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Compactifying the soft fuzzy product generalized topological space

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Abstract. In this paper, the soft fuzzy product generalized topology is introduced. Properties of product associated maps are studied. In this connection, soft fuzzy $G_\delta$pre quotient product maps with relevant properties are discussed. Moreover, compactification on soft fuzzy product generalized topological space is established.

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1. Introduction

The fuzzy concept has penetrated almost all branches of Mathematics since the introduction of the concept of fuzzy set by Zadeh [8]. Fuzzy sets have applications in many fields such as information [4] and control [5]. The theory of fuzzy topological spaces was introduced and developed by C. L. Chang [2].

The idea of Fuzzy soft sets, introduced and developed by P. K. Maji, R. Biswas, A. R. Roy [3]. The notions of Soft fuzzy set over a poset I and soft fuzzy topological space was introduced by Ismail U. Tiryaki [6].

In this paper, a new class of set, called Soft fuzzy set for $\mathcal{Q}(X_1 \times X_2)$ is established. A product map which is associated with the some product map is defined and some of the properties are studied. In this connection, a compactification on the soft fuzzy product generalized topological space is established.

2. Preliminaries

Definition 2.1 ([6]). Let X be a nonempty set. Let μ be a fuzzy subset of X such that $\mu : X \to [0,1]$ and M be any crisp subset of X. Then, the ordered pair (μ, M) is called a *soft fuzzy set* in X. The family of all soft fuzzy subsets of X,will be denoted by *SF(X)*.

Definition 2.2 ([6]). Let X be a non-empty set. Then, the *complement* of a soft fuzzy set (μ, M) is defined as $(\mu, M)' = (1 - \mu, X|M)$

Definition 2.3 ([6]). Let X be a non-empty set and the soft fuzzy sets A and B be in the form,

$$A = \{(\mu, M) : \mu(x) \in I^X, \forall x \in X, M \subseteq X\}$$
$$B = \{(\lambda, N) : \lambda(x) \in I^X, \forall x \in X, N \subseteq X\}$$

Then,

(1) $A \sqsubseteq B \Leftrightarrow \mu(x) \leq \lambda(x), \forall x \in X, M \subseteq N.$
(2) $A = B \Leftrightarrow \mu(x) = \lambda(x), \forall x \in X, M = N.$
(3) $A \sqcap B \Leftrightarrow \mu(x) \wedge \lambda(x), \forall x \in X, M \cap N.$
(4) $A \sqcup B \Leftrightarrow \mu(x) \vee \lambda(x), \forall x \in X, M \cup N.$

Proposition 2.4 ([6]). *Let $f : X \to Y$ be a function. If (λ, N) is a soft fuzzy set in Y, then its pre-image under f, denoted $f^{-1}(\lambda, N)$ is defined as,*

$$f^{-1}(\lambda, N) = (\lambda \circ f, f^{-1}(N)$$

where, $f^{-1}(N) = \{x \in X : f(x) = y, for\, y \in N\}$.

Proposition 2.5 ([6]). *Let $f : X \to Y$ be a function. If (μ, M) is a soft fuzzy set in X, then its image under f, denoted $f(\mu, M)$ is defined as,*

$$f(\mu, M) = (\gamma, L)$$

where, $\gamma(y) = f(\mu)(y) = sup\{\mu(x) : x \in f^{-1}(y)\}$

$$L = \{f(x) : x \in M\}.$$

Definition 2.6 ([6]). A soft fuzzy topology on a non-empty set X is a family T of soft fuzzy sets in X satisfying the following axioms:

(1) $(0, \phi), (1, X) \in T$.
(2) For any family of soft fuzzy sets $(\lambda_j, N_j) \in T, j \in J, \sqcup_{j\in J}(\lambda_j, N_j) \in T$.
(3) For any finite number of soft fuzzy sets $(\lambda_j, N_j) \in T, j = 1, 2, 3, ...n$, $\sqcap_{j=1}^{n}(\lambda_j, N_j) \in T$. Then, the pair (X, T) is called a *soft fuzzy topological space* (in short, SFTS).

Any soft fuzzy set in T is said to be a *soft fuzzy open set* (in short, *SFOS*) in X. The complement of SFOS in a SFTS (X, T) is called as a *soft fuzzy closed set*, denoted *SFCS* in X.

Example 2.7. Let $X = \{a, b, c\}$ and the soft fuzzy topology on X is given by $T = \{(0, \phi), (1, X), (\mu_1, M_1), (\mu_2, M_2), (\mu_3, M_3), (\mu_4, M_4)\}$ where each $(\mu_i, M_i)(i = 1, 2, 3, 4)$ is a soft fuzzy set defined as follows $\mu_1 : X \to I \ni \mu_1(a) = 0.2$, $\mu_1(b) = 0.3$, $\mu_1(c) = 0.7$ and $M_1 = \{a\} \subset X$; $\mu_2 : X \to I \ni \mu_2(a) = 0.4$, $\mu_2(b) = 0.5$, $\mu_2(c) = 0.2$ and $M_2 = \{a, b\} \subset X$; $\mu_3 : X \to I \ni \mu_3(a) = 0.4$, $\mu_3(b) = 0.5$, $\mu_3(c) = 0.7$ and $M_3 = \{a, b\} \subset X$; $\mu_4 : X \to I \ni \mu_4(a) = 0.2$, $\mu_4(b) = 0.3$, $\mu_4(c) = 0.2$ and $M_4 =$

$\{a\} \subset X$. The members of T are called soft fuzzy open set and their complements are called soft fuzzy closed sets. The pair (X, T) is a soft fuzzy topological space.

Definition 2.8 ([1]). The *product* $\lambda \times \mu$ *of a fuzzy set* λ of X and a fuzzy set μ of Y is a fuzzy set of $X \times Y$, defined by $(\lambda \times \mu) < x, y > = min(\lambda(x), \mu(y))$, for each $< x, y > \in X \times Y$.

Example 2.9. Let $X = \{a, b, c\}$ and λ be a fuzzy set of X which is defined by $\lambda : X \to I \ni \lambda(a) = 0.4$, $\lambda(b) = 0.7$, $\lambda(c) = 0.3$. Let $Y = \{p, q\}$ and μ be a fuzzy set of Y which is defined by $\mu : Y \to I \ni \mu(p) = 0.3$, $\mu(q) = 0.8$. Now $\lambda \times \mu$ is a fuzzy set of $X \times Y$ and is defined as follows $\lambda \times \mu : X \times Y \to I \ni (\lambda \times \mu)(< a, p >) = min(\lambda(a), \mu(p)) = 0.3$, $(\lambda \times \mu)(< a, q >) = min(\lambda(a), \mu(q)) = 0.4$, $(\lambda \times \mu)(< b, p >) = min(\lambda(b), \mu(p)) = 0.3$, $(\lambda \times \mu)(< b, q >) = min(\lambda(b), \mu(q)) = 0.7$, $(\lambda \times \mu)(< c, p >) = min(\lambda(c), \mu(p)) = 0.3$, $(\lambda \times \mu)(< c, q >) = min(\lambda(c), \mu(q)) = 0.3$.

Definition 2.10 ([1]). The *product* $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ *of mappings* $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, defined by $(f_1 \times f_2) < x_1, x_2 > = (f_1(x_1), f_2(x_2))$, for each $< x_1, x_2 > \in X_1 \times X_2$.

Definition 2.11 ([7]). Let (X,T) be a soft fuzzy topological space. Let (λ, N) be any soft fuzzy set. Then (λ, N) is said to be *soft fuzzy* G_δ *pre open set* if $(\lambda, N) = (\mu, M) \sqcap (\gamma, L)$, where (μ, M) is soft fuzzy G_δ set and (γ, L) is soft fuzzy pre open set. The complement of a soft fuzzy G_δ pre open set is soft fuzzy F_σ pre closed.

Example 2.12. Let (X, T) be a soft fuzzy topological space. Let $X = \{a, b, c\}$ and $T = \{(0, \phi), (1, X), (\mu_1, M_1), (\mu_2, M_2), (\mu_3, M_3), (\mu_4, M_4)\}$ where each (μ_i, M_i)(i = 1,2,3,4) is a soft fuzzy set defined as follows $\mu_1 : X \to I \ni \mu_1(a) = 0.2$, $\mu_1(b) = 0.3$, $\mu_1(c) = 0.7$ and $M_1 = \{a\} \subset X$; $\mu_2 : X \to I \ni \mu_2(a) = 0.4$, $\mu_2(b) = 0.5$, $\mu_2(c) = 0.2$ and $M_2 = \{a, b\} \subset X$; $\mu_3 : X \to I \ni \mu_3(a) = 0.4$, $\mu_3(b) = 0.5$, $\mu_3(c) = 0.7$ and $M_3 = \{a, b\} \subset X$; $\mu_4 : X \to I \ni \mu_4(a) = 0.2$, $\mu_4(b) = 0.3$, $\mu_4(c) = 0.2$ and $M_4 = \{a\} \subset X$. Now (λ_1, M_1) defined by $\lambda_1 : X \to I \ni \lambda_1(a) = 0.1$, $\lambda_1(b) = 0.6$, $\lambda_1(c) = 0.4$, and $M_1 = \{a\}$ is a soft fuzzy pre open set and (μ_1, M_1) is a soft fuzzy G_δ set. Now $(\mu_1, M_1) \sqcap (\lambda_1, M_1) = (\gamma_1, M_1)$ where $\gamma_1(a) = 0.1$, $\gamma_1(b) = 0.3$, $\gamma_1(c) = 0.4$ and $M_1 = \{a\}$ is a soft fuzzy G_δ pre open set in (X, T).

Definition 2.13 ([7]). Let (X,T) and (Y,S) be any two soft fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is said to be *soft fuzzy* G_δ*-pre continuous*, if the inverse image of every soft fuzzy open set in (Y,S) is soft fuzzy G_δ pre open in (X,T).

Example 2.14. Let (X, T) be a soft fuzzy topological space. Let $X = \{a, b, c\}$ and $T = \{(0, \phi), (1, X), (\mu_1, M_1), (\mu_2, M_2), (\mu_3, M_3), (\mu_4, M_4)\}$ where each (μ_i, M_i)(i = 1,2,3,4) is a soft fuzzy set defined as follows $\mu_1 : X \to I \ni \mu_1(a) = 0.2$, $\mu_1(b) = 0.3$, $\mu_1(c) = 0.7$ and $M_1 = \{a\} \subset X$; $\mu_2 : X \to I \ni \mu_2(a) = 0.4$, $\mu_2(b) = 0.5$, $\mu_2(c) = 0.2$ and $M_2 = \{a, b\} \subset X$; $\mu_3 : X \to I \ni \mu_3(a) = 0.4$, $\mu_3(b) = 0.5$, $\mu_3(c) = 0.7$ and $M_3 = \{a, b\} \subset X$; $\mu_4 : X \to I \ni \mu_4(a) = 0.2$, $\mu_4(b) = 0.3$, $\mu_4(c) = 0.2$ and $M_4 = \{a\} \subset X$. Let (Y, S) be a soft fuzzy topological space. Let $Y = \{p, q, r\}$ and $S = \{(0, \phi), (1, Y), (\lambda_1, N_1)\}$ where (λ_1, N_1) is a soft fuzzy set defined as follows $\lambda_1 : Y \to I \ni \lambda_1(p) = 0.1$, $\lambda_1(q) = 0.6$, $\lambda_1(r) = 0.4$ and $N_1 = \{p\}$. Let $f : (X, T) \to$

(Y, S) defined by $f(a) = p$, $f(b) = q$, $f(c) = r$. Since the inverse image of every soft fuzzy open set in (Y, S) is soft fuzzy $G_\delta$pre open in (X, T). Thus f is a soft fuzzy $G_\delta$pre continuous.

Definition 2.15 ([7])**.** Let (X,T) and (Y,S) be any two soft fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is said to be *soft fuzzy G_δ-pre irresolute*, if the inverse image of every soft fuzzy G_δ pre open set in (Y,S) is soft fuzzy G_δ pre open in (X,T).

3. On soft fuzzy product map

Definition 3.1. Let $< x_1, x_2 >\in X_1 \times X_2$ and $\lambda : X_1 \times X_2 \to [0, 1]$. Define,

$$< x_1, x_2 >_\lambda (< y_1, y_2 >) = \begin{cases} \lambda\ (0 < \lambda \leq 1) & \text{if } < x_1, x_2 >=< y_1, y_2 >; \\ 0 & \text{otherwise.} \end{cases}$$

Then, the soft fuzzy set $(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})$ is called as the *soft fuzzy point* (inshort, *SFP*) in $SF(X_1 \times X_2)$, with *support,* $< x_1, x_2 >$ and *value,* λ.

Definition 3.2. *Soft fuzzy product set* $(\lambda_1, N_1) \times (\lambda_2, N_2)$ is defined as

$$(\lambda_1, N_1) \times (\lambda_2, N_2) = (\lambda_1 \times \lambda_2, N_1 \times N_2).$$

Example 3.3. Let $X = \{a, b, c\}$ and (λ, N) be a soft fuzzy set of X which is defined by $\lambda : X \to I \ni \lambda(a) = 0.4$, $\lambda(b) = 0.7$, $\lambda(c) = 0.3$ and $N = \{a, b\}$. Let $Y = \{p, q\}$ and (μ, M) be a soft fuzzy set of Y which is defined by $\mu : Y \to I \ni \mu(p) = 0.3$, $\mu(q) = 0.8$ and $M =\{p\}$. Now $(\lambda\times\mu, N\times M)$ is a soft fuzzy product set of $X\times Y$ and is defined as follows $\lambda\times\mu : X\times Y \to I \ni (\lambda\times\mu)(< a, p >) = min(\lambda(a), \mu(p)) = 0.3$, $(\lambda\times\mu)(< a, q >) = min(\lambda(a), \mu(q)) = 0.4$, $(\lambda\times\mu)(< b, p >) = min(\lambda(b), \mu(p)) = 0.3$, $(\lambda\times\mu)(< b, q >) = min(\lambda(b), \mu(q)) = 0.7$, $(\lambda\times\mu)(< c, p >) = min(\lambda(c), \mu(p)) = 0.3$, $(\lambda \times \mu)(< c, q >) = min(\lambda(c), \mu(q)) = 0.3$ and $N \times M = \{< a, p >, < b, p >\}$.

Definition 3.4. Let (X_1, T_1) and (X_2, T_2) be any two soft fuzzy topological spaces. The collection $\mathcal{B} = \{(\lambda_1 \times \lambda_2, N_1 \times N_2) : (\lambda_1, N_1) \in T_1, (\lambda_2, M_2) \in T_2$ and $N_1 \times N_2 \subseteq X_1 \times X_2\}$ forms a soft fuzzy open base of a soft fuzzy topology in $X_1 \times X_2$.

The soft fuzzy topology in $X_1 \times X_2$, induced by $\mathcal{B}$ is called as the *soft fuzzy product topology* of T_1 and T_2, denoted by $T_1 \times T_2$.

The ordered pair $(X_1 \times X_2, T_1 \times T_2)$, which means the product of (X_1, T_1) and (X_2, T_2), is called the *soft fuzzy product topological space.*

Example 3.5. Let $X = \{a, b, c\}$ and $T = \{(\lambda_1, N_1), (\lambda_2, N_2), (\lambda_3, N_3)\}$ where (λ_i, N_i) (i=1, 2,3) is defined as follows $\lambda_1 : X \to I \ni \lambda_1(a) = 0$, $\lambda_1(b) = 0$, $\lambda_1(c) = 0$ and $N_1 = \phi$; $\lambda_2 : X \to I \ni \lambda_2(a) = 1$, $\lambda_2(b) = 1$, $\lambda_2(c) = 1$ and $N_2 = X$; $\lambda_3 : X \to I \ni \lambda_3(a) = 0.3$, $\lambda_3(b) = 0.4$, $\lambda_3(c) = 0.2$ and $N_3 = \{a, b\}$. Let $Y = \{p, q\}$ and $S = \{(\mu_1, M_1), (\mu_2, M_2), (\mu_3, M_3)\}$ where (μ_i, M_i) (i=1, 2,3) is defined as follows $\mu_1 : Y \to I \ni \mu_1(p) = 0$, $\mu_1(q) = 0$ and $M_1 = \phi$; $\mu_2 : Y \to I \ni \mu_2(p) = 1$, $\mu_2(q) = 1$ and $M_2 = Y$; $\mu_3 : Y \to I \ni \mu_3(p) = 0.4$, $\mu_3(q) = 0.2$ and $M_3 = Y$. $\mathcal{B} = \{(\lambda_1\times\mu_1, N_1\times M_1), (\lambda_1\times\mu_2, N_1\times M_2), (\lambda_1\times\mu_3, N_1\times M_3), (\lambda_2\times\mu_1, N_2\times M_1), (\lambda_2\times \mu_2, N_2 \times M_2), (\lambda_2 \times \mu_3, N_2 \times M_3), (\lambda_3 \times \mu_1, N_3 \times M_1), (\lambda_3 \times \mu_2, N_3 \times M_2), (\lambda_3 \times \mu_3, N_3 \times M_3)\}$ where $(\lambda_1 \times \mu_1, N_1 \times M_1)$ is defined by $\lambda_1 \times \mu_1 : X \times Y \to I \ni (\lambda_1 \times \mu_1)(< a, p >) = 0$, $(\lambda_1 \times \mu_1)(< a, q >) = 0$, $(\lambda_1 \times \mu_1)(< b, p >) = 0$, $(\lambda_1 \times \mu_1)(< b, q >) = 0$, $(\lambda_1 \times \mu_1)(< c, p >) = 0$, $(\lambda_1 \times \mu_1)(< c, q >) = 0$

and $N_1 \times M_1 = \phi$. $(\lambda_1 \times \mu_2, N_1 \times M_2)$ is defined by $\lambda_1 \times \mu_2 : X \times Y \to I \ni (\lambda_1 \times \mu_2)(< a,p >) = 0$, $(\lambda_1 \times \mu_2)(< a,q >) = 0$, $(\lambda_1 \times \mu_2)(< b,p >) = 0$, $(\lambda_1 \times \mu_2)(< b,q >) = 0$, $(\lambda_1 \times \mu_2)(< c,p >) = 0$, $(\lambda_1 \times \mu_2)(< c,q >) = 0$ and $N_1 \times M_2 = \phi$. $(\lambda_1 \times \mu_3, N_1 \times M_3)$ is defined by $\lambda_1 \times \mu_3 : X \times Y \to I \ni (\lambda_1 \times \mu_3)(< a,p >) = 0$, $(\lambda_1 \times \mu_3)(< a,q >) = 0$, $(\lambda_1 \times \mu_3)(< b,p >) = 0$, $(\lambda_1 \times \mu_3)(< b,q >) = 0$, $(\lambda_1 \times \mu_3)(< c,p >) = 0$, $(\lambda_1 \times \mu_3)(< c,q >) = 0$ and $N_1 \times M_3 = \phi$. $(\lambda_2 \times \mu_1, N_2 \times M_1)$ is defined by $\lambda_2 \times \mu_1 : X \times Y \to I \ni (\lambda_2 \times \mu_1)(< a,p >) = 0$, $(\lambda_2 \times \mu_1)(< a,q >) = 0$, $(\lambda_2 \times \mu_1)(< b,p >) = 0$, $(\lambda_2 \times \mu_1)(< b,q >) = 0$, $(\lambda_2 \times \mu_1)(< c,p >) = 0$, $(\lambda_2 \times \mu_1)(< c,q >) = 0$ and $N_2 \times M_1 = \phi$. $(\lambda_2 \times \mu_2, N_2 \times M_2)$ is defined by $\lambda_2 \times \mu_2 : X \times Y \to I \ni (\lambda_2 \times \mu_2)(< a,p >) = 1$, $(\lambda_2 \times \mu_2)(< a,q >) = 1$, $(\lambda_2 \times \mu_2)(< b,p >) = 1$, $(\lambda_2 \times \mu_2)(< b,q >) = 1$, $(\lambda_2 \times \mu_2)(< c,p >) = 1$, $(\lambda_2 \times \mu_2)(< c,q >) = 1$ and $N_2 \times M_2 = X \times Y$. $(\lambda_2 \times \mu_3, N_2 \times M_3)$ is defined by $\lambda_2 \times \mu_3 : X \times Y \to I \ni (\lambda_2 \times \mu_3)(< a,p >) = 0.4$, $(\lambda_2 \times \mu_3)(< a,q >) = 0.2$, $(\lambda_2 \times \mu_3)(< b,p >) = 0.4$, $(\lambda_2 \times \mu_3)(< b,q >) = 0.2$, $(\lambda_2 \times \mu_3)(< c,p >) = 0.4$, $(\lambda_2 \times \mu_3)(< c,q >) = 0.2$ and $N_2 \times M_3 = X \times Y$. $(\lambda_3 \times \mu_1, N_3 \times M_1)$ is defined by $\lambda_3 \times \mu_1 : X \times Y \to I \ni (\lambda_3 \times \mu_1)(< a,p >) = 0$, $(\lambda_3 \times \mu_1)(< a,q >) = 0$, $(\lambda_3 \times \mu_1)(< b,p >) = 0$, $(\lambda_3 \times \mu_1)(< b,q >) = 0$, $(\lambda_3 \times \mu_1)(< c,p >) = 0$, $(\lambda_3 \times \mu_1)(< c,q >) = 0$ and $N_3 \times M_1 = \phi$. $(\lambda_3 \times \mu_2, N_3 \times M_2)$ is defined by $\lambda_3 \times \mu_2 : X \times Y \to I \ni (\lambda_3 \times \mu_2)(< a,p >) = 0.3$, $(\lambda_3 \times \mu_2)(< a,q >) = 0.3$, $(\lambda_3 \times \mu_2)(< b,p >) = 0.4$, $(\lambda_3 \times \mu_2)(< b,q >) = 0.4$, $(\lambda_3 \times \mu_2)(< c,p >) = 0.2$, $(\lambda_3 \times \mu_2)(< c,q >) = 0.2$ and $N_3 \times M_2 = \{< a,p >, < a,q >, < b,p >, < b,q >\}$.$(\lambda_3 \times \mu_3, N_3 \times M_3)$ is defined by $\lambda_3 \times \mu_3 : X \times Y \to I \ni (\lambda_3 \times \mu_3)(< a,p >) = 0.3$, $(\lambda_3 \times \mu_3)(< a,q >) = 0.2$, $(\lambda_3 \times \mu_3)(< b,p >) = 0.4$, $(\lambda_3 \times \mu_3)(< b,q >) = 0.2$, $(\lambda_3 \times \mu_3)(< c,p >) = 0.2$, $(\lambda_3 \times \mu_3)(< c,q >) = 0.2$ and $N_3 \times M_3 = \{< a,p >, < a,q >, < b,p >, < b,q >\}$. The soft fuzzy topology induced by $\mathcal{B}$ is called as soft fuzzy product topological of T and S denoted by $T \times S$. The pair $(X \times Y, T \times S)$ is a soft fuzzy product topological space.

Definition 3.6. Let $(X_1 \times X_2, T_1 \times T_2)$ be a soft fuzzy product topological space. The family of all soft fuzzy $G_\delta$pre open sets are denoted by $SFG_\delta preOS$. *Soft fuzzy $G_\delta$pre structure* $st(T_1 \times T_2)$ is the collection of all soft fuzzy $G_\delta$pre open sets satisfies the following conditions:

(1) $(0, \phi), (1, X) \in st(T_1 \times T_2)$.

(2) For any family of soft fuzzy sets $(\lambda_j, N_j) \in st(T_1 \times T_2), j \in J$, $\sqcup_{j \in J}(\lambda_j, N_j) \in st(T_1 \times T_2)$.

(3) For any finite number of soft fuzzy sets $(\lambda_j, N_j) \in st(T_1 \times T_2), j = 1, 2, 3, ...n$, $\sqcap_{j=1}^{n}(\lambda_j, N_j) \in st(T_1 \times T_2)$. Then, the pair $(X_1 \times X_2, st(T_1 \times T_2))$is called as a *soft fuzzy product $G_\delta$pre space*, (in short,$SFPG_\delta pre_{st}S$)

Definition 3.7. Let (X, τ) be a topological space. A subset A of X is said to be *$G_\delta$pre open* if $A = B \cap C$ where B and C are G_δ and pre open set respectively.

Definition 3.8. Let $(X_1 \times X_2, \tau_1 \times \tau_2)$ be a topological space and $I = [0, 1]$ equipped with the usual topology, *a lower semi $G_\delta$pre continuous* pair* (μ, M), where $\mu : (X_1 \times X_2, \tau_1 \times \tau_2) \to I$ with $G_\delta$pre open set $\mu^{-1}((\alpha, 1])$ and $M \subseteq X_1 \times X_2$ is a $G_\delta$pre open set in $\tau_1 \times \tau_2$ for all $\alpha \in$[0,1].

Definition 3.9. A soft fuzzy product $G_\delta$pre space $(X_1 \times X_2, st(T_1 \times T_2))$ is said to be a *weakly induced soft fuzzy product $G_\delta$pre structure*, which is the soft fuzzy product $G_\delta$pre space induced by a topological space $(X_1 \times X_2, \tau_1 \times \tau_2)$ if the following conditions hold :

(a) $\tau_1 \times \tau_2 = \{A \subset X_1 \times X_2 \mid (\chi_A, A) \in st(T_1 \times T_2\}$

(b) Every $(\mu, M) \in st(T_1 \times T_2)$ is a lower semi $G_\delta$pre continuous* pair.

Definition 3.10. Let $PrTop$ be the category of all the product topological spaces and the continuous maps. Let $SFPrG_\delta prest$ be the category of all the soft fuzzy product $G_\delta$pre space and SF$G_\delta$pre continuous maps. Define a *functor*, $\omega : PrTop \to SFPrG_\delta prest$ which associates to any product topological space, $(X_1 \times X_2, T_1 \times T_2)$, the soft fuzzy product $G_\delta$pre space $(X_1 \times X_2, \omega(T_1 \times T_2))$ where $\omega(T_1 \times T_2)$ is the totality of all lower semi $G_\delta$pre continuous* pair. Then, $\omega(T_1 \times T_2)$ is called as the *weakly induced soft fuzzy $G_\delta$pre structure* by $(X_1 \times X_2, T_1 \times T_2)$.

Definition 3.11. Let $(X_1 \times X_2, T_1 \times T_2)$, $(Y_1 \times Y_2, S_1 \times S_2)$ be any two soft fuzzy product topological spaces. A surjective map $f:(X_1 \times X_2, T_1 \times T_2) \to (Y_1 \times Y_2, S_1 \times S_2)$ is said to be a *soft fuzzy $G_\delta$pre quotient product map* if the inverse image of every soft fuzzy $G_\delta$pre open set in $(Y_1 \times Y_2, S_1 \times S_2)$ is soft fuzzy $G_\delta$pre open in $(X_1 \times X_2, T_1 \times T_2)$.

Proposition 3.12. *For mappings $f_i : X_i \to Y_i$ and soft fuzzy sets (λ_i, M_i) of Y_i, $(i = 1, 2)$; we have $(f_1 \times f_2)^{-1}(\lambda_1 \times \lambda_2, M_1 \times M_2) = f_1^{-1}(\lambda_1, M_1) \times f_2^{-1}(\lambda_2, M_2)$.*

Proof. Proof is clear. □

Proposition 3.13. *For mappings $f_i : X_i \to Y_i$ and soft fuzzy sets (λ_i, M_i) of Y_i, $(i = 1, 2)$; we have $(f_1 \times f_2)(\lambda_1 \times \lambda_2, M_1 \times M_2) \sqsubseteq f_1(\lambda_1, M_1) \times f_2(\lambda_2, M_2)$.*

Proof. Proof is clear. □

4. Properties of the product associated map on soft fuzzy product $G_\delta$pre space

Definition 4.1. Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two maps. Let $X_1 \times X_2$ and $Y_1 \times Y_2$ be two product sets and $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ be a product map. Then, define the soft fuzzy product associated map $\widetilde{f_1 \times f_2}$ as $\widetilde{f_1 \times f_2}(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\}) = f_1 \times f_2(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})$, for each soft fuzzy point $(< x_1, x_2 > \lambda, \{< x_1, x_2 >\})$ in $X_1 \times X_2$.

Proposition 4.2. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two onto maps. Let $X_1 \times X_2$ and $Y_1 \times Y_2$ be two product sets. If $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a product onto map, then for each soft fuzzy point $(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})$ in $X_1 \times X_2$, $\widetilde{f_1 \times f_2}(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})$ is the soft fuzzy point in $Y_1 \times Y_2$ that takes the value λ in $f_1 \times f_2(< x_1, x_2 >)$.*

Proof. For $0 < \lambda \leq 1$,

$$\begin{aligned}
\widetilde{f_1 \times f_2}(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\}) &= f_1 \times f_2(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\}) \\
&= (f_1 \times f_2 < x_1, x_2 >_\lambda, f_1 \times f_2(\{< x_1, x_2 >\})) \\
&= (f_1 \times f_2 < x_1, x_2 >_\lambda, \{< f_1(x_1), f_2(x_2) >\})
\end{aligned}$$

where, $f_1 \times f_2(< x_1, x_2 >_\lambda)(< y_1, y_2 >)$

$$= \begin{cases} sup_{<x,y> \in (f_1\times f_2)^{-1}(<y_1,y_2>)} & < x_1, x_2 >_\lambda (< x, y >) \\ & \text{if } (f_1 \times f_2)^{-1}(< y_1, y_2 >) \neq \phi; \\ 0 & \text{otherwise.} \end{cases}$$

$$f_1 \times f_2(< x_1, x_2 >_\lambda)(< y_1, y_2 >) = \begin{cases} \lambda \; (0 < \lambda \leq 1) & \text{if } (f_1 \times f_2)^{-1}(< y_1, y_2 >) \neq \phi; \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\widetilde{f_1 \times f_2}(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})$ is the soft fuzzy point in $Y_1 \times Y_2$ that takes the value λ in $f_1 \times f_2(< x_1, x_2 >)$. □

Proposition 4.3. *Let $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$, $g_1 : Y_1 \to Z_1$ and $g_2 : Y_2 \to Z_2$ be any maps. Let $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ and $g_1 \times g_2 : Y_1 \times Y_2 \to Z_1 \times Z_2$ be the two product onto maps. Then, $\widetilde{(g_1 \times g_2) \circ (f_1 \times f_2)} = (\widetilde{g_1 \times g_2}) \circ (\widetilde{f_1 \times f_2})$.*

Proof. By using the Proposition 4.2, we have for each soft fuzzy point $(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})$ in $X_1 \times X_2$

$$\begin{aligned} &\widetilde{(g_1 \times g_2) \circ (f_1 \times f_2)}(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\}) \\ &\qquad = (g_1 \times g_2) \circ (f_1 \times f_2)(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\}) \\ &\qquad = (g_1 \times g_2)((f_1 \times f_2)(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})) \\ &\qquad = (g_1 \times g_2)((\widetilde{f_1 \times f_2})(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})) \\ &\qquad = (\widetilde{g_1 \times g_2})((\widetilde{f_1 \times f_2})(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})) \\ &\qquad = (\widetilde{g_1 \times g_2}) \circ (\widetilde{f_1 \times f_2})(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\}) \end{aligned}$$

Thus, $\widetilde{(g_1 \times g_2) \circ (f_1 \times f_2)} = (\widetilde{g_1 \times g_2}) \circ (\widetilde{f_1 \times f_2})$. □

Proposition 4.4. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two onto maps. Let $X_1 \times X_2$ and $Y_1 \times Y_2$ be two product sets. Let $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a product onto map. If $(\widetilde{f_1 \times f_2})$ is the identity map, then $(f_1 \times f_2)$ is also the identity map.*

Proof. Since $(\widetilde{f_1 \times f_2})$ is the identity map, $(\widetilde{f_1 \times f_2})(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})$ $= ((x_1, x_2)_\lambda, \{(x_1, x_2)\})$, for each soft fuzzy point $(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})$. Now, by the definition of the soft fuzzy product associated map, $(\widetilde{f_1 \times f_2})(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\}) = (f_1 \times f_2)(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\}) = ((f_1 \times f_2)(< x_1, x_2 >_\lambda), \{(f_1(x_1), f_2(x_2))\})$. This implies that, $((f_1 \times f_2)(< x_1, x_2 >_\lambda), \{< f_1(x_1), f_2(x_2) >\})$ is the soft fuzzy point which takes the value λ in $(f_1 \times f_2) < x_1, x_2 >$ and $(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})$ is also the soft fuzzy point which takes the value λ in $< x_1, x_2 >_\lambda$. Thus, $(f_1 \times f_2)(< x_1, x_2 >) = < x_1, x_2 >$, for each $< x_1, x_2 > \in X_1 \times X_2$. Thus, $(f_1 \times f_2)$ is the identity map. □

Proposition 4.5. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two maps.*

(1) *If $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a product onto map, then $(\widetilde{f_1 \times f_2})$ is also the product onto map.*

(2) *If $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a product one- to-one map, then $\widetilde{(f_1 \times f_2)}$ is also the product one-to-one map.*

Proof. (1) For each $(< y_1, y_2 >_\alpha, \{< y_1, y_2 >\})$ soft fuzzy point in $Y_1 \times Y_2$, we have $< y_1, y_2 >\in Y_1 \times Y_2$, then there exists atleast $< x_1, x_2 >\in X_1 \times X_2$ such that $(f_1 \times f_2) < x_1, x_2 > = < y_1, y_2 >$. Now, $\widetilde{(f_1 \times f_2)}(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\}) = (f_1 \times f_2)(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\}) = ((f_1 \times f_2)(< x_1, x_2 >_\alpha), (f_1 \times f_2)(\{< x_1, x_2 >\}))$ which takes the value α in $(f_1 \times f_2) < x_1, x_2 >$ and since $(f_1 \times f_2) < x_1, x_2 > = < y_1, y_2 >$, this shows that $((f_1 \times f_2)(< x_1, x_2 >_\alpha), (f_1 \times f_2)(\{< x_1, x_2 >\})) = (< y_1, y_2 >_\alpha, \{< y_1, y_2 >\})$. Therefore, $\widetilde{(f_1 \times f_2)}(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\}) = (< y_1, y_2 >_\alpha, \{< y_1, y_2 >\})$. Thus, $\widetilde{(f_1 \times f_2)}$ is the product onto map.

(2) If $(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\}), (< x_1', x_2' >_\beta, \{< x_1', x_2' >\})$ are the two soft fuzzy points in $X_1 \times X_2$ such that $\widetilde{(f_1 \times f_2)}(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\}) = \widetilde{(f_1 \times f_2)}(< x_1', x_2' >_\beta, \{< x_1', x_2' >\})$. This implies that $(f_1 \times f_2)(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\}) = (f_1 \times f_2)(< x_1', x_2' >_\beta, \{< x_1', x_2' >\})$. This shows $(f_1 \times f_2) < x_1, x_2 >= (f_1 \times f_2) < x_1', x_2' >$ and $\alpha = \beta$. Since $(f_1 \times f_2)$ is a one-to-one map, $< x_1, x_2 >=< x_1', x_2' >$ and $\alpha = \beta$, it follows $(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\}) = (< x_1', x_2' >_\beta, \{< x_1', x_2' >\})$. Thus, $\widetilde{(f_1 \times f_2)}$ is one-to-one. □

Proposition 4.6. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two onto maps. If $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a product one- to-one map, then $(\widetilde{f_1 \times f_2})^{-1} = \widetilde{(f_1 \times f_2)}^{-1}$.*

Proof. For each soft fuzzy point $(< y_1, y_2 >_\alpha, \{< y_1, y_2 >\})$ in $Y_1 \times Y_2$ and by the hypothesis, there exists a unique $< x_1, x_2 >\in X_1 \times X_2$ such that $(f_1 \times f_2) < x_1, x_2 > = < y_1, y_2 >$. It is enough to show that $(\widetilde{f_1 \times f_2})^{-1}(< y_1, y_2 >_\alpha, \{< y_1, y_2 >\}) = (< x_1, x_2 >_\alpha, \{< x_1, x_2 >\})$. Otherwise, let $(\widetilde{f_1 \times f_2})^{-1}(< y_1, y_2 >_\alpha, \{< y_1, y_2 >\}) = (< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})$ and $\alpha \neq \lambda$. Then, $(f_1 \times f_2)(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\}) = \widetilde{(f_1 \times f_2)}((\widetilde{f_1 \times f_2})^{-1}\ (< y_1, y_2 >_\alpha, \{< y_1, y_2 >\})) = \widetilde{(f_1 \times f_2)}(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\})$. Since $(f_1 \times f_2)$ is a one-to-one map, $\widetilde{(f_1 \times f_2)}$ is one-to-one. Thus, $(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\}) = (< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})$. But, $\lambda \neq \alpha$. Thus, it leads to a contradiction. Therefore, $(\widetilde{f_1 \times f_2})^{-1}(< y_1, y_2 >_\alpha, \{< y_1, y_2 >\})$ is uniquely the soft fuzzy point in $X_1 \times X_2$ which takes the value α in $f^{-1} < y_1, y_2 >$. This implies that, $\widetilde{(f_1 \times f_2)}^{-1}(< y_1, y_2 >_\alpha, \{< y_1, y_2 >\}) = (\widetilde{f_1 \times f_2})^{-1}(< y_1, y_2 >_\alpha, \{< y_1, y_2 >\})$. Hence, $(\widetilde{f_1 \times f_2})^{-1} = \widetilde{(f_1 \times f_2)}^{-1}$. □

Proposition 4.7. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two maps. Let $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ be a product map.*

(a) *If $\widetilde{(f_1 \times f_2)}$ is onto, then $(f_1 \times f_2)$ is also onto.*

(b) *If $\widetilde{(f_1 \times f_2)}$ is one-to-one, then $(f_1 \times f_2)$ is also one-to-one.*

Proof. (a) For each $< y_1, y_2 >\in Y_1 \times Y_2$, let $(< y_1, y_2 >_1, \{< y_1, y_2 >\})$ be the soft fuzzy point in $Y_1 \times Y_2$ which takes the value 1 in $< y_1, y_2 >$. By the hypothesis, there exists a soft fuzzy point $(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\})$ in $X_1 \times X_2$ such that $\widetilde{f_1 \times f_2}(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\}) = (< y_1, y_2 >_1, \{< y_1, y_2 >\})$. Then, $f_1 \times f_2$ $(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\}) = (< y_1, y_2 >_1, \{< y_1, y_2 >\})$ and $(f_1 \times f_2)^{-1}$ $(< y_1, y_2 >) \neq \phi$. Hence, $(f_1 \times f_2)$ is a onto map.

(b) Let $< x_1, x_2 >, \widetilde{< x_1', x_2'} > \in X_1 \times X_2$ with $(f_1 \times f_2)$ $(< x_1, x_2 >) = (f_1 \times f_2)(< x_1', x_2' >)$. Now, $(f_1 \times f_2)(< x_1, x_2 >_1, \{< x_1, x_2 >\}) = (f_1 \times f_2)(< x_1, x_2 >_1, \{< x_1, x_2 >\}) = ((f_1 \times f_2)(< x_1, x_2)_1 >, (f_1 \times f_2)(\{< x_1, x_2 >\}))$, where, for $\lambda = 1$

$$(f_1 \times f_2)(< x_1, x_2 >_1)(< y_1, y_2 >)$$

$$= \begin{cases} sup_{<x,y> \in (f_1 \times f_2)^{-1}(<y_1,y_2>)} & < x_1, x_2 >_\lambda (< x, y >) \\ & \text{if } (f_1 \times f_2)^{-1}(< y_1, y_2 >) \neq \phi; \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 1 & \text{if } < x, y > = < x_1, x_2 > \text{ and } (f_1 \times f_2)^{-1}(< y_1, y_2 >) \neq \phi; \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 1 & \text{if } (f_1 \times f_2)(< x_1', x_2' >) = < y_1, y_2 >; \\ 0 & \text{otherwise.} \end{cases}$$

This implies that, $(\widetilde{f_1 \times f_2})(< x_1, x_2 >_1, \{< x_1, x_2 >\}) = (\widetilde{f_1 \times f_2})$ $(< x_1', x_2' >_1, \{< x_1', x_2' >\})$. Since $(\widetilde{f_1 \times f_2})$ is a one-to-one map, $(< x_1', x_2' >_1, \{< x_1', x_2' >\}) = (< x_1, x_2 >_1, \{< x_1, x_2 >\})$. This implies that, $< x_1, x_2 > = < x_1', x_2' >$. Thus, $(f_1 \times f_2)$ is a one-to-one map. □

Definition 4.8. Let (X, τ) and (Y, σ) be any two topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *$G_\delta$pre irresolute*, if the inverse image of every G_δ pre open set in (Y, σ) is G_δ pre open in (X, τ).

Proposition 4.9. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two maps. Let $(X_1 \times X_2,\ T_1 \times T_2)$ and $(Y_1 \times Y_2, S_1 \times S_2)$ be any two product topological spaces. If $f_1 \times f_2 : (X_1 \times X_2,\ T_1 \times T_2) \to (Y_1 \times Y_2, S_1 \times S_2)$ is a $G_\delta$pre irresolute map iff $\widetilde{f_1 \times f_2} : (X_1 \times X_2, \omega(T_1 \times T_2)) \to (Y_1 \times Y_2, \omega(S_1 \times S_2))$ is a soft fuzzy $G_\delta$pre irresolute.*

Proof. For each soft fuzzy $G_\delta$pre open set (μ, M) in $(Y_1 \times Y_2, \omega(S_1 \times S_2))$, we have $\mu^{-1}((\alpha, 1])$ is a $G_\delta$pre open set in $(S_1 \times S_2)$ for all $\alpha \in [0, 1]$ and by hypothesis $(f_1 \times f_2)^{-1}(\mu^{-1}(\alpha, 1])$ is a $G_\delta$pre open set in $T_1 \times T_2$. Then, $(\mu \circ (f_1 \times f_2))^{-1}(\alpha, 1]$ is a $G_\delta$pre open set in $T_1 \times T_2$, and also $(f_1 \times f_2)^{-1}(M) \subseteq X_1 \times X_2$ is a $G_\delta$pre open set in $T_1 \times T_2$. Therefore, $((\mu \circ\ (f_1 \times f_2)), (f_1 \times f_2)^{-1}(M))$ is a soft fuzzy $G_\delta$pre open set in $(X_1 \times X_2, \omega(T_1 \times T_2))$. Now,

$$\begin{aligned} (\widetilde{f_1 \times f_2})^{-1}(\mu, M) &= (f_1 \times f_2)^{-1}(\mu, M) \\ &= ((f_1 \times f_2)^{-1}(\mu), (f_1 \times f_2)^{-1}(M)) \\ &= ((\mu \circ (f_1 \times f_2)), (f_1 \times f_2)^{-1}(M)) \end{aligned}$$

Thus, $(\widetilde{f_1 \times f_2})^{-1}(\mu, M)$ is a soft fuzzy $G_\delta$pre open set in $(X_1 \times X_2, \omega(T_1 \times T_2))$. Hence, $(\widetilde{f_1 \times f_2})$ is a soft fuzzy $G_\delta$pre irresolute function.

Conversely, A is a $G_\delta$pre open set in $(S_1 \times S_2)$ iff (χ_A, A) is a soft fuzzy $G_\delta$pre open set in $((Y_1 \times Y_2), \omega(S_1 \times S_2))$. Now,

$$\begin{aligned}\widetilde{(f_1 \times f_2)}^{-1}(\chi_A, A) &= (f_1 \times f_2)^{-1}(\chi_A, A)\\ &= ((f_1 \times f_2)^{-1}(\chi_A), (f_1 \times f_2)^{-1}(A))\\ &= (\chi_A \circ (f_1 \times f_2), (f_1 \times f_2)^{-1}(A))\\ &= (\chi_{(f_1 \times f_2)^{-1}(A)}, (f_1 \times f_2)^{-1}(A))\end{aligned}$$

That is, $\widetilde{(f_1 \times f_2)}^{-1}(\chi_A, A) = (\chi_{(f_1 \times f_2)^{-1}(A)}, (f_1 \times f_2)^{-1}(A))$ is the soft fuzzy $G_\delta$pre open set in $(X_1 \times X_2, \omega(T_1 \times T_2))$. Hence, $(f_1 \times f_2)^{-1}(A)$ is a soft fuzzy $G_\delta$pre open set in $T_1 \times T_2$. Therefore, $f_1 \times f_2$ is a soft fuzzy $G_\delta$pre irresolute map. □

Definition 4.10. Let $(X_1 \times X_2, T_1 \times T_2)$, $(Y_1 \times Y_2, S_1 \times S_2)$ be any two topological spaces. A surjective map $f : (X_1 \times X_2, T_1 \times T_2) \to (Y_1 \times Y_2, S_1 \times S_2)$ is said to be a *$G_\delta$pre quotient product map* if the inverse image of every $G_\delta$pre open set in $(Y_1 \times Y_2, S_1 \times S_2)$ is $G_\delta$pre open in $(X_1 \times X_2, T_1 \times T_2)$.

Proposition 4.11. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two maps. Let $(X_1 \times X_2,\ T_1 \times T_2)$ and $(Y_1 \times Y_2, S_1 \times S_2)$ be any two product topological spaces. Then, $f_1 \times f_2 : (X_1 \times X_2, T_1 \times T_2) \to (Y_1 \times Y_2, S_1 \times S_2)$ is a $G_\delta$pre quotient product map iff $\widetilde{f_1 \times f_2} : (X_1 \times X_2, \omega(T_1 \times T_2)) \to (Y_1 \times Y_2, \omega(S_1 \times S_2))$ is also a soft fuzzy $G_\delta$pre quotient product map.*

Proof. A is a $G_\delta$pre open in $(S_1 \times S_2)$ iff (χ_A, A) is a soft fuzzy $G_\delta$pre open set in $((Y_1 \times Y_2), \omega(S_1 \times S_2))$. Now,

$$\begin{aligned}\widetilde{(f_1 \times f_2)}^{-1}(\chi_A, A) &= (f_1 \times f_2)^{-1}(\chi_A, A)\\ &= ((f_1 \times f_2)^{-1}(\chi_A), (f_1 \times f_2)^{-1}(A))\\ &= (\chi_A \circ (f_1 \times f_2), (f_1 \times f_2)^{-1}(A))\\ &= (\chi_{(f_1 \times f_2)^{-1}(A)}, (f_1 \times f_2)^{-1}(A))\end{aligned}$$

That is, $\widetilde{(f_1 \times f_2)}^{-1}(\chi_A, A) = (\chi_{(f_1 \times f_2)^{-1}(A)}, (f_1 \times f_2)^{-1}(A))$ is the soft fuzzy $G_\delta$pre open set in $(X_1 \times X_2, \omega(T_1 \times T_2))$. Hence, $(f_1 \times f_2)^{-1}(A)$ is a soft fuzzy $G_\delta$pre open in $T_1 \times T_2$. Therefore, $f_1 \times f_2$ is a $G_\delta$pre quotient product map.

Conversely, let (μ, M) be a soft fuzzy $G_\delta$pre open set in $(Y_1 \times Y_2, \omega(S_1 \times S_2))$ iff $\mu^{-1}(\alpha, 1]$ is $G_\delta$pre open in $S_1 \times S_2$ and $M \subseteq Y_1 \times Y_2$ is $G_\delta$pre open in $S_1 \times S_2$, for all $\alpha \in [0, 1]$. By the hypothesis, $(f_1 \times f_2)^{-1}(\mu^{-1}(\alpha, 1]) \in T_1 \times T_2$ and $(f_1 \times f_2)^{-1}(M) \in T_1 \times T_2$ $(f_1 \times f_2)^{-1}(M) \subseteq X_1 \times X_2$ for each $\alpha \in [0, 1]$. That is, $(\widetilde{(f_1 \times f_2)}^{-1}(\mu), \widetilde{(f_1 \times f_2)}^{-1}(M)) = \widetilde{(f_1 \times f_2)}^{-1}(\mu, M)$ is a soft fuzzy $G_\delta$pre open set in $(X_1 \times X_2, \omega(T_1 \times T_2))$. Hence, $\widetilde{f_1 \times f_2}$ is also a soft fuzzy $G_\delta$pre quotient product map. □

Definition 4.12. Let $(X_1 \times X_2, T_1 \times T_2)$ and $(Y_1 \times Y_2, S_1 \times S_2)$ be any two topological spaces. $(X_1 \times X_2, T_1 \times T_2)$ is said to be *$G_\delta$pre homeomorphic* to $(Y_1 \times Y_2, S_1 \times S_2)$, if $f : (X_1 \times X_2, T_1 \times T_2) \to (Y_1 \times Y_2, S_1 \times S_2)$ is one to one, onto, f and f^{-1} is $G_\delta$pre irresolute.

Definition 4.13. Let $(X_1 \times X_2, T_1 \times T_2)$ and $(Y_1 \times Y_2, S_1 \times S_2)$ be any two soft fuzzy topological spaces. $(X_1 \times X_2, T_1 \times T_2)$ is said to be soft fuzzy $G_\delta$pre homeomorphic to $(Y_1 \times Y_2, S_1 \times S_2)$, if $f : (X_1 \times X_2, T_1 \times T_2) \to (Y_1 \times Y_2, S_1 \times S_2)$ is one to one, onto, f and f^{-1} is soft fuzzy $G_\delta$pre irresolute.

Proposition 4.14. *Let $(X_1 \times X_2, \omega(T_1 \times T_2))$ and $(Y_1 \times Y_2, \omega(S_1 \times S_2))$ be two weakly induced soft fuzzy product $G_\delta$pre spaces, and $(\widetilde{f_1 \times f_2})$ be a soft fuzzy $G_\delta$pre irresolute map from $(X_1 \times X_2, \omega(\widetilde{T_1 \times T_2}))$ onto $(Y_1 \times Y_2, \omega(S_1 \times S_2))$. If there exists a soft fuzzy $G_\delta$pre irresolute map $(\widetilde{g_1 \times g_2})$ from $(Y_1 \times Y_2, \omega(S_1 \times S_2))$ to $(X_1 \times X_2, \omega(T_1 \times T_2))$ such that $(\widetilde{f_1 \times f_2}) \circ (\widetilde{g_1 \times g_2}) = 1_{Y_1 \times Y_2}$, then $(Y_1 \times Y_2, \omega(S_1 \times S_2))$ is soft fuzzy $G_\delta$pre homeomorphic with $(X_1 \times X_2) \mid R$, where R is the equivalence relation.*

Proof. Since $(\widetilde{f_1 \times f_2}) \circ (\widetilde{g_1 \times g_2}) = 1_{Y_1 \times Y_2}$, then by using the above propositions, we have $(f_1 \times f_2) \circ (g_1 \times g_2) = 1_{Y_1 \times Y_2}$. Then, the map $h_1 \times h_2 : (X_1 \times X_2) \mid R \to Y_1 \times Y_2$ induced by $\widetilde{f}$ is a soft fuzzy $G_\delta$pre homeomorphism. Finally, by the above all propositions , $\widetilde{h_1 \times h_2}$ is clearly a soft fuzzy $G_\delta$pre homeomorphism. □

5. Compactification on $\mathcal{Q}(X_1 \times X_2)$

Definition 5.1. Let $R_{<x_1,x_2>}$ be an equivalence relation. Then,

$$X_1 \times X_2 \mid R_{<x_1,x_2>} = \{[<x_1,x_2>],[<y_1,y_2>] \mid <x_1,x_2>/R<z_1,z_2>, \\ <y_1,y_2> R <z_1,z_2>, \forall <z_1,z_2> \in X_1 \times X_2\}$$

is a *quotient set* on $X_1 \times X_2$.

Definition 5.2. Let $X_1 \times X_2$ be a product space. Let $(X_1 \times X_2) \mid R$ be a quotient set on $(X_1 \times X_2)$ with R, an equivalence relation. Then, the collection of all quotient sets on $X_1 \times X_2$ is denoted by $\mathcal{Q}(X_1 \times X_2)$.

Definition 5.3. Let $(X_1 \times X_2, st(T_1 \times T_2))$ be a soft fuzzy product $G_\delta$pre space and A be a subset of $X_1 \times X_2$. If χ_A is a characteristic function of A in $X_1 \times X_2$, then

$$st(T_1 \times T_2)_A = \{(\lambda, N) \sqcap (\chi_A, A) : (\lambda, N) \in st(T_1 \times T_2)\}$$

is called as a *soft fuzzy product $G_\delta$pre substructure*. Now, the pair $(A, st(T_1 \times T_2)_A)$ is called as a *soft fuzzy product $G_\delta$pre subspace*.

Let $(X_1 \times X_2, T_1 \times T_2)$ be a non compact soft fuzzy product $G_\delta$pre space. Associated with each $(\mu, M) \in st(T_1 \times T_2)$, we define $(\mu, M)^* = (\mu^*, M^*) \in SF(\mathcal{Q}(X_1 \times X_2))$. For each $X_1 \times X_2 \mid R \in \mathcal{Q}(X_1 \times X_2)$, $\mu^*(X_1 \times X_2 \mid R)$

$$= \begin{cases} \mu(<x_1,x_2>) & \text{if } \exists <x_1,x_2> \in X_1 \times X_2, \\ & \text{with } X_1 \times X_2 \mid R = X_1 \times X_2 \mid R_{<x_1,x_2>}; \\ \bigvee_{[<x_1,x_2>] \in X_1 \times X_2 \mid R} \mu(<x_1,x_2>) & \text{otherwise.} \end{cases}$$

$$M^* = \begin{cases} \phi & \text{if } M = \phi; \\ \mathcal{Q}(X_1 \times X_2) & \text{if } M = X_1 \times X_2; \\ X_1 \times X_2 \mid R_{<x_1,x_2>} & \text{if } <x_1,x_2> \in M \subset X_1 \times X_2. \end{cases}$$

Proposition 5.4. *Under the previous conditions the following identities hold.*
(i) $(0, \phi)^* = (0, \phi)$.
(ii) $(1, X_1 \times X_2)^* = (1, \mathcal{Q}(X_1 \times X_2))$.

Proposition 5.5. *Under the previous conditions the collection*

$$\mathcal{B}^* = \{(\mu, M)^* : (\mu, M) \in T_1 \times T_2\}$$

is a base for some soft fuzzy product generalized topology on $\mathcal{Q}(X_1 \times X_2)$.

Proof. (i) For $(\mu_i, M_i) \in T_1 \times T_2$ for all $i \in I$ and $X_1 \times X_2 \mid R \in Q(X_1 \times X_2)$, we have $(\bigsqcup_{i\in I}(\mu_i, M_i))^* = (\bigvee_{i\in I}(\mu_i), \bigcup_{i\in I}(M_i))^* = (\bigvee_{i\in I}(\mu_i)^*, \bigcup_{i\in I}(M_i)^*)$
$(\bigvee_{i\in I}(\mu_i))^*(X_1 \times X_2 \mid R)$

$$= \begin{cases} \bigvee_{i\in I}(\mu_i)(< x_1, x_2 >) & \text{if } \exists < x_1, x_2 >\in X_1 \times X_2 \text{ with,} \\ & X_1 \times X_2 \mid R{=}X_1 \times X_2 \mid R_{<x_1,x_2>}; \\ \bigvee_{[<x_1,x_2>]\in X_1\times X_2 \mid R} \left(\bigvee_{i\in I}(\mu_i\right)(< x_1, x_2 >) & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \bigvee_{i\in I}(\mu_i(< x_1, x_2 >)) & \text{if } \exists < x_1, x_2 >\in X_1 \times X_2 \text{ with,} \\ & X_1 \times X_2 \mid R{=}X_1 \times X_2 \mid R_{<x_1,x_2>}; \\ \bigvee_{i\in I}\left(\bigvee_{[<x_1,x_2>]\in X_1\times X_2 \mid R} \mu_i(< x_1, x_2 >)\right), & \text{otherwise.} \end{cases}$$

$$= \bigvee_{i\in I}(\mu_i^*(X_1 \times X_2 \mid R)).$$

$$(\bigcup_{i\in I} M_i)^* = \begin{cases} \phi & \text{if } \bigvee_{i\in I} M_i = \phi; \\ \mathcal{Q}(X_1 \times X_2) & \text{if } \bigvee_{i\in I} M_i = X_1 \times X_2; \\ X_1 \times X_2 \mid R_{<x_1,x_2>} & \text{if } < x_1, x_2 >\in \bigvee_{i\in I} M_i \subset X_1 \times X_2. \end{cases}$$

Then for some i

$$(\bigcup_{i\in I} M_i)^* = \begin{cases} \phi & \text{if } M_i = \phi; \\ \mathcal{Q}(X_1 \times X_2) & \text{if } M_i = X_1 \times X_2; \\ X_1 \times X_2 \mid R_{<x_1,x_2>} & \text{if } < x_1, x_2 >\in M_i \subset X_1 \times X_2. \end{cases}$$

$$(\bigcup_{i\in I} M_i)^* = \bigcup_i M_i^*$$

Therefore $(\bigsqcup_{i\in I}(\mu_i, M_i))^* = \bigsqcup_{i\in I}((\mu_i, M_i)^*)$. From the above Proposition $\mathcal{B}^*$ forms a base for $\mathcal{Q}(X_1 \times X_2)$. □

Definition 5.6. The soft fuzzy generalized topology generated by the base $\mathcal{B}^*$ is denoted by $(T_1 \times T_2)^* = T_1^* \times T_2^*$.

Definition 5.7. A soft fuzzy product generalized topology on a non-empty set $\mathcal{Q}(X_1 \times X_2)$ is a family $(T_1 \times T_2)^*$ of soft fuzzy sets in $\mathcal{Q}(X_1 \times X_2)$ satisfying the following axioms:
(1) $(0, \phi), (1, \mathcal{Q}(X_1 \times X_2)) \in (T_1 \times T_2)^*$.
(2) For any family of soft fuzzy sets $(\lambda_j, N_j) \in (T_1 \times T_2)^*, j \in J, \Rightarrow \sqcup_{j\in J}(\lambda_j, N_j) \in (T_1 \times T_2)^*$. Then, the pair $(\mathcal{Q}(X_1 \times X_2), (T_1 \times T_2)^*)$ is called as a *soft fuzzy product generalized topological space*, (in short,*SFPGTS*)

Any soft fuzzy set in $(T_1 \times T_2)^*$ is said to be a *soft fuzzy product* $(T_1 \times T_2)^*$ *open set* in $\mathcal{Q}(X_1 \times X_2)$.

The complement of SFGOS in a SFPGTS $(\mathcal{Q}(X_1 \times X_2), (T_1 \times T_2)^*)$ is called as a *soft fuzzy product* $(T_1 \times T_2)^*$ *closed set* in $\mathcal{Q}(X_1 \times X_2)$.

Definition 5.8. Let $\mathfrak{q} : X_1 \times X_2 \to Q(X_1 \times X_2)$ defined by

$$\mathfrak{q}(< x_1, x_2 >) = X_1 \times X_2 \mid R_{<x_1,x_2>}$$

for each $< x_1, x_2 >\in X_1 \times X_2$.

Proposition 5.9. *Under the previous conditions,* $\mathfrak{q}(X_1 \times X_2)$ *is soft fuzzy dense in* $(\mathcal{Q}(X_1 \times X_2), (T_1 \times T_2)^*)$*, that is*

$$cl_{(T_1 \times T_2)^*}(\mathfrak{q}(1_{X_1 \times X_2}, X_1 \times X_2)) = (1_{\mathcal{Q}(X_1 \times X_2)}, \mathcal{Q}(X_1 \times X_2)).$$

Proof. Given $(\mu, M) \in SF(X_1 \times X_2)$, we have $\mathfrak{q}(\mu, M) \in SF(\mathcal{Q}(X_1 \times X_2))$. Then for each $(\mu, M) \in SF(X_1 \times X_2)$. Now $\mathfrak{q}(\mu, M) = (\mathfrak{q}(\mu), \mathfrak{q}(M))$

$\mathfrak{q}(\mu)(X_1 \times X_2 \mid R)$

$$= \begin{cases} sup_{<x_1,x_2>\in \mathfrak{q}^{-1}(X_1 \times X_2 | R)} \mu(< x_1, x_2 >) & \text{if } \mathfrak{q}^{-1}(X_1 \times X_2 \mid R) \neq \phi; \\ 0, & \text{if } \mathfrak{q}^{-1}(X_1 \times X_2 \mid R) = \phi. \end{cases}$$

$$= \begin{cases} \mu(< x_1, x_2 >) & \text{if } \exists < x_1, x_2 >\in X_1 \times X_2 \text{ such that,} \\ & \ X_1 \times X_2 \mid R = X_1 \times X_2 \mid R_{<x_1,x_2>}; \\ 0, & \text{if } \forall < x_1, x_2 >\in X_1 \times X_2, \\ & \ X_1 \times X_2 \mid R \neq X_1 \times X_2 \mid R_{<x_1,x_2>}. \end{cases}$$

$\mathfrak{q}(M) = \{\mathfrak{q}(< x_1, x_2 >), \forall < x_1, x_2 >\in M\}$.
Now $cl_{(T_1 \times T_2)^*}(\mathfrak{q}(1_{X_1 \times X_2}, X_1 \times X_2))$

$$= \begin{cases} cl_{(T_1 \times T_2)^*}(1_{\mathcal{Q}(X_1, X_2)}, \mathcal{Q}(X_1 \times X_2)) & \text{if } \exists < x_1, x_2 >\in X_1 \times X_2 \text{ such that,} \\ & X_1 \times X_2 \mid R = X_1 \times X_2 \mid R_{<x_1,x_2>}; \\ cl_{(T_1 \times T_2)^*}(0, \phi) & \ \text{if } \forall < x_1, x_2 >\in X_1 \times X_2, \\ & \ X_1 \times X_2 \mid R \neq X_1 \times X_2 \mid R_{<x_1,x_2>}. \end{cases}$$

Let $(\lambda, N) = \bigsqcup_{j\in J}(\mu_j, M_j)^* = cl_{(T_1 \times T_2)^*}(\mathfrak{q}(1_{X_1 \times X_2}, X_1 \times X_2))$. Since $\mathfrak{q}(1_{X_1 \times X_2}, X_1 \times X_2) \sqsubseteq (\lambda, N)$, $\mathfrak{q}(1_{X_1 \times X_2})(X_1 \times X_2 \mid R_{<x_1,x_2>}) \leq \lambda(X_1 \times X_2 \mid R_{<x_1,x_2>})$ and $\mathfrak{q}(X_1 \times X_2) \subseteq N$ for each $< x_1, x_2 >\in X_1 \times X_2$. That is $1 \leq \lambda(X_1 \times X_2 \mid R_{<x_1,x_2>})$, $\mathfrak{q}(X_1 \times X_2) \subseteq N$. Thus for each $< x_1, x_2 >\in X_1 \times X_2$, $\lambda(X_1 \times X_2 \mid R_{<x_1,x_2>}) = 1$, $\mathfrak{q}(X_1 \times X_2) \subseteq N$. For each $< x_1, x_2 >\in X_1 \times X_2$ and $j \in J$, $\mu_j^*(X_1 \times X_2 \mid R_{<x_1,x_2>}) = 1$, $\mathcal{Q}(X_1 \times X_2) \supseteq M_j^* \supseteq \mathfrak{q}(X_1 \times X_2)$. $\bigvee_{j\in J} \mu_j^*(X_1 \times X_2 \mid R_{<x_1,x_2>}) = 1$, $\bigcup_{j\in J} M_j^* = \mathcal{Q}(X_1 \times X_2)$. This implies $\mu_j(< x_1, x_2 >) = 1, M_j = X_1 \times X_2$. Thus $(\mu_j, M_j) = (1_{X_1 \times X_2}, X_1 \times X_2)$. Therefore $(\lambda, N)= \bigsqcup_{j\in J}(1_{X_1 \times X_2}, X_1 \times X_2)^* = (1_{\mathcal{Q}(X_1 \times X_2)}, \mathcal{Q}(X_1 \times X_2))$. Hence $\mathfrak{q}(X_1 \times X_2)$ is soft fuzzy dense in $(\mathcal{Q}(X_1 \times X_2), (T_1 \times T_2)^*)$. □

Definition 5.10. Let $(X_1 \times X_2, st(T_1 \times T_2))$ and $(\mathcal{Q}(X_1 \times X_2), (T_1 \times T_2)^*)$ be any soft fuzzy product $G_\delta$pre space and soft fuzzy product generalized topological space respectively. A function $f : (X_1 \times X_2, st(T_1 \times T_2)) \to (\mathcal{Q}(X_1 \times X_2), (T_1 \times T_2)^*)$ is said to be soft fuzzy $*$ continuous, if for each $(\mu^*, M^*) \in (T_1 \times T_2)^*$, $f^{-1}(\mu^*, M^*) \in st(T_1 \times T_2)$.

Definition 5.11. Let $(X_1 \times X_2, st(T_1 \times T_2))$ and $(\mathcal{Q}(X_1 \times X_2), (T_1 \times T_2)^*)$ be any soft fuzzy product $G_\delta$pre space and soft fuzzy product generalized topological space respectively. A function $f : (X_1 \times X_2, st(T_1 \times T_2)) \to (\mathcal{Q}(X_1 \times X_2), (T_1 \times T_2)^*)$ is said to be soft fuzzy $*$ open, if for each $(\mu, M) \in st(T_1 \times T_2)$, $f(\mu, M) \in (T_1 \times T_2)^*$.

Proposition 5.12. *The function* $\mathfrak{q}$ *is a soft fuzzy embedding of* $X_1 \times X_2$ *into* $\mathcal{Q}(X_1 \times X_2)$.

Proof. (i) $\mathfrak{q}$ is a soft fuzzy one to one function:

If $< x_1, x_2 > \neq < y_1, y_2 >$, we have $R_{<x_1,x_2>} \neq R_{<y_1,y_2>}$. Let $(< x_1, x_2) >_\alpha, \{< x_1, x_2 >\}) \neq (< y_1, y_2 >_\beta, \{< y_1, y_2 >\})$ be two soft fuzzy points.

(a) If $< x_1, x_2 > \neq < y_1, y_2 >$ for each $X_1 \times X_2 \mid R \in \mathcal{Q}(X_1 \times X_2)$. we have

$\mathfrak{q}(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\}) = (X_1 \times X_2 \mid R_{<x_1,x_2>_\alpha}, \{X_1 \times X_2 \mid R_{<x_1,x_2>}\})$.

Similarly $\mathfrak{q}(< y_1, y_2 >_\beta, \{< y_1, y_2 >\})$ and it is clear that

$$q(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\}) \neq \mathfrak{q}(< y_1, y_2 >_\beta, \{< y_1, y_2 >\}).$$

(b) If $< x_1, x_2 > = < y_1, y_2 >$, then $\alpha \neq \beta$ and therefore $\mathfrak{q}(< x_1, x_2 >_\alpha, \{< x_1, x_2 >\}) \neq \mathfrak{q}(< y_1, y_2 >_\beta, \{< y_1, y_2 >\})$. Hence $\mathfrak{q}$ is soft fuzzy one to one.

(ii) $\mathfrak{q}$ is soft fuzzy $*$ continuous:

For each $(\mu, M)^* \in (T_1 \times T_2)^*$ and $< x_1, x_2 > \in X_1 \times X_2$, we have

$$\begin{aligned} \mathfrak{q}^{-1}(\mu, M)^* &= \mathfrak{q}^{-1}(\mu^*, M^*) \\ &= (\mathfrak{q}^{-1}(\mu^*), \mathfrak{q}^{-1}(M^*)) \\ &= (\mu^* \circ \mathfrak{q}, \mathfrak{q}^{-1}(M^*)) \end{aligned}$$

where

$$\begin{aligned} \mu^* \circ \mathfrak{q}(< x_1, x_2 >) &= \mu^*(\mathfrak{q}(< x_1, x_2 >)) \\ &= \mu^*(X_1 \times X_2 \mid R_{<x_1,x_2>}) \\ &= \mu(< x_1, x_2 >) \end{aligned}$$

and $\mathfrak{q}^{-1}(M^*) = M$. Thus $\mathfrak{q}^{-1}(\mu, M)^* = (\mu, M) \in st(T_1 \times T_2)$. Hence $\mathfrak{q}$ is soft fuzzy $*$ continuous.

(iii) $\mathfrak{q}$ is a soft fuzzy $*$ open function on $\mathcal{Q}(X_1 \times X_2)$:

For each $X_1 \times X_2 \mid R \in \mathcal{Q}(X_1 \times X_2)$ and $(\mu, M) \in T_1 \times T_2$.

$$\begin{aligned} (\mu^*, M^*) \sqcap (\chi_{\mathfrak{q}(X_1 \times X_2)}, \{\mathfrak{q}(X_1 \times X_2)\}) &= (\mu^* \wedge \chi_{\mathfrak{q}(X_1 \times X_2)}, M^* \cap \{\mathfrak{q}(X_1 \times X_2)\}) \\ &= (\mathfrak{q}(\mu), \mathfrak{q}(M)) \\ &= \mathfrak{q}(\mu, M) \in (T_1 \times T_2)^* \end{aligned}$$

Thus $\mathfrak{q}$ is a soft fuzzy $*$ open function. Hence $\mathfrak{q}$ is a soft fuzzy embedding of $X_1 \times X_2$ into $\mathcal{Q}(X_1 \times X_2)$. □

Definition 5.13. A soft fuzzy product generalized topological space is said to be a *soft fuzzy product generalized compact space* if whenever $\sqcup_{i \in I}(\lambda_i, M_i) = (1, \mathcal{Q}(X_1 \times X_2))$, each (λ_i, M_i) is soft fuzzy product $(T_1 \times T_2)^*$ open , $i \in I$, there is a finite subset J of I with $\sqcup_{j \in J}(\lambda_i, M_i) = (1, \mathcal{Q}(X_1 \times X_2))$.

Proposition 5.14. *The soft fuzzy product generalized topological space* $(\mathcal{Q}(X_1 \times X_2),$ $(T_1 \times T_2)^*)$ *is soft fuzzy product generalized compact space.*

Proof. Let

$$\mathfrak{F} = \{(\lambda_i^*, N_i^*) \in (T_1 \times T_2)^* : (\lambda_i, N_i) \in st(T_1 \times T_2) fori \in J\}$$

be a soft fuzzy product $(T_1 \times T_2)^*$ open cover of $\mathcal{Q}(X_1 \times X_2)$. That is

$$\sqcup_{i \in J}(\lambda_i^*, N_i^*) = (1, \mathcal{Q}(X_1 \times X_2)).$$

By definition of (λ_i^*, N_i^*), $\sqcup_{i \in F}(\lambda_i^*, N_i^*) \sqsubseteq (1, \mathcal{Q}(X_1 \times X_2))$ for some finite subfamily F of J. Thus $\mathfrak{F}$ has a soft fuzzy finite subcover. Hence $(\mathcal{Q}(X_1 \times X_2), (T_1 \times T_2)^*)$ is soft fuzzy product generalized compact space. □

Proposition 5.15. *The soft fuzzy product generalized topological space*

$$(\mathcal{Q}(X_1 \times X_2),\ (T_1 \times T_2)^*)$$

is a compactification of a soft fuzzy product G_δ*pre space.*

Proof. Proof of the proposition is obtained from Propositions 5.4, 5.5, 5.9, 5.12 and 5.14. □

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