5\}$ be a set of nodes $G = (\sigma, \mu)$ is a vague graph, where $\sigma = (t_\sigma, f_\sigma)$ is a vague set on N given by

$$\sigma = \{(u_1, 0.5, 0.2), (u_2, 0.6, 0.3), (u_3, 0.25, 0.4), (u_4, 0.2, 0.7), (u_5, 0.1, 0.6)\}.$$

Vague relation on N is given by

$$\mu = \{(u_1u_2, 0.5, 0.3), (u_2u_3, 0.25, 0.4), (u_1u_5, 0.1, 0.7), (u_3u_4, 0.2, 0.7),$$

$$(u_4u_5, 0.1, 0.7), (u_1u_3, 0.2, 0.4), (u_2u_4, 0.1, 0.8), (u_5u_3, 0.2, 0.5),$$

$$(u_2u_5, 0.1, 0.7), (u_1u_4, 0.2, 0.7)\}.$$

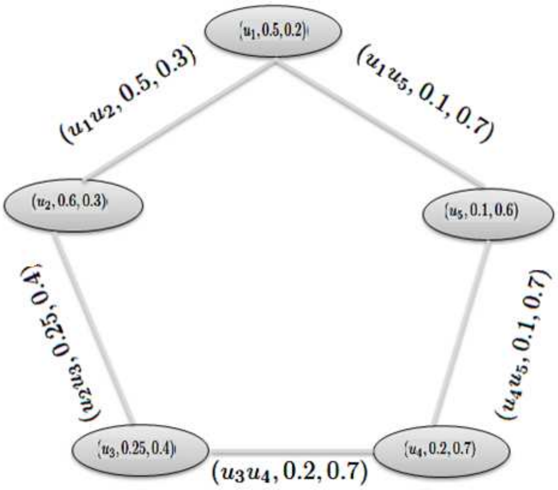

Figure 2. Vague graph G

Definition 3.3. A vague graph $G = (\sigma, \mu)$ is a strong vague graph if $V_\mu(uv) = min\{V_\sigma(u), V_\sigma(v)\}$, $\forall uv \in \mu^*$, i.e.,

$$t_\mu(uv) = min\{t_\sigma(u), t_\sigma(v)\},$$
$$f_\mu(uv) = max\{f_\sigma(u), f_\sigma(v)\}, \qquad \forall uv \in \mu^*.$$

Example 3.4. Let $N = \{u_1, u_2, u_3\}$ be a set of nodes. $G = (\sigma, \mu)$ is a vague graph, where $\sigma = (t_\sigma, f_\sigma)$ is a strong vague set on N given by

$$\sigma = \{(u_1, 0.25, 0.65), (u_2, 0.1, 0.85), (u_3, 0.15, 0.6)\},$$

and

$$\mu = \{(u_1u_2, 0.1, 0.85), (u_1u_3, 0.15, 0.65), (u_2u_3, 0.1, 0.85)\}.$$

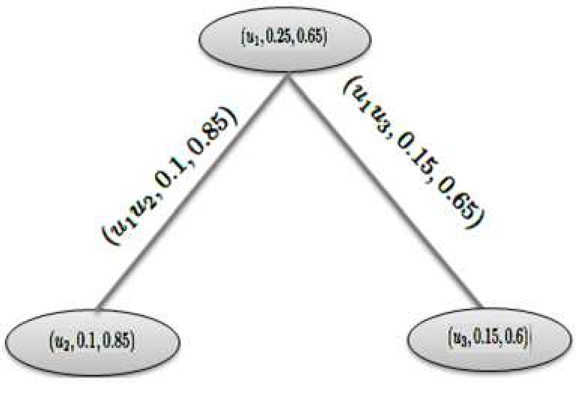

Figure 3. Strong vague graph G

Remark 3.5. Let $G = (\sigma, \mu)$ be a vague graph. We denote the underlying (crisp) graph of $G = (\sigma, \mu)$ by $G^* = (\sigma^*, \mu^*)$ where σ^* is referred to as the non-empty set

N of nodes and $\mu^* = E \subseteq N \times N$. i.e.,

$$\sigma^* = \{u \in N \mid V_\sigma(u) > 0\}, i.e., u \in \sigma^* \Leftrightarrow V_\sigma(u) > 0$$

$$\mu^* = \{uv \in N \times N \mid V_\mu(uv) > 0\}, i.e., uv \in \mu^* \Leftrightarrow V_\mu(uv) > 0$$

Definition 3.6. A vague graph is complete if

$$\begin{cases} V_\mu(xy) = min\{V_\sigma(x), V_\sigma(y)\} \\ t_\mu(xy) = min\{t_\sigma(x), t_\sigma(y)\} & \forall x, y \in \sigma^* \\ f_\mu(xy) = max\{f_\sigma(x), f_\sigma(y)\}. \end{cases}$$

Example 3.7. Let $N = \{u_1, u_2, u_3\}$ be a set of nodes. $G = (\sigma, \mu)$ is a vague graph, where $\sigma = (t_\sigma, f_\sigma)$ is a complete vague set on N given by

$$\sigma = \{(u_1, 0.3, 0.6), (u_2, 0.15, 0.75), (u_3, 0.1, 0.5)\},$$

and

$$\mu = \{(u_1u_2, 0.15, 0.75), (u_2u_3, 0.1, 0.75), (u_1u_3, 0.1, 0.6)\}.$$

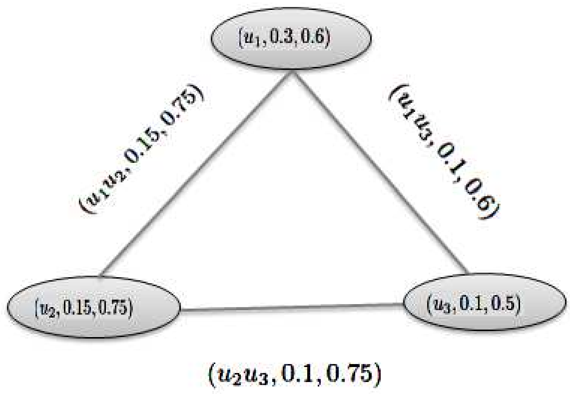

Figure 4. Complete vague graph G

Clearly the complete vague graph is strong vague graph.

Definition 3.8. Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two vague graphs. A homomorphism $f : G_1 \to G_2$ is a mapping $h : N_1 \to N_2$ such that:

(a) $V_{\sigma_1}(x_1) \le V_{\sigma_2}(h(x_1)), i.e., t_{\sigma_1}(x_1) \le t_{\sigma_2}(h(x_1)), f_{\sigma_1}(x_1) \ge f_{\sigma_2}(h(x_1))$.

(b) $V_{\mu_1}(x_1y_1) \le V_{\mu_2}(h(x_1)h(y_1)), i.e.,$

$$t_{\mu_1}(x_1y_1) \le t_{\mu_2}(h(x_1)h(y_1)),$$

$$f_{\mu_1}(x_1y_1) \ge f_{\mu_2}(h(x_1)h(y_1)). \qquad \forall x_1, y_1 \in N_1$$

A bijective homomorphism with the property

(c) $V_\sigma(x_1) = V_{\sigma_2}(h(x_1)), i.e., t_\sigma(x_1) = t_{\sigma_2}(h(x_1)), f_{\sigma_1}(x_1) = f_{\sigma_2}(h(x_1))$

is called a weak isomorphism.

A bijective homomorphism preserving the weights of the arcs but not necessarily the weights of the nodes, i.e. a bijective homomorphism $f : G_1 \to G_2$ such that:

(d) $V_{\mu_1}(x_1y_1) = V_{\mu_2}(h(x_1)h(y_1)), i.e.,$

$$t_{\mu_1}(x_1y_1) = t_{\mu_2}(h(x_1)h(y_1)),$$

$$f_{\mu_1}(x_1y_1) = f_{\mu_2}(h(x_1)h(y_1)). \qquad \forall x_1, y_1 \in N_1$$

is called a co-weak isomorphism. A bijective mapping $h : G_1 \to G_2$ satisfying (c) and (d) is called an isomorphism.

Theorem 3.9. *In a complete vague graph $G = (\sigma, \mu)$, we have:*

(i) $\sum_{x \neq y} t_\mu(xy) = \sum_{x \neq y} (t_\sigma(x) \wedge t_\sigma(y))$

(ii) $\sum_{x \neq y} f_\mu(xy) = \sum_{x \neq y} (f_\sigma(x) \vee f_\sigma(y))$.

Proof. It is clear. □

Theorem 3.10. *Isomorphism between vague graphs is an equivalence relation.*

Proof. It follows from the definitions. □

Definition 3.11. A vague graph G is said to be self weak complementary vague graph if G is weak isomorphic with it's complement $\overline{G}$, i.e., there is bijection mapping $h : G \to \overline{G}$, such that $V_\sigma(x) = V_{\overline{\sigma}}(h(x))$, for all $x \in N$, and $V_\mu(xy) \leq V_{\overline{\mu}}(h(x)h(y))$, $\forall x, y \in N$, i.e.

$$\begin{cases} t_\sigma(x) = t_{\overline{\sigma}}(h(x)) \\ f_\sigma(x) = f_{\overline{\sigma}}(h(x)) \end{cases}$$

and $\forall x, y \in N$

$$\begin{cases} t_\mu(xy) \leq t_{\overline{\mu}}(h(x)h(y) \\ f_\mu(xy) \geq f_{\overline{\mu}}(h(x)h(y)). \end{cases}$$

Theorem 3.12. *Weak isomorphism between vague graphs satisfies the partial relation.*

Proof. (i) Reflexive: Let G_1 be a vague graph with underlying set N_1. Consider the identity map $h : N_1 \to N_1$ that $h(v) = v$, $\forall v \in N_1$. This h is a bijective map satisfying:

$$\begin{cases} t_{\sigma_1}(v) = t_{\sigma_1}(h(v)) \\ \\ f_{\sigma_1}(v) = f_{\sigma_1}(h(v)) \end{cases} \qquad \forall v \in N_1$$

and (3.12.1)

$$\begin{cases} t_{\mu_1}(vu) \geq_{\mu_2} (h(v)h(u)) \\ \\ f_{\mu_1}(vu) = f_{\mu_2}(h(v)h(u)). \end{cases} \qquad \forall v, u \in N_1$$

Hence h is a weak isomorphism of the vague graph G_1 to itself.

(ii) Antisymmetric: Let h be a weak isomorphism between G_1 and G_2 and g be a weak isomorphism between G_2 and G_1, i.e. $h : N_1 \to N_2$ is a bijective map, $h(v) = v'$, $\forall v \in N_1, v' \in N_2$ satisfying:

$$\begin{cases} t_{\sigma_1}(v) = t_{\sigma_2}(h(v)) \\ \\ f_{\sigma_1}(v) = f_{\sigma_2}(h(v)) \end{cases} \qquad \forall v \in N_1$$

and (3.12.2)

$$\begin{cases} t_{\mu_2}(v'u') \leq t_{\mu_1}(g(v')g(u')) \\ \\ f_{\mu_2}(v'u') \geq f_{\mu_1}(g(v')g(u')). \end{cases} \qquad \forall v', u' \in N_2$$

The inequalities (3.12.1) and (3.12.2) hold good on the finite sets N_1 and N_2 only when G_1 and G_2 have the same number of edges and the corresponding edges have same weight. Hence G_1 and G_2 are identical.

(iii) Transitive: Let $h : N_1 \to N_2$ and $g : N_2 \to N_3$ be weak isomorphism of the vague graphs G_1 onto G_2 and G_2 onto G_3, respectively. Then goh is a 1-1, onto map from N_1 to N_3, where $goh(v) = g(h(v))$, $\forall v \in N_1$. As h is a weak isomorphism, $h(v) = v'$, $\forall v \in N_1, v' \in N_2$,

$$\begin{cases} t_{\sigma_1}(v) = t_{\sigma_2}(h(v)) \\ \\ f_{\sigma_1}(v) = f_{\sigma_2}(h(v)) \end{cases} \qquad \forall v \in N_1 \quad (3.12.3)$$

and

$$\begin{cases} t_{\mu_1}(vu) \leq t_{\mu_2}(h(v)h(u)) \\ \\ f_{\mu_1}(vu) \geq_{\mu_2} (h(v)h(u)). \end{cases} \qquad \forall v, u \in N_1 \quad (3.12.4)$$

As g is a weak isomorphism from N_2 to N_3, we have, $g(v') = v''$, $\forall v'' \in N_3, v' \in N_2$,

$$\begin{cases} t_{\sigma_2}(v') = t_{\sigma_3}(g(v')) \\ \\ f_{\sigma_2}(v') = f_{\sigma_3}(g(v')) \end{cases} \qquad \forall v' \in N_2$$

and (3.12.5)

$$\begin{cases} t_{\mu_2}(v'u') \leq t_{\mu_3}(g(v')g(u')) \\ \\ f_{\mu_2}(v'u') \geq f_{\mu_3}(g(v')g(u')). \end{cases} \qquad \forall v', u' \in N_2$$

From (3.12.3) and (3.12.5):

$$\begin{cases} t_{\sigma_1}(v) = t_{\sigma_2}(v') = t_{\sigma_3}(g(v')) = t_{\sigma_3}(g(h(v))) \\ \\ f_{\sigma_1}(v) = f_{\sigma_2}(v') = f_{\sigma_3}(g(v')) = f_{\sigma_3}(g(h(v))). \end{cases} \qquad (3.12.6)$$

From (3.12.3), (3.12.4), (3.12.5) and (3.12.6), we have:

$$\begin{cases} t_{\mu_1}(vu) \leq t_{\mu_2}(h(v)h(u)) \leq t_{\mu_3}(g(h(v))g(h(u))) \\ f_{\mu_1}(vu) \geq f_{\mu_2}(h(v)h(u)) \geq f_{\mu_3}(g(h(v))g(h(u))) \end{cases} \qquad \forall v, u \in N_1.$$

Therefore goh is a weak isomorphism between G_1 and G_3, i.e. weak isomorphism satisfies transitivity. Hence weak isomorphism between vague graphs is a partial order relation. □

Theorem 3.13. *Let $G = (\sigma, \mu)$ be a self weak complementary vague graph, then $\sum_{x \neq y} V_\mu(xy) \leq \frac{1}{2} \sum_{x \neq y} min\{(V_\sigma(x), V_\sigma(y))\}$, i.e.*

$$\begin{cases} \sum_{x \neq y} t_\mu(xy) \leq \frac{1}{2} \sum_{x \neq y} (t_\sigma(x) \wedge t_\sigma(y)) \\ \sum_{x \neq y} f_\mu(xy) \geq \frac{1}{2} \sum_{x \neq y} (f_\sigma(x) \vee f_\sigma(y)). \end{cases}$$

Proof. G is a self weak complementary vague graph. Hence G is weak-isomorphic with $\overline{G}$. So, there exists an isomorphism $h : N \to \overline{N}$, a bijective mapping satisfying:

$$\begin{cases} t_\sigma(x) = t_{\overline{\sigma}}(h(x)) = t_\sigma(h(x)) \\ f_\sigma(x) = f_{\overline{\sigma}}(h(x)) = f_\sigma(h(x)) \\ t_\mu(xy) \leq t_{\overline{\mu}}(h(x)h(y)) \\ f_\mu(xy) \geq f_{\overline{\mu}}(h(x)h(y)). \end{cases}$$

Using the definition of complement in the above inequality,

$$\begin{cases} t_{\overline{\mu}}(h(x)h(y)) = t_\sigma(h(x)) \wedge t_\sigma(h(y)) - t_\mu(h(x)h(y)) \\ f_{\overline{\mu}}(h(x)h(y)) = f_\mu(h(x)h(y)) - f_\sigma(h(x)) \vee f_\sigma(h(y)), \end{cases}$$

$$\begin{cases} t_\mu(xy) \leq t_{\overline{\mu}}(h(x)h(y)) = t_\sigma(h(x)) \wedge t_\sigma(h(y)) - t_\mu(h(x)h(y)) \\ \qquad\qquad\qquad\qquad\quad = t_\sigma(x) \wedge t_\sigma(y) - t_\mu(h(x), h(y)) \\ f_\mu(xy) \geq f_{\overline{\mu}}(h(x)h(y)) = f_\mu(h(x)h(y)) - f_\sigma(h(x)) \vee f_\sigma(h(y)) \\ \qquad\qquad\qquad\qquad\quad = f_\mu(h(x)h(y)) - f_\sigma(x) \vee f_\sigma(y), \end{cases}$$

$$\Rightarrow \begin{cases} t_\mu(xy) + t_\mu(h(x)h(y)) \leq t_\sigma(x) \wedge t_\sigma(y) \\ f_\mu(xy) + f_\sigma(x) \vee f_\sigma(y) \geq f_\mu(h(x)h(y)), \\ f_\mu(xy) + f_\sigma(x) \vee f_\sigma(y) \geq f_\mu(h(x)h(y)) \geq f_\sigma(h(x)) \vee f_\sigma(h(y)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = f_\sigma(x) \vee f_\sigma(y) \end{cases}$$

$$\Rightarrow \begin{cases} \sum\limits_{x\neq y} t_\mu(xy) + \sum\limits_{x\neq y} t_\mu(h(x)h(y)) \leq \sum\limits_{x\neq y}(t_\sigma(x) \wedge t_\sigma(y)) \\ \sum\limits_{x\neq y} f_\mu(xy) + \sum\limits_{x\neq y} f_\mu(xy) \geq \sum\limits_{x\neq y} f_\mu(xy) + \sum\limits_{x\neq y} f_\sigma(x) \vee f_\sigma(y) \\ \qquad\qquad\qquad\qquad\qquad \geq \sum_{x\neq y}(f_\sigma(x) \vee f_\sigma(y)), \end{cases}$$

$$\Rightarrow \begin{cases} 2\sum\limits_{x\neq y} t_\mu(xy) \leq \sum\limits_{x\neq y}(t_\sigma(x) \wedge t_\sigma(y)) \\ 2\sum\limits_{x\neq y} f_\mu(xy) \geq \sum\limits_{x\neq y}(f_\sigma(x) \vee f_\sigma(y)). \end{cases}$$

$$\Rightarrow \begin{cases} \sum\limits_{x\neq y} t_\mu(xy) \leq \frac{1}{2}\sum\limits_{x\neq y}(t_\sigma(x) \wedge t_\sigma(y)) \\ \sum\limits_{x\neq y} f_\mu(xy) \geq \frac{1}{2}\sum\limits_{x\neq y}(f_\sigma(x) \vee f_\sigma(y)). \end{cases}$$ □

Theorem 3.14. *Let $G = (\sigma, \mu)$ be a vague graph. If*

$$\begin{cases} t_\mu(xy) \leq \frac{1}{2}(t_\sigma(x) \wedge t_\sigma(y)) \\ f_\mu(xy) \geq \frac{3}{2}(f_\sigma(x) \vee f_\sigma(y)), \end{cases} \qquad \forall x, y \in N$$

then G is a self weak complementary vague graph.

Proof. Consider the identity map $h : N \to N$, $\begin{cases} V_\sigma(x) = V_\sigma(h(x)) \\ t_\sigma(x) = t_\sigma(h(x)) \\ f_\sigma(x) = f_\sigma(h(x)). \end{cases} \quad \forall x \in N$

By definition of $V_{\overline{\mu}}(x)$, $t_{\overline{\mu}}(x)$ and $f_{\overline{\mu}}$, we have:

$$\begin{cases} t_{\overline{\mu}}(xy) = t_\sigma(x) \wedge t_\sigma(y) - t_\mu(xy) \\ \\ f_{\overline{\mu}}(xy) = f_\mu(xy) - f_\sigma(x) \vee f_\sigma(y), \end{cases} \quad \forall x, y \in N$$

$$\begin{cases} t_{\overline{\mu}}(xy) \geq t_\sigma(x) \wedge t_\sigma(y) - \frac{1}{2}(t_\sigma(x) \wedge t_\sigma(y)) = \frac{1}{2}(t_\sigma(x) \wedge t_\sigma(y)) \geq t_\mu(xy) \\ f_{\overline{\mu}}(xy) \leq \frac{3}{2}(f_\sigma(x) \vee f_\sigma(y)) - f_\sigma(x) \vee f_\sigma(y) = \frac{1}{2}(f_\sigma(x) \vee f_\sigma(y)) \leq f_\mu(xy), \end{cases}$$

i.e. $\begin{cases} t_\mu(xy) \leq t_{\overline{\mu}}(h(x)h(y)) \\ f_\mu(xy) \geq f_{\overline{\mu}}(h(x)h(y)). \end{cases}$

This is complete the proof. □

Theorem 3.15. *Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be vague graphs. Then $G_1 \cong G_2$ if and only if $\overline{G_1} \cong \overline{G_2}$.*

Proof. It is obvious. □

Definition 3.16. The size of $G = (\sigma, \mu)$ is defined to be $S(G) = (S_t(G), S_f(G)) = \sum_{x,y \in N} V_\mu(xy)$, i.e., $S_t(G) = \sum_{x,y \in N} t_\mu(xy)$ and $S_f(G) = \sum_{x,y \in N} f_\mu(xy)$.

Theorem 3.17. *If $G = (\sigma, \mu)$ be vague graph, we have:*

$$S_t(G) + S_t(\overline{G}) \leq 2 \sum_{x \neq y} min\{t_\sigma(x), t_\sigma(y)\}.$$

Proof. (i) : $t_\mu(xy) \leq min\{t_\sigma(x), t_\sigma(y)\}$ (3.17.7)

$$t_{\overline{\mu}}(xy) = min\{t_\sigma(x), t_\sigma(y)\} - t_\mu(xy)$$

$$\Rightarrow t_{\overline{\mu}}(xy) + t_\mu(xy) = min\{t_\sigma(x), t_\sigma(y)\}. \quad (3.17.8)$$

Also, $t_{\overline{\mu}}(xy) \leq min\{t_\sigma(x), t_\sigma(y)\}$. (3.17.9)

From (3.17.7), (3.17.8) and (3.17.9), we have:

$min\{t_\sigma(x), t_\sigma(y)\} = t_\mu(xy) + t_{\overline{\mu}}(xy) \leq 2\ min\{t_\sigma(x), t_\sigma(y)\}$,

i.e. $t_\mu(xy) + t_{\overline{\mu}}(xy) \leq 2\ min\{t_\sigma(x), t_\sigma(y)\}$.

$$\Rightarrow \sum_{x \neq y} t_\mu(xy) + \sum_{x \neq y} t_{\overline{\mu}}(xy) \leq 2 \sum_{x \neq y} min\{t_\sigma(x), t_\sigma(y)\},$$

$$\Rightarrow S_t(G) + S_t(\overline{G}) \leq 2 \sum_{x \neq y} min\{t_\sigma(x), t_\sigma(y)\}.$$ □

Remark 3.18. If there is a weak isomorphism between two vague graphs G_1 and G_2, then there need not to be a weak isomorphism between $\overline{G_1}$ and $\overline{G_2}$.

Consider the following example.

Example 3.19. Suppose that $N_1 = \{u_1, u_2, u_3, u_4\}$ be a set of nodes $G_1 = (\sigma_1, \mu_1)$ is a vague graph by

$$\sigma_1 = \{(u_1, 0.2, 0.4), (u_2, 0.3, 0.5), (u_3, 0.3, 0.5), (u_4, 0.2, 0.6)\}.$$

Vague relation on N_1 is given by

$\mu_1 = \{(u_1u_2, 0.1, 0.5), (u_1u_3, 0.1, 0.6), (u_3u_2, 0.2, 0.6), (u_2u_4, 0.1, 0.7), (u_3u_4, 0.1, 0.7)\}.$

Let $N_2 = \{v_1, v_2, v_3, v_4\}$ be a set of nodes $G_1 = (\sigma_1, \mu_1)$ is a vague graph by

$$\sigma_2 = \{(v_1, 0.2, 0.4), (v_2, 0.3, 0.5), (v_3, 0.3, 0.5), (v_4, 0.2, 0.6)\}.$$

Vague relation on N_2 is given by

$$\mu_2 = \{(v_1v_2, 0.15, 0.5), (v_2v_3, 0.25, 0.55), (v_1v_3, 0.15, 0.55), (v_2v_4, 0.1, 0.65), (v_3v_4, 0.1, 0.65)\}.$$

There is a weak isomorphism $h : N_1 \to N_2$ such that $h(u_i) = v_i, 1 \le i \le 4$ and we have

$\overline{\mu_1} = \{(u_1u_2, 0.1, 0), (u_1u_3, 0.1, 0.1), (u_3u_2, 0.1, 0.1), (u_2u_4, 0.1, 0.1), (u_3u_4, 0.1, 0.1)\}.$

$$\overline{\mu_2} = \{(v_1v_2, 0.05, 0), (v_2v_3, 0.05, 0.05), (v_1v_3, 0.05, 0.05), (v_2v_4, 0.1, 0.05), (v_3v_4, 0.1, 0.05)\}.$$

Hence there is not a weak isomorphism between $\overline{G_1}$ and $\overline{G_2}$.

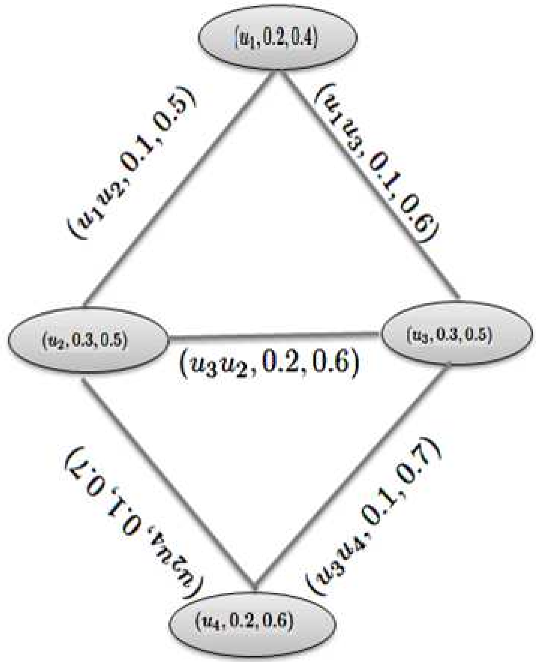

Figure 5. Vague graph G_1

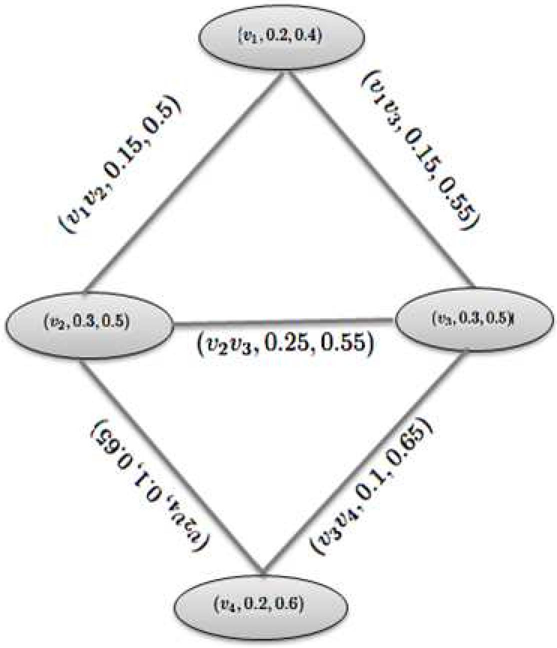

Figure 6. Vague graph G_2

Remark 3.20. If there is a co-weak isomorphism between vague graphs G_1, G_2, then there need not to be a homomorphism between $\overline{G_1}$ and $\overline{G_2}$.

The following example proves the above sentence.

Example 3.21. Suppose that $N_1 = \{u_1, u_2, u_3\}$ be a set of nodes $G_1 = (\sigma_1, \mu_1)$ is a vague graph by

$$\sigma_1 = \{(u_1, 0.3, 0.1), (u_2, 0.5, 0.4), (u_3, 0.4, 0.1)\}.$$

Vague relation on N_1 is given by

$$\mu_1 = \{(u_1u_2, 0.2, 0.6), (u_1u_3, 0.3, 0.3), (u_3u_2, 0.3, 0.6)\}.$$

Let $N_2 = \{v_1, v_2, v_3\}$ be a set of nodes $G_1 = (\sigma_1, \mu_1)$ is a vague graph by

$$\sigma_2 = \{(v_1, 0.4, 0.05), (v_2, 0.6, 0.3), (v_3, 0.5, 0.05)\}.$$

Vague relation on N_2 is given by

$$\mu_2 = \{(v_1v_3, 0.3, 0.3), (v_2v_3, 0.3, 0.6), (v_1v_2, 0.2, 0.6)\}.$$

There is a co-weak isomorphism $g : N_1 \to N_2$ such that $g(u_i) = v_i, 1 \leq i \leq 3$ and we have

$$\overline{\mu_1} = \{(u_1u_2, 0.1, 0.2), (u_1u_3, 0, 0.2), (u_3u_2, 0.1, 0.2)\}.$$

$$\overline{\mu_2} = \{(v_1v_3, 0.1, 0.25), (v_2v_3, 0.2, 0.3), (v_1v_2, 0.2, 0.3)\}.$$

Hence there is not a homomorphism between $\overline{G_1}$ and $\overline{G_2}$.

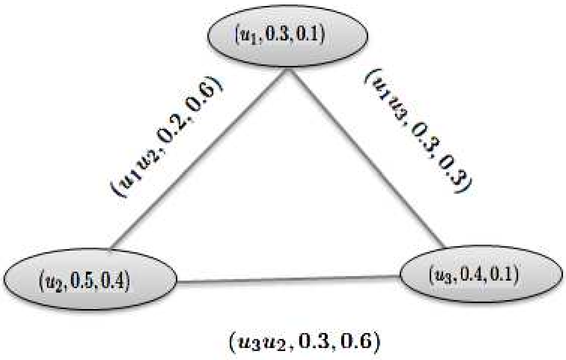

Figure 7. Vague graph G_1

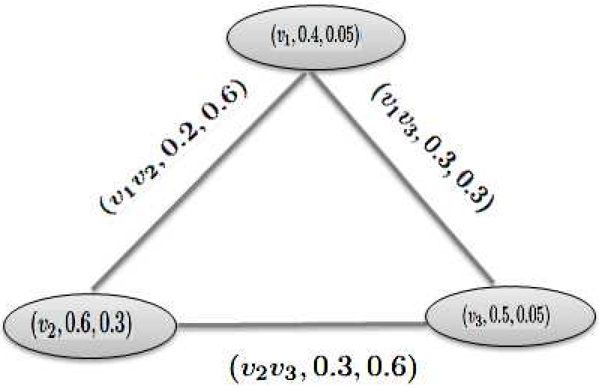

Figure 8. Vague graph G_2

4. Conclusions

Graph theory is an extremely useful tool in solving the combinatorial problems in different areas including geometry, algebra, number theory, topology, operations research, optimization, and computer science. The concept of vague sets is due to Gau and Buehrer who studied the concept with the aim of interpreting the real life problems in better way than the existing mechanisms such as Fuzzy sets. In this paper, we have introduced weak isomorphism, co-weak isomorphism, and self-weak complementary on vague graphs and investigate some of their properties.

Acknowledgements. The authors would like to thank the reviewers for useful comments and suggestions.

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