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New types of intuitionistic fuzzy interior ideals of ordered semigroups

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Abstract. In many applied disciplines like computer science, coding theory and formal languages, the use of fuzzified algebraic structures play a remarkable role. In this paper, we introduced some new types of generalization of an intuitionistic fuzzy interior ideal, called (α, β)-intuitionistic interior ideals of an ordered semigroup S. The important milestone of this paper is to link the ordinary interior ideals and $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideals. The notion of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal is introduced and characterizations of ordered semigroups in terms of $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideals and $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideals are given.

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1. Introduction

The use of fuzzy sets (extension of ordinary sets) in real world problem involving uncertainties are considered to be the most powerful tool compare to ordinary sets. This pioneering concept was first introduced by Zadeh [25] in 1965 which opened a new era of research around the world. The fuzzification of algebraic structures in a variety of applied subjects such as control engineering, fuzzy automata and error-correcting codes is of great interest for the researchers. The importance of such algebraic structures can be seen from the latest research which has been carried out in the last few years. In [6], Zadeh's idea of fuzzy sets was applied to generalize some of the basic concepts of general topology. Rosenfeld constituted in [18, 19] a similar application to the elementary theory of groupoids and groups and defined fuzzy subgroupoids and ideals. The notions of intuitionistic fuzzy sets (IFS) and intuitionistic L-fuzzy sets (ILFS) were introduced in [1, 2, 3, 4, 5] as a generalization of the notion of fuzzy sets (FS). The concept of fuzzy interior ideals in semigroups was introduced in [7, 17]. Generalizing the concept of fuzzy interior ideals, Kim and Jun [16] in 2001 defined intuitionistic fuzzy interior ideals of semigroups and investigated some properties of such ideals. Later on in 2006, Jun and Song [8] used the idea of quasi-coincidence of a fuzzy point with a fuzzy set and introduced (α, β)-fuzzy interior ideals of semigroups as a generalization of the concept of fuzzy interior ideal in semigroups. In an attempt to show the similarity between the theory of ordered semigroups and the theory of fuzzy ordered semigroups, Kehayopulu and Tsingelis [10] proved that in regular and intra-regular ordered semigroups the fuzzy ideals and the fuzzy interior ideals coincide. Khan and Shabir [12] introduced the concept of (α, β)-fuzzy interior ideal in ordered semigroups which is a generalization of a fuzzy interior ideal in ordered semigroup and investigated some related properties of ordered semigroups in terms of this notion. They also proved that the concepts of an $(\in, \in \vee q)$-fuzzy interior ideal and $(\in, \in \vee q)$-fuzzy ideal coincide in case of regular (resp. intra-regular) ordered semigroups. In [13], the characterizations of ordered semigroups in terms of $(\in, \in \vee q)$-fuzzy interior ideals are given. Sardar et al. [20] fuzzified the notions of interior ideals and characteristic interior ideals of a Γ-semigroup, and investigate some of their basic properties which leads to the characterization of a simple Γ-semigroup in terms of fuzzy interior ideals. For further study on interior ideals one can refer to [8, 11, 21, 24]. Shabir and Khan [22] introduced the notion of intuitionistic fuzzy interior ideals in ordered semigroups and provided classified ordered semigroups on the basis of intuitionistic fuzzy interior ideals.

Jun [9] generalized the concept of quasi-coincident with relation (q) in BCK-algebra and introduced a new notion denoted by (q_k) where $k \in [0, 1)$. Furthermore, Shabir et al. [23] characterized semigroups in terms of $(\in, \in \vee q_k)$-fuzzy ideals, where in [14, 15] this concept was applied to ordered semigroups and investigated several fundamental results of ordered semigroups in terms of this notion.

In this paper the notions of $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideals and $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideals are introduced. Moreover, the linkage between ordinary interior ideals and fuzzy interior ideals of type $(\in, \in \vee q_k)$ is an important achievement of this paper. Also, ordered semigroups are characterized by the properties of $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideals and $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideals.

2. Preliminaries

A semigroup $(S, .)$ at the same time a poset which satisfied the condition that, $a \leq b \longrightarrow ax \leq bx$ and $xa \leq xb$ for all $x \in S$ is known to be an *ordered semigroup* (or *po-semigroup*) denoted by a structure $(S, ., \leq)$. Throughout the paper S will denote an ordered semigroup unless otherwise stated. For any subsets A, B of S, we denote by,

$$AB := \{ab | a \in A, \text{ and } b \in B\}.$$

If A is a *subset* of S, then we denote by $(A]$ the subset of S defined as follows:

$$(A] := \{t \in S | t \leq a \text{ for some } a \in A\}.$$

If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. The operator "( ]" is a closure operator, and therefore,

· extensive (that is, $A \subseteq (A]$),

· isoton (that is, $A \subseteq B$ implies $(A] \subseteq (B]$),

· idempotent (that is, $((A]] \subseteq (A]$ and therefore $((A]] = (A]$).

Let $(S, ., \leq)$ be an ordered semigroup. A non-empty subset A of S is called a *subsemigroup* of S if $A^2 \subseteq A$.

A non-empty subset A of S is called *left* (resp. *right*) *ideal* of S if

(i) $(\forall a \in S)(\forall b \in A)$ $(a \leq b \longrightarrow a \in A)$,

(ii) $SA \subseteq A$ (resp. $AS \subseteq A$).

A non-empty subset A of S is called an ideal if it is both a left and a right ideal of S.

A non-empty subset A of an ordered semigroup S is called an *interior ideal* of S if

(i) $(\forall a \in S)(\forall b \in A)$ $(a \leq b \longrightarrow a \in A)$,

(ii) $A^2 \subseteq A$,

(iii) $SAS \subseteq A$.

An ordered semigroup S is *regular* [12] if for every $a \in S$ there exists, $x \in S$ such that $a \leq axa$, or equivalently, we have (i) $a \in (aSa]$ $\forall a \in S$ and (ii) $A \subseteq (ASA]$ $\forall A \subseteq S$. An ordered semigroup S is called *left* (resp. *right*) *regular* [12] if for every $a \in S$ there exists $x \in S$, such that $a \leq xa^2$(resp. $a \leq a^2x$), or equivalently, (i) $a \in (Sa^2]$(resp. $a \in (a^2S]$) $\forall a \in S$ and (ii) $A \subseteq (SA^2]$(resp. $A \subseteq (A^2S]$) $\forall A \subseteq S$.

The study of fuzzification of algebraic structures has been started in the pioneering paper of Rosenfeld [19] in 1971. Rosenfeld introduced the notion of fuzzy groups and successfully extended many results from groups in the theory of fuzzy groups.

An intuitionistic fuzzy sets (briefly IFS) A in a non-empty set X is an object having the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in X\}$, where the function $\mu_A : X \longrightarrow [0, 1]$ and $\gamma_A : X \longrightarrow [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$. For the sack of simplicity we will use the symbol $A = \langle x, \mu_A, \gamma_A\rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in X\}$.

Let S be an ordered semigroup and $A = \langle x, \mu_A, \gamma_A\rangle$ be an IFS of S. Then $A = \langle x, \mu_A, \gamma_A\rangle$ is called *an intuitionistic fuzzy subsemigroup* of S if

$$(\forall x, y \in S)(\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \text{and } \gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}).$$

Definition 2.1 ([22])**.** Let $A = \langle x, \mu_A, \gamma_A\rangle$ be an intuitionistic fuzzy subset of an ordered semigroup S. Then $A = \langle x, \mu_A, \gamma_A\rangle$ is called an intuitionistic fuzzy interior ideal of S if A satisfies the following conditions for all $x, y, z \in S$:

(C_1) $(\forall x \leq y)$ $\mu_A(x) \geq \mu_A(y)$ and $\gamma_A(x) \leq \gamma_A(y)$,

(C_2) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,

(C_3) $\mu_A(xyz) \geq \mu_A(y)$ and $\gamma_A(xyz) \leq \gamma_A(y)$.

The μ_A-level cut and γ_A-level cut of an intuitionistic fuzzy subset $A = \langle x, \mu_A, \gamma_A \rangle$ of an ordered semigroup S respectively are denoted and defined as

$$U(\mu_A; t) = \{x \in S \mid \mu_A(x) \geq t\}$$

and

$$L(\gamma_A; s) = \{x \in S \mid \gamma_A(x) \leq s\}$$

where $t \in (0, 1]$ and $s \in [0, 1)$. The (μ_A, γ_A)-level (t, s)-cut is defined as

$$C_{(t,s)}(A) = \{x \in S \mid \mu_A(x) \geq t \text{ and } \gamma_A(x) \leq s\}.$$

It is clear that $C_{(t,s)}(A) = U(\mu_A; t) \cup L(\gamma_A; s)$.

Theorem 2.2. *Let $A = \langle x, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy subset of S. Then $A = \langle x, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy interior ideal of S if and only if*

$$(\forall t \in (0, 1],\ s \in [0, 1)) \left(C_{(t,s)}(A) \neq \phi \Rightarrow C_{(t,s)}(A) \text{ is an interior ideal of } S \right).$$

Definition 2.3. Let S be an ordered semigroup. Consider $t \in (0, 1]$, $s \in [0, 1)$ and $x \in S$ such that $0 < t + s \leq 1$. An ordered intuitionistic fuzzy point (OIFP) is defined as:

$$x_t(a) = \begin{cases} t, & \text{if } a \in (x], \\ 0, & \text{if } a \notin (x]. \end{cases} \qquad x_s(a) = \begin{cases} 1 - t, & \text{if } a \in (x], \\ 1, & \text{if } a \notin (x], \end{cases}$$

denoted by $[x; (t, s)]$, $x \in S$ is the support of $[x; (t, s)]$, t (resp. s) is the degree of membership (resp. non-membership) of $[x; (t, s)]$.

Let $\phi \neq A \subseteq S$. Then the characteristic function $\chi_A =< x, \mu_{\chi_A}(x), \gamma_{\chi_A}(x) >$ of A is defined by

$$\mu_{\chi_A}(x) : S \longrightarrow [0, 1], x \mapsto \mu_{\chi_A}(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

and

$$\gamma_{\chi_A}(x) : S \longrightarrow [0, 1], x \mapsto \gamma_{\chi_A}(x) := \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$$

Lemma 2.4 ([22]). *A nonempty subset A of S is an intuitionistic fuzzy interior ideal of S if and only if the characteristic function $\chi_A =< x, \mu_{\chi_A}(x), \gamma_{\chi_A}(x) >$ of A is an intuitionistic fuzzy interior ideal of S.*

3. $(\in, \in \vee q_k)$-Intuitionistic fuzzy interior ideals

In this section, we define a generalized form of intuitionistic fuzzy interior ideals of an ordered semigroup S and introduce $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideals S, where $k \in [0, 1)$. An OIFP $[x; (t, s)]$ in said to belongs to (resp. quasi-coincident with) an intuitionistic fuzzy subset $A =< x, \mu_A(x), \gamma_A(x) >$ of S, written as $[x; (t, s)] \in A$ (resp. $[x; (t, s)]qA$) if $\mu_A(x) \geq t$ and $\gamma_A(x) \leq s$ (resp. $\mu_A(x) + t > 1$ and $\gamma_A(x) + s < 1$). If $[x; (t, s)] \in A$ or $[x; (t, s)]qA$), then we write $[x; (t, s)] \in \vee qA$.

Generalizing this concept of a fuzzy point $[x; (t, s)]$ quasi-coincident with an intuitionistic fuzzy set $A =< x, \mu_A(x), \gamma_A(x) >$, we define $[x; (t, s)]q_kA$ as $\mu_A(x) + t + k > 1$ and $\gamma_A(x) + s + k < 1$, where $k \in [0, 1)$.

By $[x; (t, s)]\overline{q_k}A$ we mean that $[x; (t, s)]q_kA$ does not hold.

Note that the case when $[x;(t,s)] \in \wedge q_k A$ is omitted. Since for an intuitionistic fuzzy subset $A =< x, \mu_A(x), \gamma_A(x) >$ of an ordered semigroup S with $\mu_A(x) \leq \frac{1-k}{2}$ and $\gamma_A(x) \geq \frac{1-k}{2}$ for all $x \in S$, the set $\{[x;(t,s)] \mid [x;(t,s)] \in \wedge q_k A\}$ is empty. This is because, if $[x;(t,s)] \in \wedge q_k A$ for some $t \in (0,1]$ and $s \in [0,1)$, then $\mu_A(x) \geq t$, $\mu_A(x) + t + k > 1$ and $\gamma_A(x) \leq s$ and $\gamma_A(x) + s + k < 1$. Therefore

$$1 < \mu_A(x) + t + k \leq \mu_A(x) + \mu_A(x) + k = 2\mu_A(x) + k$$

and

$$1 > \gamma_A(x) + s + k \geq \gamma_A(x) + \gamma_A(x) + k = 2\gamma_A(x) + k.$$

From this we see that $\mu_A(x) > \frac{1-k}{2}$ and $\gamma_A(x) < \frac{1-k}{2}$ which is a contradiction. Hence, $\{[x;(t,s)] \mid [x;(t,s)] \in \wedge q_k A\} = \varnothing$.

Definition 3.1. An intuitionistic fuzzy subset $A =< x, \mu_A(x), \gamma_A(x) >$ of S is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideal of S, if for all $x, y, z \in S$, $t, t_1, t_2 \in (0,1]$ and $s, s_1, s_2 \in [0,1)$, A satisfies the following condition:
(C_4) $x \leq y, [y;(t,s)] \in A \rightarrow [x;(t,s)] \in \vee q_k A$,
(C_5) $[x;(t_1,s_1)] \in A$, $[y;(t_2,s_2)] \in A \rightarrow [xy; \min\{t_1,t_2\}, \max\{s_1,s_2\}] \in \vee q_k A$,
(C_6) $[y;(t,s)] \in A \rightarrow [xyz;(t,s)] \in \vee q_k A$.

Example 3.2. Consider an ordered semigroup $S = \{a,b,c,d,e\}$ with order relations $a \leq c \leq e$, $a \leq d \leq e$, $b \leq d$ and $b \leq e$ and the following multiplication table:

.	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

(1) Define an IFS $A =< x, \mu_A, \gamma_A >$ by

$$\mu_A \quad : \quad S \rightarrow [0,1] \mid \mu_A(x) = \begin{cases} 0.50 \text{ if } x = a \\ 0.45 \text{ if } x = b \\ 0.65 \text{ if } x = c \\ 0.55 \text{ if } x = d \\ 0.40 \text{ if } x = e \end{cases}$$

and

$$\gamma_A \quad : \quad S \rightarrow [0,1] \mid \gamma_A(x) = \begin{cases} 0.35 \text{ if } x = a \\ 0.20 \text{ if } x = b \\ 0.30 \text{ if } x = c \\ 0.40 \text{ if } x = d \\ 0.40 \text{ if } x = e \end{cases}.$$

Then $A =< x, \mu_A, \gamma_A >$ is a $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideals of S for all $t, t_1, t_2 \in (0,1]$ and $s, s_1, s_2 \in [0,1)$ and $k = 0.2$.

Lemma 3.3. *An intuitionistic fuzzy subset $A =< x, \mu_A(x), \gamma_A(x) >$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideal of S if and only if it satisfies the following conditions for all $x, y, z \in S$:*
(C_7) $x \leq y \rightarrow \mu_A(x) \geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\}$ and $\gamma_A(x) \leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\}$,

(C_8) $\mu_A(xy) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$ *and* $\gamma_A(xy) \leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\}$,

(C_9) $\mu_A(xyz) \geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\}$ *and* $\gamma_A(xyz) \leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\}$, *for all* $x, y, z \in S$.

Proof. Consider $A =< x, \mu_A(x), \gamma_A(x) >$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideal of S. Suppose $x, y \in S$ be such that $x \leq y$. Let $\mu_A(y) < \frac{1-k}{2}$ and $\gamma_A(y) > \frac{1-k}{2}$. If $\mu_A(x) < \mu_A(y)$ and $\gamma_A(x) > \gamma_A(y)$, then there exist some $t \in (0, \frac{1-k}{2}]$ and $s \in [\frac{1-k}{2}, 1)$ such that $\mu_A(x) < t \leq \mu_A(y)$ and $\gamma_A(x) > s \geq \gamma_A(y)$. We see that $[y; (t, s)] \in A$ but $[x; (t, s)] \overline{\in} A$, also $\mu_A(x) + t + k < 1$ and $\gamma_A(x) + s + k > 1$, follows that $[x; (t, s)] \overline{\in \vee q_k} A$, a contradiction. Hence, $\mu_A(x) \geq \mu_A(y)$ and $\gamma_A(x) \leq \gamma_A(y)$. And if $\mu_A(y) \geq \frac{1-k}{2}$ and $\gamma_A(y) \leq \frac{1-k}{2}$, then $\left[y; \left(\frac{1-k}{2}, \frac{1-k}{2}\right)\right] \in A$ and consequently $\left[x; \left(\frac{1-k}{2}, \frac{1-k}{2}\right)\right] \in \vee q_k A$, follows that $\left[x; \left(\frac{1-k}{2}, \frac{1-k}{2}\right)\right] \in A$ or $\left[x; \left(\frac{1-k}{2}, \frac{1-k}{2}\right)\right] q_k A$, that is $\mu_A(x) \geq \frac{1-k}{2}$ and $\gamma_A(x) \leq \frac{1-k}{2}$ or $\mu_A(x) + \frac{1-k}{2} + k > 1$ and $\gamma_A(x) + \frac{1-k}{2} + k < 1$. Hence, $\mu_A(x) \geq \frac{1-k}{2}$ and $\gamma_A(x) \leq \frac{1-k}{2}$ other wise $\mu_A(x) + \frac{1-k}{2} + k < 1$ and $\gamma_A(x) + \frac{1-k}{2} + k > 1$, a contradiction. Consequently

$$\mu_A(x) \geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \text{ and } \gamma_A(x) \leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\},$$

for all $x, y \in S$ with $x \leq y$.

Let us take $x, y \in S$ such that

$$\min\left\{\mu_A(x), \mu_A(y)\right\} < \frac{1-k}{2} \text{ and } \max\left\{\gamma_A(x), \gamma_A(y)\right\} > \frac{1-k}{2}.$$

We claim that

$$\mu_A(xy) \geq \min\left\{\mu_A(x), \mu_A(y)\right\} \text{ and } \gamma_A(xy) \leq \max\left\{\gamma_A(x), \gamma_A(y)\right\}.$$

If not, then for some $t \in (0, \frac{1-k}{2}]$ and $s \in [\frac{1-k}{2}, 1)$ we have

$$\mu_A(xy) < t \leq \min\left\{\mu_A(x), \mu_A(y)\right\} \text{ and } \gamma_A(xy) > s \geq \max\left\{\gamma_A(x), \gamma_A(y)\right\}.$$

Follows that $[x; (t, s)] \in A$ and $[y; (t, s)] \in A$ but $[xy; (t, s)] \overline{\in} A$, also $\mu_A(xy) + t + k < 1$, $\gamma_A(xy) + s + k > 1$, that is $[xy; (t, s)] \overline{q_k} A$ and so $[xy; (t, s)] \overline{\in \vee q_k} A$ a contradiction. Hence

$$\mu_A(xy) \geq \min\left\{\mu_A(x), \mu_A(y)\right\} \text{ and } \gamma_A(xy) \leq \max\left\{\gamma_A(x), \gamma_A(y)\right\}.$$

Now if

$$\min\left\{\mu_A(x), \mu_A(y)\right\} \geq \frac{1-k}{2} \text{ and } \max\left\{\gamma_A(x), \gamma_A(y)\right\} \leq \frac{1-k}{2},$$

then $\left[x; \left(\frac{1-k}{2}, \frac{1-k}{2}\right)\right] \in A$ and $\left[y; \left(\frac{1-k}{2}, \frac{1-k}{2}\right)\right] \in A$. By ($C_5$)

$$\left[xy; \left(\frac{1-k}{2}, \frac{1-k}{2}\right)\right] = \left[xy; \min\left\{\frac{1-k}{2}, \frac{1-k}{2}\right\}, \max\left\{\frac{1-k}{2}, \frac{1-k}{2}\right\}\right] \in \vee q_k A,$$

and hence $\mu_A(xy) \geq \frac{1-k}{2}$ and $\gamma_A(xy) \leq \frac{1-k}{2}$ or $\mu_A(xy) + \frac{1-k}{2} + k > 1$ and $\gamma_A(xy) + \frac{1-k}{2} + k < 1$. If $\mu_A(xy) < \frac{1-k}{2}$ and $\gamma_A(xy) > \frac{1-k}{2}$, then $\mu_A(xy) + \frac{1-k}{2} + k < 1$ and $\gamma_A(xy) + \frac{1-k}{2} + k > 1$, this is impossible. Therefore

$$\mu_A(xy) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \text{ and } \gamma_A(xy) \leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\},$$

for all $x, y \in S$.

We take $a, b, c \in S$ such that $\mu_A(b) < \frac{1-k}{2}$ and $\gamma_A(b) > \frac{1-k}{2}$ and claim that

$$\mu_A(abc) \geq \mu_A(b) \text{ and } \gamma_A(abc) \leq \gamma_A(b).$$

If this is not true, then for some $t_a \in (0, \frac{1-k}{2}]$ and $s_a \in [\frac{1-k}{2}, 1)$ we have

$$\mu_A(abc) < t_a \leq \mu_A(b) \text{ and } \gamma_A(abc) > s_a \geq \gamma_A(b).$$

Follows that $[b; (t_a, s_a)] \in A$ but $[abc; (t_a, s_a)] \overline{\in} A$, also $\mu_A(abc)+t_a+k < 1$, $\gamma_A(abc)+s_a+k > 1$, i.e., $[abc; (t_a, s_a)] \overline{q_k} A$ and so $[abc; (t_a, s_a)] \overline{\in \vee q_k} A$ a contradiction. Hence, $\mu_A(abc) \geq \mu_A(b)$ and $\gamma_A(abc) \leq \gamma_A(b)$. Now if $\mu_A(b) \geq \frac{1-k}{2}$ and $\gamma_A(b) \leq \frac{1-k}{2}$, then $\left[b; \left(\frac{1-k}{2}, \frac{1-k}{2}\right)\right] \in A$ and by (C_6)

$$\left[abc; \left(\frac{1-k}{2}, \frac{1-k}{2}\right)\right] = \left[abc; \min\left\{\frac{1-k}{2}, \frac{1-k}{2}\right\}, \max\left\{\frac{1-k}{2}, \frac{1-k}{2}\right\}\right] \in \vee q_k A,$$

and hence $\mu_A(abc) \geq \frac{1-k}{2}$ and $\gamma_A(abc) \leq \frac{1-k}{2}$ or $\mu_A(abc)+\frac{1-k}{2}+k > 1$ and $\gamma_A(abc)+\frac{1-k}{2}+k < 1$. If $\mu_A(abc) < \frac{1-k}{2}$ and $\gamma_A(abc) > \frac{1-k}{2}$, then $\mu_A(abc)+\frac{1-k}{2}+k < 1$ and $\gamma_A(abc)+\frac{1-k}{2}+k > 1$, this is impossible. Therefore

$$\mu_A(x, y, z) \geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \text{ and } \gamma_A(xyz) \leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\},$$

for all $x, y, z \in S$.

Conversely, let $A =< x, \mu_A(x), \gamma_A(x) >$ be an intuitionistic fuzzy subset of S that satisfies conditions (C_7), (C_8) and (C_9). Let $x, y \in S$ with $x \leq y$, $t \in (0, 1]$, $s \in [0, 1)$ and $[y; (t, s)] \in A$. Then $\mu_A(y) \geq t$ and $\gamma_A(y) \leq s$ and so by (C_7)

$$\begin{aligned} \mu_A(x) &\geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \\ &\geq \min\left\{t, \frac{1-k}{2}\right\} \\ &= \begin{cases} t & \text{if } t \leq \frac{1-k}{2} \\ \frac{1-k}{2} & \text{if } t > \frac{1-k}{2} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(x) &\leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\} \\ &\leq \max\left\{s, \frac{1-k}{2}\right\} \\ &= \begin{cases} s & \text{if } s \geq \frac{1-k}{2}, \\ \frac{1-k}{2} & \text{if } s < \frac{1-k}{2}. \end{cases} \end{aligned}$$

It follows that $\mu_A(x) \geq t$ and $\gamma_A(x) \leq s$ or $\mu_A(x)+t+k > 1$ and $\gamma_A(x)+s+k < 1$ (since $t > \frac{1-k}{2}$ and $s < \frac{1-k}{2}$) and so $[x; (t, s)] \in A$ or $[x; (t, s)]\, q_k A$, hence $[x; (t, s)] \in \vee q_k A$.

Let $x, y \in S$, $t_1, t_2 \in (0, 1]$ and $s_1, s_2 \in [0, 1)$ such that $[x; (t_1, s_1)] \in A$ and $[y; (t_2, s_2)] \in A$. Then $\mu_A(x) \geq t_1$, $\gamma_A(x) \leq s_1$ and $\mu_A(y) \geq t_2$, $\gamma_A(y) \leq s_2$. This follows from (C_8) that

$$\begin{aligned}\mu_A(xy) &\geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}\\ &\geq \min\left\{t_1, t_2, \frac{1-k}{2}\right\}\\ &= \begin{cases}\min\{t_1,t_2\} & \text{if } \min\{t_1,t_2\} \leq \frac{1-k}{2}\\ \frac{1-k}{2} & \text{if } \min\{t_1,t_2\} > \frac{1-k}{2}\end{cases}\end{aligned}$$

and

$$\begin{aligned}\gamma_A(xy) &\leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\}\\ &\leq \max\left\{s_1, s_2, \frac{1-k}{2}\right\},\\ &= \begin{cases}\max\{s_1,s_2\} & \text{if } \max\{s_1,s_2\} \geq \frac{1-k}{2}\\ \frac{1-k}{2} & \text{if } \max\{s_1,s_2\} < \frac{1-k}{2}\end{cases}.\end{aligned}$$

Thus we have $[xy; \min\{t_1,t_2\}, \max\{s_1,s_2\}] \in A$ or $[xy; \min\{t_1,t_2\}, \max\{s_1,s_2\}]\, q_k A$, hence $[xy; \min\{t_1,t_2\}, \max\{s_1,s_2\}] \in \vee q_k A$.

Let $x, y, z \in S$, $t \in (0,1]$ and $s \in [0,1)$ such that $[y;(t,s)] \in A$. Then $\mu_A(y) \geq t$, $\gamma_A(y) \leq s$. This follows from (C_9) that

$$\begin{aligned}\mu_A(xyz) &\geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\}\\ &\geq \min\left\{t, \frac{1-k}{2}\right\}\\ &= \begin{cases}t & \text{if } t \leq \frac{1-k}{2}\\ \frac{1-k}{2} & \text{if } t > \frac{1-k}{2}\end{cases},\end{aligned}$$

and

$$\begin{aligned}\gamma_A(xyz) &\leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\}\\ &\leq \max\left\{s, \frac{1-k}{2}\right\}\\ &= \begin{cases}s & \text{if } s \geq \frac{1-k}{2}\\ \frac{1-k}{2} & \text{if } s < \frac{1-k}{2}\end{cases}.\end{aligned}$$

Thus we have $[xyz;(t,s)] \in A$ or $[xyz;(t,s)]\, q_k A$, hence $[xyz; \min\{t_1,t_2\}, \max\{s_1,s_2\}] \in \vee q_k A$.

Consequently, $A =< x, \mu_A(x), \gamma_A(x) >$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideal of S. □

Definition 3.4. Let $A =< x, \mu_A(x), \gamma_A(x) >$ be an intuitionistic fuzzy subset of S. Then A is an $(\in, \in)$-intuitionistic fuzzy interior ideal of S if it satisfies the following conditions for all $x, y, z \in S$:

(C_{10}) $x \leq y, [y;(t,s)] \in A \rightarrow [x;(t,s)] \in A$,

(C_{11}) $[x;(t_1,s_1)] \in A$, $[y;(t_2,s_2)] \in A \rightarrow [xy; \min\{t_1,t_2\}, \max\{s_1,s_2\}] \in A$,

(C_{12}) $[y;(t,s)] \in A \to [xyz;(t,s)] \in A$.

Theorem 3.5. *Let $A =< x, \mu_A(x), \gamma_A(x) >$ be an intuitionistic fuzzy interior ideal of S. Then A is an $(\in, \in)$-intuitionistic fuzzy interior ideal of S.*

Proof. Let A is an intuitionistic fuzzy interior ideal of S. Let$x, y \in S$ such that $x \leq y$ and $[y;(t,s)] \in A$. Then $\mu_A(y) \geq t$ and $\gamma_A(y) \leq s$. And by (C_1) we have $\mu_A(x) \geq \mu_A(y) \geq t$ and $\gamma_A(x) \leq \gamma_A(y) \leq s$ and hence $[x;(t,s)] \in A$. Next we suppose $[x;(t_1,s_1)] \in A$ and $[y;(t_2,s_2)] \in A$, then $\mu_A(x) \geq t_1$, $\gamma_A(x) \leq s_1$ and $\mu_A(y) \geq t_2$, $\gamma_A(y) \leq s_2$. By (C_2) we see that

$$\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \geq \min\{t_1, t_2\},$$

and

$$\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\} \leq \max\{s_1, s_2\},$$

follows that $[xy; \min\{t_1,t_2\}, \max\{s_1,s_2\}] \in A$.

Finally $[y;(t,s)] \in A$, then $\mu_A(y) \geq t$ and $\gamma_A(y) \leq s$ and by (C_3) we have $\mu_A(xyz) \geq \mu_A(y) \geq t$ and $\gamma_A(xyz) \leq \gamma_A(y) \leq s$ shows that $[xyz;(t,s)] \in A$. □

From the above Theorem it is clear that every intuitionistic fuzzy interior ideal is an $(\in, \in)$-intuitionistic fuzzy interior ideal and it can be simply shown that every $(\in, \in)$-intuitionistic fuzzy interior ideal of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideal of S. The following example shows that every $(\in, \in \vee q_k)$-intuitionistic fuzzy interior-ideal of S may not be an intuitionistic fuzzy interior ideal of S.

Example 3.6. Consider the ordered semigroup S and IFS $A =< x, \mu_A, \gamma_A >$ as defined in Example 1, then $A =< x, \mu_A, \gamma_A >$ is an $(\in, \in \vee q_k)-$intuitionistic fuzzy interior ideals of S but $A =< x, \mu_A, \gamma_A >$ is not an $(\in, \in)-$ intuitionistic fuzzy interior ideals of S because for $b, c, d \in S$, we have $\mu_A(c) = 0.65 > 0.60 = t$, $\gamma_A(c) = 0.30 < 0.35 = s$ that is $< c;(t,s) > \in A$ and $\mu_A(bcd) = \mu_A(d) = 0.55 < 0.60 = t$, $\gamma_A(d) = 0.40 > 0.35 = s$ i.e. $< bcd;(t,s) > \overline{\in} A$.

In the following theorem we give condition for an $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideal to be and $(\in, \in)$-intuitionistic fuzzy interior ideal.

Theorem 3.7. *Let $A = \langle x, \mu_A, \gamma_A \rangle$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideal of S. If $\mu_A(x) < \frac{1-k}{2}$ and $\gamma_A(x) > \frac{1-k}{2}$ for all $x \in S$, then $A = \langle x, \mu_A, \gamma_A \rangle$ is an $(\in, \in)$-intuitionistic fuzzy interior ideal of S.*

Proof. Let $A = \langle x, \mu_A, \gamma_A \rangle$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideal of S and $\mu_A(x) < \frac{1-k}{2}$ and $\gamma_A(x) > \frac{1-k}{2}$ for all $x \in S$. Take $t \in (0,1]$, $s \in [0,1)$ and $x, y \in S$ such that $x \leq y$ and $[y;(t,s)] \in A$, then $\mu_A(y) \geq t$ and $\gamma_A(y) \leq s$. Also from (C_7) of Lemma (3.3), we have

$$\begin{aligned} \mu_A(x) &\geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \\ &= \mu_A(y) \\ &\geq t, \end{aligned}$$

and

$$\begin{aligned}\gamma_A(x) &\leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\} \\ &= \gamma_A(y) \\ &\leq s.\end{aligned}$$

Hence, $[x;(t,s)] \in A$.

Let $t_1, t_2 \in (0,1]$, $s_1, s_2 \in [0,1)$ and $[x;(t_1,s_1)] \in A$, $[y;(t_2,s_2)] \in A$, then $\mu_A(x) \geq t_1$, $\gamma_A(x) \leq s_1$ and $\mu_A(y) \geq t_2$, $\gamma_A(y) \leq s_2$ and from (C_8) of Lemma (3.3),

$$\begin{aligned}\mu_A(xy) &\geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \\ &= \min\{\mu_A(x), \mu_A(y)\} \\ &\geq \min\{t_1, t_2\},\end{aligned}$$

similarly $\gamma_A(xy) \leq \max\{s_1, s_2\}$, hence $[xy; \min\{t_1,t_2\}, \max\{s_1,s_2\}] \in A$.

Let $x, y, z \in S$ with $[y;(t,s)] \in A$, then $\mu_A(y) \geq t$, $\gamma_A(y) \leq s$ and from (C_9) of Lemma (3.3) we see that $\mu_A(xyz) \geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} = \mu_A(y) \geq t$ and $\gamma_A(xyz) \leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\} = \gamma_A(y) \leq s$, hence $[xyz;(t,s)] \in A$. Consequently, $A = \langle x, \mu_A, \gamma_A\rangle$ is an $(\in, \in)$-intuitionistic fuzzy interior ideal of S. □

Theorem 3.8. *For an intuitionistic fuzzy subset $A = \langle x, \mu_A, \gamma_A\rangle$ of S, the following conditions are equivalent for all $t \in (0, \frac{1-k}{2}]$ and $s \in [\frac{1-k}{2}, 1)$:*

(C_{13}) $A = \langle x, \mu_A, \gamma_A\rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideal of S.

(C_{14}) $C_{(t,s)}(A) \neq \phi \Rightarrow C_{(t,s)}(A)$ is an interior ideal of S.

Proof. Assume that $A = \langle x, \mu_A, \gamma_A\rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideal of S and $C_{(t,s)}(A) \neq \phi$. Let $x, y \in S$ with $x \leq y$ such that $y \in C_{(t,s)}(A)$ for some $t \in (0, \frac{1-k}{2}]$ and $s \in [\frac{1-k}{2}, 1)$. Then $\mu_A(y) \geq t$ and $\gamma_A(y) \leq s$ and by hypothesis using (C_7) of Lemma (3.3), we have

$$\mu_A(x) \geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} = t,$$

and

$$\gamma_A(x) \leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\} \leq \max\left\{s, \frac{1-k}{2}\right\} = s,$$

This shows that $x \in C_{(t,s)}(A)$.

Let $x, y \in C_{(t,s)}(A)$, then $\mu_A(x) \geq t$, $\gamma_A(x) \leq s$ and $\mu_A(y) \geq t$, $\gamma_A(y) \leq s$. (C_8) of Lemma (3.3) implies that

$$\begin{aligned}\mu_A(xy) &\geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \\ &\geq \min\left\{t, \frac{1-k}{2}\right\} = t\end{aligned}$$

and

$$\begin{aligned}\gamma_A(xy) &\leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} \\ &\leq \max\left\{s, \frac{1-k}{2}\right\} = s.\end{aligned}$$

Thus $xy \in C_{(t,s)}(A)$.

Next we suppose $x, y, z \in S$ with $y \in C_{(t,s)}(A)$, then $\mu_A(y) \geq t$, $\gamma_A(y) \leq s$, then from Lemma 3.3 (C_9)

$$\mu_A(xyz) \geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} = t,$$

and

$$\gamma_A(xyz) \leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\} \leq \max\left\{s, \frac{1-k}{2}\right\} = s,$$

Thus $xyz \in C_{(t,s)}(A)$. Consequently $C_{(t,s)}(A)$ is is an interior ideal of S.

Conversely, let $A = \langle x, \mu_A, \gamma_A \rangle$ be an IFS of S such that $C_{(t,s)}(A) \neq \phi$ and is an interior ideal of S for all $t \in (0, \frac{1-k}{2}]$ and $s \in [\frac{1-k}{2}, 1)$. If there exist $a, b \in S$ with $a \leq b$, we claim that $\mu_A(a) \geq \min\left\{\mu_A(b), \frac{1-k}{2}\right\}$ and $\gamma_A(a) \leq \max\left\{\gamma_A(b), \frac{1-k}{2}\right\}$. If not then

$$\mu_A(a) < t_a \leq \min\left\{\mu_A(b), \frac{1-k}{2}\right\}$$

and

$$\gamma_A(a) > s_a \geq \max\left\{\gamma_A(b), \frac{1-k}{2}\right\}$$

for some $t_a \in (0, \frac{1-k}{2}]$ and $s_a \in [\frac{1-k}{2}, 1)$. Follows that $b \in C_{(t_a,s_a)}(A)$ but $a \notin C_{(t_a,s_a)}(A)$ a contradiction. Therefore

$$\mu_A(x) \geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \text{ and } \gamma_A(x) \leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\},$$

for all $x, y \in S$ with $x \leq y$.

Assume that there exist $a, b \in S$ such that

$$\mu_A(ab) < \min\left\{\mu_A(a), \mu_A(b), \frac{1-k}{2}\right\}$$

and

$$\gamma_A(ab) > \max\left\{\gamma_A(a), \gamma_A(b), \frac{1-k}{2}\right\},$$

then

$$\mu_A(ab) < t_0 \leq \min\left\{\mu_A(a), \mu_A(b), \frac{1-k}{2}\right\}$$

and

$$\gamma_A(ab) > s_0 \geq \max\left\{\gamma_A(a), \gamma_A(b), \frac{1-k}{2}\right\},$$

for some $t_0 \in (0, \frac{1-k}{2}]$ and $s_0 \in [\frac{1-k}{2}, 1)$. It follows that $a \in C_{(t_0,s_0)}(A)$ and $b \in C_{(t_0,s_0)}(A)$ but $ab \notin C_{(t_0,s_0)}(A)$, a contradiction. Therefore

$$\mu_A(xy) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$$
and
$$\gamma_A(xy) \leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\},$$

for all $x, y \in S$.

Consider

$$\mu_A(abc) < \min\left\{\mu_A(b), \frac{1-k}{2}\right\} \text{ and } \gamma_A(ab) > \max\left\{\gamma_A(b), \frac{1-k}{2}\right\},$$

for some $a, b, c \in S$.

Then there exist $t_1 \in (0, \frac{1-k}{2}]$ and $s_1 \in [\frac{1-k}{2}, 1)$ such that

$$\mu_A(abc) < t_1 \leq \min\left\{\mu_A(b), \frac{1-k}{2}\right\}$$
and
$$\gamma_A(abc) > s_1 \geq \max\left\{\gamma_A(b), \frac{1-k}{2}\right\},$$

for some $t_1 \in (0, \frac{1-k}{2}]$ and $s_1 \in [\frac{1-k}{2}, 1)$. It follows that $b \in C_{(t_1,s_1)}(A)$ but $abc \notin C_{(t_1,s_1)}(A)$, a contradiction. Therefore

$$\mu_A(xyz) \geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \text{ and } \gamma_A(xyz) \leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\},$$

for all $x, y, z \in S$. An thus $A = \langle x, \mu_A, \gamma_A\rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideal of S. □

For any IFS $A = \langle x, \mu_A, \gamma_A\rangle$ of S, $t \in (0, 1]$ and $s \in [0, 1)$, we consider two subsets:

$$Q^k_{(t,s)}(A) := \{x \in S |\, [x; (t, s)]\, q_k A\},$$

$$[A]^k_{(t,s)} := \{x \in S |\, [x; (t, s)] \in \vee q_k A\}.$$

It is obvious that $[A]^k_{(t,s)} = C_{(t_1,s_1)}(A) \cup Q^k_{(t,s)}(A)$.

Proposition 3.9. *Let $A = \langle x, \mu_A, \gamma_A\rangle$ is an $(\in, \in \vee q_k)$ intuitionistic fuzzy interior ideal of S, then $Q_{(t,s)}(A) \neq \phi \Rightarrow Q_{(t,s)}(A)$ is an interior ideal of S for all $t \in (0, 1]$, $s \in [0, 1)$.*

Proof. Consider $A = \langle x, \mu_A, \gamma_A\rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideal of S. Let $t \in (0, 1]$ and $s \in [0, 1)$ such that $Q_{(t,s)}(A) \neq \phi$. Let $x, y \in S$ with $x \leq y$ such that $y \in Q_{(t,s)}(A)$. Then $\mu_A(y) + t + k > 1$ and $\gamma_A(y) + s + k < 1$ and by hypothesis using Lemma (3.3) (C_7), we have

$$\begin{aligned}\mu_A(x) &\geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \\ &= \begin{cases} \frac{1-k}{2} & \text{if } \mu_A(y) \geq \frac{1-k}{2} \\ \mu_A(y) & \text{if } \mu_A(y) < \frac{1-k}{2} \end{cases} \\ &> 1-t-k\end{aligned}$$

and

$$\begin{aligned}\gamma_A(x) &\leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\} \\ &= \begin{cases} \frac{1-k}{2} & \text{if } \gamma_A(y) \leq \frac{1-k}{2} \\ \gamma_A(y) & \text{if } \gamma_A(y) > \frac{1-k}{2} \end{cases} \\ &< 1-s-k\end{aligned}$$

It follows that $x \in Q^k_{(t,s)}(A)$.

Let $x, y \in Q^k_{(t,s)}(A)$, then $\mu_A(x)+t+k > 1$, $\gamma_A(x)+s+k < 1$ and $\mu_A(y)+t+k > 1$, $\gamma_A(y) + s + k < 1$. Lemma (3.3) (C_8), implies that

$$\begin{aligned}\mu_A(xy) &\geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \\ &= \left\{\begin{array}{ll} \frac{1-k}{2} & \text{if} \min\{\mu_A(x), \mu_A(y)\} \geq \frac{1-k}{2} \\ \min\{\mu_A(x), \mu_A(y)\} & \text{if } \min\{\mu_A(x), \mu_A(y)\} < \frac{1-k}{2} \end{array}\right\} \\ &> 1-t-k,\end{aligned}$$

and

$$\begin{aligned}\gamma_A(xy) &\leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} \\ &= \left\{\begin{array}{ll} \frac{1-k}{2} & \text{if } \max\{\gamma_A(x), \gamma_A(y)\} < \frac{1-k}{2} \\ \max\{\gamma_A(x), \gamma_A(y)\} & \text{if } \max\{\gamma_A(x), \gamma_A(y)\} \geq \frac{1-k}{2} \end{array}\right\} \\ &< 1-s-k.\end{aligned}$$

Thus $xy \in Q^k_{(t,s)}(A)$.

Next we suppose $x, y, z \in S$ such that $y \in Q^k_{(t,s)}(A)$, then $\mu_A(y) + t + k > 1$, $\gamma_A(y) + s + k < 1$. Lemma (3.3) (C_6), implies that

$$\begin{aligned}\mu_A(xyz) &\geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \\ &= \left\{\begin{array}{ll} \frac{1-k}{2} & \text{if } \mu_A(y) \geq \frac{1-k}{2} \\ \mu_A(y) & \text{if } \mu_A(y) < \frac{1-k}{2} \end{array}\right\} \\ &> 1-t-k,\end{aligned}$$

and

$$\begin{aligned}\gamma_A(xyz) &\leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\} \\ &= \left\{\begin{array}{ll} \frac{1-k}{2} & \text{if } \gamma_A(y) < \frac{1-k}{2} \\ \gamma_A(y) & \text{if } \gamma_A(y) \geq \frac{1-k}{2} \end{array}\right\} \\ &< 1-s-k.\end{aligned}$$

Thus $xyz \in Q^k_{(t,s)}(A)$. Consequently $Q^k_{(t,s)}(A)$ is an interior ideal of S. □

Theorem 3.10. *An IFS $A = \langle x, \mu_A, \gamma_A\rangle$ of S is an $(\in, \in \vee q_k)$ intuitionistic fuzzy interior ideal of S if and only if $[A]^k_{(t,s)} \neq \phi \Rightarrow [A]^k_{(t,s)}$ is an interior ideal of S for all $t \in (0,1]$, $s \in [0,1)$.*

Proof. Assume that $A = \langle x, \mu_A, \gamma_A\rangle$ is an $(\in, \in \vee q_k)$ intuitionistic fuzzy interior ideal of S and $[A]^k_{(t,s)} \neq \phi$ for all $t \in (0,1]$, $s \in [0,1)$. Let $x, y \in S$ such that $x \leq y$ and $y \in [A]^k_{(t,s)}$ then $y \in C_{(t,s)}(A)$ or $y \in Q^k_{(t,s)}(A)$, follows that $\mu_A(y) \geq t$ and $\gamma_A(y) \leq s$ or $\mu_A(y) + t + k > 1$ and $\gamma_A(y) + s + k < 1$. Using (C_7) of Lemma (3.3), we get

$$\mu_A(x) \geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \text{ and } \gamma_A(x) \leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\} \tag{3.1}$$

We consider two cases:
(i) $\mu_A(y) \leq \frac{1-k}{2}$ and $\gamma_A(y) \geq \frac{1-k}{2}$.
(ii) $\mu_A(y) > \frac{1-k}{2}$ and $\gamma_A(y) < \frac{1-k}{2}$.

Combining case (i) and (3.1) we get $\mu_A(x) \geq \mu_A(y)$ and $\gamma_A(x) \leq \gamma_A(y)$. Now if $\mu_A(y) \geq t$ and $\gamma_A(y) \leq s$, then $\mu_A(x) \geq \mu_A(y) \geq t$ and $\gamma_A(x) \leq \gamma_A(y) \leq s$, this implies $x \in C_{(t,s)}(A)$. And if $\mu_A(y) + t + k > 1$ and $\gamma_A(y) + s + k < 1$, then $\mu_A(x) + t + k \geq \mu_A(y) + t + k > 1$ and $\gamma_A(x) + s + k \leq \gamma_A(y) + s + k < 1$, this shows that $x \in Q^k_{(t,s)}(A)$. Hence, we see that $x \in C_{(t,s)}(A)$ or $x \in Q^k_{(t,s)}(A)$ and so we say that $x \in C_{(t,s)}(A) \cup Q^k_{(t,s)}(A) = [A]^k_{(t,s)}$.

Now combining case (ii) and (3.1) we get $\mu_A(x) \geq \frac{1-k}{2}$ and $\gamma_A(x) \leq \frac{1-k}{2}$. If $t \leq \frac{1-k}{2}$ and $s \geq \frac{1-k}{2}$, then $\mu_A(x) \geq \frac{1-k}{2} \geq t$ and $\gamma_A(x) \leq \frac{1-k}{2} \leq s$ implies $x \in C_{(t,s)}(A)$. But if $t > \frac{1-k}{2}$ and $s < \frac{1-k}{2}$, then $\mu_A(x) + t + k > 1$ and $\gamma_A(x) + s + k < 1$, this shows that $x \in Q^k_{(t,s)}(A)$ and hence $x \in C_{(t,s)}(A) \cup Q^k_{(t,s)}(A) = [A]^k_{(t,s)}$.

Let $x, y \in [A]^k_{(t,s)}$, then $x, y \in C_{(t,s)}(A)$ or $x, y \in Q^k_{(t,s)}(A)$, that is $\mu_A(x) \geq t$ and $\gamma_A(x) \leq s$ or $\mu_A(x) + t + k > 1$ and $\gamma_A(x) + s + k < 1$ and $\mu_A(y) \geq t$ and $\gamma_A(y) \leq s$ or $\mu_A(y) + t + k > 1$ and $\gamma_A(y) + s + k < 1$. Here we consider four cases:
(i) $\mu_A(x) \geq t$, $\gamma_A(x) \leq s$ and $\mu_A(y) \geq t$, $\gamma_A(y) \leq s$,
(ii) $\mu_A(x) \geq t$, $\gamma_A(x) \leq s$ and $\mu_A(y) + t + k > 1$, $\gamma_A(y) + s + k < 1$,
(iii) $\mu_A(x) + t + k > 1$, $\gamma_A(x) + s + k < 1$ and $\mu_A(y) \geq t$, $\gamma_A(y) \leq s$,
(iv) $\mu_A(x) + t + k > 1$, $\gamma_A(x) + s + k < 1$ and $\mu_A(y) + t + k > 1$, $\gamma_A(y) + s + k < 1$.

For case (i) using (C_8) of Lemma (3.3), we get

$$\begin{aligned}\mu_A(xy) &\geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} \\ &= \left\{\begin{array}{ll} \frac{1-k}{2} & \text{if } t > \frac{1-k}{2} \\ t & \text{if } t \leq \frac{1-k}{2} \end{array}\right\} \\ &> \left\{\begin{array}{ll} 1-t-k & \text{if } t > \frac{1-k}{2} \\ t & \text{if } t \leq \frac{1-k}{2} \end{array}\right\}\end{aligned}$$

and

$$\begin{aligned}\gamma_A(xy) &\leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} \leq \max\left\{s, \frac{1-k}{2}\right\} \\ &= \left\{\begin{array}{ll} \frac{1-k}{2} & \text{if } s < \frac{1-k}{2} \\ s & \text{if } s \geq \frac{1-k}{2} \end{array}\right\} \\ &< \left\{\begin{array}{ll} 1-s-k & \text{if } s < \frac{1-k}{2} \\ s & \text{if } s \geq \frac{1-k}{2} \end{array}\right\}.\end{aligned}$$

Thus $xy \in C_{(t,s)}(A)$ or $xy \in Q^k_{(t,s)}(A)$. Hence, $xy \in C_{(t,s)}(A) \cup Q^k_{(t,s)}(A) = [A]^k_{(t,s)}$.

For case (ii) using (C_8) of Lemma (3.3), we get

$$\begin{aligned}\mu_A(xy) &\geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \\ &\geq \min\left\{t, 1-t-k, \frac{1-k}{2}\right\} \\ &= \left\{\begin{array}{ll} 1-t-k & \text{if } t > \frac{1-k}{2} \\ t & \text{if } t \leq \frac{1-k}{2} \end{array}\right\} (\text{if } t > \frac{1-k}{2}, \text{ then } 1-t-k < \frac{1-k}{2})\end{aligned}$$

and

$$\begin{aligned}\gamma_A(xy) &\leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} \\ &\leq \max\left\{s, 1-s-k, \frac{1-k}{2}\right\} \\ &= \left\{\begin{array}{ll} 1-s-k & \text{if } s < \frac{1-k}{2} \\ s & \text{if } s \geq \frac{1-k}{2} \end{array}\right\} (\text{if } s < \frac{1-k}{2}, \text{ then } 1-s-k > \frac{1-k}{2})\end{aligned}$$

Thus $xy \in C_{(t,s)}(A)$ or $xy \in Q^k_{(t,s)}(A)$. Hence, $xy \in C_{(t,s)}(A) \cup Q^k_{(t,s)}(A) = [A]^k_{(t,s)}$.

For case (iii) we have a similar result.

For case (iv) using (C_8) of Lemma (3.3), we get

$$\begin{aligned}\mu_A(xy) &\geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} > \min\left\{1-t-k, \frac{1-k}{2}\right\} \\ &= \left\{\begin{array}{ll} 1-t-k & \text{if } t > \frac{1-k}{2} \\ t & \text{if } t \leq \frac{1-k}{2} \end{array}\right\} (\text{if } t > \frac{1-k}{2}, \text{ then } 1-t-k < \frac{1-k}{2})\end{aligned}$$

and

$$\begin{aligned} \gamma_A(xy) &\leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} \\ &< \max\left\{1-s-k, \frac{1-k}{2}\right\} \\ &= \left\{\begin{array}{ll} 1-s-k & \text{if } s < \frac{1-k}{2} \\ s & \text{if } s \geq \frac{1-k}{2} \end{array}\right\} (\text{if } s < \frac{1-k}{2}, \text{ then } 1-s-k > \frac{1-k}{2}) \end{aligned}$$

Thus $xy \in C_{(t,s)}(A)$ or $xy \in Q^k_{(t,s)}(A)$. Hence, $xy \in C_{(t,s)}(A) \cup Q^k_{(t,s)}(A) = [A]^k_{(t,s)}$.

Next we consider $x, y, z \in S$ such that $y \in [A]^k_{(t,s)}$ then $y \in C_{(t,s)}(A)$ or $y \in Q^k_{(t,s)}(A)$, that is,

(i) $\mu_A(y) \geq t$ and $\gamma_A(y) \leq s$

or

(ii) $\mu_A(y) + t + k > 1$ and $\gamma_A(y) + s + k < 1$.

For case (i) using (C_9) of Lemma (3.3), we get

$$\begin{aligned} \mu_A(xyz) &\geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \\ &\geq \min\left\{t, \frac{1-k}{2}\right\} \\ &= \left\{\begin{array}{ll} \frac{1-k}{2} & \text{if } t > \frac{1-k}{2} \\ t & \text{if } t \leq \frac{1-k}{2} \end{array}\right\} \\ &> \left\{\begin{array}{ll} 1-t-k & \text{if } t > \frac{1-k}{2} \\ t & \text{if } t \leq \frac{1-k}{2} \end{array}\right\} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xyz) &\leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\} \\ &\leq \max\left\{s, \frac{1-k}{2}\right\} \\ &= \left\{\begin{array}{ll} \frac{1-k}{2} & \text{if } s < \frac{1-k}{2} \\ s & \text{if } s \geq \frac{1-k}{2} \end{array}\right\} \\ &< \left\{\begin{array}{ll} 1-s-k & \text{if } s < \frac{1-k}{2} \\ s & \text{if } s \geq \frac{1-k}{2} \end{array}\right\}. \end{aligned}$$

Thus $xyz \in C_{(t,s)}(A)$ or $xyz \in Q^k_{(t,s)}(A)$. Hence, $xyz \in C_{(t,s)}(A) \cup Q^k_{(t,s)}(A) = [A]^k_{(t,s)}$.

For case (ii) using (C_9) of Lemma (3.3), we get

$$\begin{aligned}\mu_A(xyz) &\geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \\ &> \min\left\{1-t-k, \frac{1-k}{2}\right\} \\ &= \left\{\begin{array}{ll} 1-t-k & \text{if } t > \frac{1-k}{2} \\ t & \text{if } t \leq \frac{1-k}{2} \end{array}\right\} (\text{if } t > \frac{1-k}{2}, \text{ then } 1-t-k < \frac{1-k}{2})\end{aligned}$$

and

$$\begin{aligned}\gamma_A(xyz) &\leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\} \\ &< \max\left\{1-s-k, \frac{1-k}{2}\right\} \\ &= \left\{\begin{array}{ll} 1-s-k & \text{if } s < \frac{1-k}{2} \\ s & \text{if } s \geq \frac{1-k}{2} \end{array}\right\} (\text{if } s < \frac{1-k}{2}, \text{ then } 1-s-k > \frac{1-k}{2})\end{aligned}$$

Thus $xyz \in C_{(t,s)}(A)$ or $xyz \in Q^k_{(t,s)}(A)$. Hence, $xyz \in C_{(t,s)}(A) \cup Q^k_{(t,s)}(A) = [A]^k_{(t,s)}$. Consequently $[A]^k_{(t,s)}$ is an interior ideal of S.

Conversely; Let $A = \langle x, \mu_A, \gamma_A \rangle$ be an IFS of S such that $[A]^k_{(t,s)} \neq \phi$ and is a interior ideal of S for all $t \in (0,1]$ and $s \in [0,1)$. If there exist $a, b \in S$ with $a \leq b$ and $\mu_A(a) < \min\left\{\mu_A(b), \frac{1-k}{2}\right\}$ and $\gamma_A(a) > \max\left\{\gamma_A(b), \frac{1-k}{2}\right\}$, then

$$\begin{aligned}\mu_A(a) &< t_a \leq \min\left\{\mu_A(b), \frac{1-k}{2}\right\} \\ &\text{and} \\ \gamma_A(a) &> s_a \geq \max\left\{\gamma_A(b), \frac{1-k}{2}\right\}\end{aligned}$$

for some $t_a \in (0,1]$ and $s_a \in [0,1)$. Follows that $b \in C_{(t,s)}(A) \subseteq [A]^k_{(t,s)}$but $a \notin C_{(t,s)}(A)$. Also $\mu_A(a) + t_a + k < 1$ and $\gamma_A(a) + s_a + k > 1$ follows that $a \notin Q^k_{(t,s)}(A)$ i.e. $a \notin [A]^k_{(t,s)}$ a contradiction. Therefore

$$\mu_A(x) \geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \text{ and } \gamma_A(x) \leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\},$$

for all $x, y \in S$ with $x \leq y$.

Assume that there exist $a, b \in S$ such that

$$\mu_A(ab) < \min\left\{\mu_A(a), \mu_A(b), \frac{1-k}{2}\right\} \text{ and } \gamma_A(ab) > \max\left\{\gamma_A(a), \gamma_A(b), \frac{1-k}{2}\right\},$$

then

$$\begin{aligned}\mu_A(ab) &< t \leq \min\left\{\mu_A(a), \mu_A(b), \frac{1-k}{2}\right\} \\ &\text{and} \\ \gamma_A(ab) &> s \geq \max\left\{\gamma_A(a), \gamma_A(b), \frac{1-k}{2}\right\},\end{aligned}$$

for some $t \in (0,1]$ and $s \in [0,1)$. Shows that $a \in C_{(t,s)}(A) \subseteq [A]^k_{(t,s)}$ and $b \in C_{(t,s)}(A) \subseteq [A]^k_{(t,s)}$ but $ab \notin C_{(t,s)}(A)$, also $\mu_A(ab)+t+k < 1$ and $\gamma_A(ab)+s_a+k > 1$ follows that $ab \notin Q^k_{(t,s)}(A)$ i.e. $ab \notin [A]^k_{(t,s)}$ a contradiction. Therefore

$$\mu_A(xy) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \text{ and } \gamma_A(xy) \leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\},$$

for all $x, y \in S$.

Next we let

$$\mu_A(abc) < \min\left\{\mu_A(b), \frac{1-k}{2}\right\} \text{ and } \gamma_A(abc) > \max\left\{\gamma_A(b), \frac{1-k}{2}\right\},$$

for some $a, b, c \in S$.

Then there exist $t_0 \in (0,1]$ and $s_0 \in [0,1)$ such that

$$\begin{aligned} \mu_A(abc) &< t_0 \leq \min\left\{\mu_A(b), \frac{1-k}{2}\right\} \\ &\text{and} \\ \gamma_A(abc) &> s_0 \geq \max\left\{\gamma_A(b), \frac{1-k}{2}\right\}. \end{aligned}$$

It follows that $b \in [A]^k_{(t,s)}$ but $abc \notin [A]^k_{(t,s)}$, a contradiction. Therefore

$$\mu_A(xyz) \geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \text{ and } \gamma_A(xyz) \leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\},$$

forall $x, y, z \in S$.

Therefore $A = \langle x, \mu_A, \gamma_A \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy interior ideal of S. □

Definition 3.11. Let $A = \langle x, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy subset of S. Then $A = \langle x, \mu_A, \gamma_A \rangle$ is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-intuitionistic fuzzy interior ideal of S if for all $x, y, z \in S$, $t, t_1, t_2 \in (0,1]$ and $s, s_1, s_2 \in [0,1)$ the following three conditions are satisfied for all $x, y, z \in S$:

(C_{15}) $(\forall x \leq y)\, [x; (t,s)]\overline{\in} A \Rightarrow [y; (t,s)]\overline{\in} \vee \overline{q_k} A$,

(C_{16}) $[xy; (t,s)]\overline{\in} A \Rightarrow [x; (t,s)]\overline{\in} \vee \overline{q_k} A, [y; (t,s)]\overline{\in} \vee \overline{q_k} A$,

(C_{17}) $[xyz; (t,s)]\overline{\in} A \Rightarrow [y; (t,s)]\overline{\in} \vee \overline{q_k} A$.

Example 3.12. Consider the ordered semigroup of example 3.2 and define an IFS $A =< x, \mu_A, \gamma_A >$ by

$$\mu_A \quad : \quad S \to [0,1] \mid \mu_A(x) = \begin{cases} 0.45 \text{ if } x = e \\ 0.50 \text{ if } x = d \\ 0.45 \text{ if } x = c \\ 0.60 \text{ if } x = b \\ 0.65 \text{ if } x = a \end{cases}$$

and

$$\gamma_A \quad : \quad S \to [0,1] \mid \gamma_A(x) = \begin{cases} 0.50 \text{ if } x = e \\ 0.40 \text{ if } x = d \\ 0.35 \text{ if } x = c \\ 0.35 \text{ if } x = b \\ 0.20 \text{ if } x = a \end{cases} .$$

Then $A = \langle x, \mu_A, \gamma_A \rangle$ is a $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-intuitionistic fuzzy interior ideal of S for $k = 0.2$.

Theorem 3.13. *An intuitionistic fuzzy subset* $A = \langle x, \mu_A, \gamma_A \rangle$ *of an ordered semigroup* S *is an* $(\overline{\in}, \overline{\in} \vee \overline{q_k})$*-intuitionistic fuzzy interior ideal of* S *if and only if the following conditions holds for all* $x, y, z \in S$*:*

(C_{18}) $x \leq y \Rightarrow \mu_A(y) \leq \max\left\{\mu_A(x), \frac{1-k}{2}\right\}$ *and* $\gamma_A(y) \geq \min\left\{\gamma_A(x), \frac{1-k}{2}\right\}$,

(C_{19}) $\max\left\{\mu_A(xy), \frac{1-k}{2}\right\} \geq \min\left\{\mu_A(x), \mu_A(y)\right\}$ *and* $\min\left\{\gamma_A(xy), \frac{1-k}{2}\right\} \leq \max\left\{\gamma_A(x), \gamma_A(y)\right\}$,

(C_{20}) $\max\left\{\mu_A(xyz), \frac{1-k}{2}\right\} \geq \mu_A(y)$ *and* $\min\left\{\gamma_A(xyz), \frac{1-k}{2}\right\} \leq \gamma_A(y)$.

Proof. Let $A = \langle x, \mu_A, \gamma_A \rangle$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-intuitionistic fuzzy interior ideal of S. Let there exist $a, b \in S$ with $a \leq b$ such that

$$\mu_A(b) > \max\left\{\mu_A(a), \frac{1-k}{2}\right\} \text{ and } \gamma_A(b) < \min\left\{\gamma_A(a), \frac{1-k}{2}\right\}.$$

Then

$$\mu_A(b) \geq t > \max\left\{\mu_A(a), \frac{1-k}{2}\right\}$$
and
$$\gamma_A(b) \leq s < \min\left\{\gamma_A(a), \frac{1-k}{2}\right\},$$

for some $t \in (0,1]$ and $s \in [0,1)$. From this we see that $[a;(t,s)]\overline{\in}A$, but $[b;(t,s)] \in A$ and $\mu_A(b) + t + k > 1$, $\gamma_A(b) + s + k < 1$, this implies $[b;(t,s)]q_kA$, a contradiction and hence

$$\mu_A(y) \leq \max\left\{\mu_A(x), \frac{1-k}{2}\right\} \text{ and } \gamma_A(y) \geq \min\left\{\gamma_A(x), \frac{1-k}{2}\right\},$$

for all $x, y \in S$ with $x \leq y$.

Consider for some $a, b \in S$ we have

$$\begin{aligned}\max\left\{\mu_A(xy), \frac{1-k}{2}\right\} &< \min\{\mu_A(x), \mu_A(y)\} \\ &\text{and} \\ \min\left\{\gamma_A(xy), \frac{1-k}{2}\right\} &> \max\{\gamma_A(x), \gamma_A(y)\}.\end{aligned}$$

Then

$$\begin{aligned}\max\left\{\mu_A(xy), \frac{1-k}{2}\right\} &< t \leq \min\{\mu_A(x), \mu_A(y)\} \\ &\text{and} \\ \min\left\{\gamma_A(xy), \frac{1-k}{2}\right\} &> s \geq \max\{\gamma_A(x), \gamma_A(y)\},\end{aligned}$$

for some $t \in (0,1]$ and $s \in [0,1)$. It follows that $[xy;(t,s)]\overline{\in}A$. But $[x;(t,s)] \in A$, $\mu_A(x) + t + k > 1$ and $\gamma_A(x) + s + k < 1$ i.e. $[x;(t,s)]q_kA$. Also $[y;(t,s)] \in A$, $\mu_A(y) + t + k > 1$ and $\gamma_A(y) + s + k < 1$ i.e. $[y;(t,s)]q_kA$, a contradiction and so condition (C_{19}) holds for all $x, y \in S$.

Consider for some $a, b, c \in S$ we have

$$\max\left\{\mu_A(xyz), \frac{1-k}{2}\right\} < \mu_A(y) \text{ and } \min\left\{\gamma_A(xyz), \frac{1-k}{2}\right\} > \gamma_A(y).$$

Then

$$\begin{aligned}\max\left\{\mu_A(xyz), \frac{1-k}{2}\right\} &< t \leq \mu_A(y) \\ &\text{and} \\ \min\left\{\gamma_A(xyz), \frac{1-k}{2}\right\} &> s \geq \gamma_A(y),\end{aligned}$$

for some $t \in (0,1]$ and $s \in [0,1)$. It follows that $[xyz;(t,s)]\overline{\in}A$. But $[y;(t,s)] \in A$, $\mu_A(y) + t + k > 1$ and $\gamma_A(y) + s + k < 1$ i.e. $[y;(t,s)]q_kA$, a contradiction and so condition (C_{20}) holds for all $x, y, z \in S$.

Conversely: Let $A = \langle x, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy subset of S satisfying conditions (C_{18}), (C_{19}) and (C_{20}). Let $x, y, z \in S$, $t, t_1, t_2 \in (0,1]$ and $s, s_1, s_2 \in [0,1)$.

Assume that $x \leq y$ and $[x;(t,s)]\overline{\in}A$ this implies $\mu_A(x) < t$ and $\gamma_A(x) > s$. Now by (C_{18}) we have

$$\begin{aligned}\mu_A(y) &\leq \max\left\{\mu_A(x), \frac{1-k}{2}\right\} \\ &< \max\left\{t, \frac{1-k}{2}\right\} \\ &= \begin{cases} t & \text{if } t \geq \frac{1-k}{2} \\ \frac{1-k}{2} & \text{if } t < \frac{1-k}{2} \end{cases},\end{aligned}$$

and

$$\begin{aligned}\gamma_A(y) &\geq \min\left\{\gamma_A(x), \frac{1-k}{2}\right\} \\ &> \min\left\{s, \frac{1-k}{2}\right\} \\ &= \begin{cases} s & \text{if } t \leq \frac{1-k}{2} \\ \frac{1-k}{2} & \text{if } s > \frac{1-k}{2} \end{cases},\end{aligned}$$

follows that $[y;(t,s)]\overline{\in}A$ and $\mu_A(y)+t+k<1$ and $\gamma_A(y)+s+k>1$ i.e. $[y;(t,s)]\overline{q_k}A$. This shows that $[y;(t,s)]\overline{\in}\vee\overline{q_k}A$.

Let $[xy;\min\{t_1,t_2\},\max\{s_1,s_2\}]\overline{\in}A$ then $\mu_A(xy)<\min\{t_1,t_2\}$ and $\gamma_A(xy)>\max\{s_1,s_2\}$. Then by condition (C_{19})

$$\begin{aligned}\min\{\mu_A(x),\mu_A(y)\} &\leq \max\left\{\mu_A(xy), \frac{1-k}{2}\right\} \\ &< \max\left\{\min\{t_1,t_2\}, \frac{1-k}{2}\right\} \\ &= \begin{cases} \min\{t_1,t_2\} & \text{if } \min\{t_1,t_2\} \geq \frac{1-k}{2} \\ \frac{1-k}{2} & \text{if } \min\{t_1,t_2\} < \frac{1-k}{2} \end{cases},\end{aligned}$$

and

$$\begin{aligned}\max\{\gamma_A(x),\gamma_A(y)\} &\geq \min\left\{\gamma_A(xy), \frac{1-k}{2}\right\} \\ &> \min\left\{\max\{s_1,s_2\}, \frac{1-k}{2}\right\} \\ &= \begin{cases} \max\{s_1,s_2\} & \text{if } \max\{s_1,s_2\} \leq \frac{1-k}{2} \\ \frac{1-k}{2} & \text{if } \max\{s_1,s_2\} > \frac{1-k}{2} \end{cases}\end{aligned}$$

Here we discuss the following two cases:

(1) If $\min\{\mu_A(x),\mu_A(y)\}=\mu_A(x)$, $\max\{\gamma_A(x),\gamma_A(y)\}=\gamma_A(x)$, $\min\{t_1,t_2\}=t_1$ and $\max\{s_1,s_2\}=s_1$,

(2) If $\min\{\mu_A(x),\mu_A(y)\}=\mu_A(y)$, $\max\{\gamma_A(x),\gamma_A(y)\}=\gamma_A(y)$, $\min\{t_1,t_2\}=t_2$ and $\max\{s_1,s_2\}=s_2$.

For the first case we have $\mu_A(x)<t_1$, $\gamma_A(x)>s_1$ or $\mu_A(x)+t_1+k<1$, $\gamma_A(x)+s_1+k>1$. That is $[x;(t_1,s_1)]\overline{\in}\vee\overline{q_k}A$.

Consider the second case we see that $\mu_A(y)<t_2$, $\gamma_A(y)>s_2$ or $\mu_A(y)+t_2+k<1$, $\gamma_A(y)+s_2+k>1$. That is $[y;(t_2,s_2)]\overline{\in}\vee\overline{q_k}A$.

Let $[xyz;(t,s)]\overline{\in}A$ then $\mu_A(xyz)<t$ and $\gamma_A(xyz)>s$. Then by condition (C_{20})

$$\begin{aligned}\mu_A(y) &\leq \max\left\{\mu_A(xyz), \frac{1-k}{2}\right\} \\ &< \max\left\{t, \frac{1-k}{2}\right\} \\ &= \begin{cases} t & \text{if } t \geq \frac{1-k}{2} \\ \frac{1-k}{2} & \text{if } t < \frac{1-k}{2} \end{cases},\end{aligned}$$

and

$$\begin{aligned}\gamma_A(y) &\geq \min\left\{\gamma_A(xyz), \frac{1-k}{2}\right\}\\ &> \min\left\{s, \frac{1-k}{2}\right\}\\ &= \begin{cases} s & \text{if } s \leq \frac{1-k}{2} \\ \frac{1-k}{2} & \text{if } s > \frac{1-k}{2} \end{cases}.\end{aligned}$$

Hence, we have $\mu_A(y) < t$, $\gamma_A(y) > s$ or $\mu_A(y) + t + k < 1$, $\gamma_A(y) + s + k > 1$. That is $[y; (t,s)] \overline{\in} \vee \overline{q_k} A$. This completes the proof. □

Theorem 3.14. *For any IFS $A = \langle x, \mu_A, \gamma_A \rangle$ of S, the following conditions are equivalent:*

(C_{21}) *$A = \langle x, \mu_A, \gamma_A \rangle$ is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ intuitionistic fuzzy interior ideal of S,*

(C_{22}) *$\left(\forall t \in (\frac{1-k}{2}, 1],\ s \in [0, \frac{1-k}{2})\right)\left(C_{(t,s)}(A) \neq \phi \Rightarrow C_{(t,s)}\right.$ is an interior ideal of S).*

Proof. Let $A = \langle x, \mu_A, \gamma_A \rangle$ be a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ intuitionistic fuzzy interior ideal of S. Let $C_{(t,s)}(A) \neq \phi$ for some $t \in (\frac{1-k}{2}, 1]$ and $s \in [0, \frac{1-k}{2})$. Let $a, b \in S$ with $a \leq b$ such that $b \in C_{(t,s)}(A)$, then $\mu_A(b) \geq t$ and $\gamma_A(b) \leq s$. Using (C_{18}), we have

$$\frac{1-k}{2} < t \leq \mu_A(b) \leq \max\left\{\mu_A(a), \frac{1-k}{2}\right\} = \mu_A(a),$$

and

$$\frac{1-k}{2} > s \geq \gamma_A(b) \geq \min\left\{\gamma_A(a), \frac{1-k}{2}\right\} = \gamma_A(a).$$

It follows that $a \in C_{(t,s)}(A)$.

Let $a, b \in S$ such that $a, b \in C_{(t,s)}(A)$, then $\mu_A(a) \geq t$, $\mu_A(b) \geq t$ and $\gamma_A(a) \leq s$, $\gamma_A(b) \leq s$. Using (C_{19}), we have

$$\max\left\{\mu_A(ab), \frac{1-k}{2}\right\} \geq \min\{\mu_A(a), \mu_A(b)\} \geq t > \frac{1-k}{2},$$

and

$$\min\left\{\gamma_A(ab), \frac{1-k}{2}\right\} \leq \max\{\gamma_A(a), \gamma_A(b)\} \leq s < \frac{1-k}{2},$$

this implies that $\mu_A(ab) = \max\left\{\mu_A(ab), \frac{1-k}{2}\right\} \geq t$ and $\gamma_A(ab) = \min\left\{\gamma_A(ab), \frac{1-k}{2}\right\} \leq s$. Therefore $xy \in C_{(t,s)}(A)$ for all $x, y \in S$.

Let $a, b, c \in S$ such that $b \in C_{(t,s)}(A)$, then $\mu_A(b) \geq t$, $\gamma_A(b) \leq s$. Using (C_{20}), we have

$$\max\left\{\mu_A(abc), \frac{1-k}{2}\right\} \geq \mu_A(b) \geq t > \frac{1-k}{2},$$

and

$$\min\left\{\gamma_A(abc), \frac{1-k}{2}\right\} \leq \gamma_A(b) \leq s < \frac{1-k}{2},$$

this implies that $\mu_A(abc) = \max\left\{\mu_A(abc), \frac{1-k}{2}\right\} \geq t$ and $\gamma_A(abc) = \min\{\gamma_A(abc), \frac{1-k}{2}\} \leq s$. Thus $xyz \in C_{(t,s)}(A)$ for all $x, y, z \in S$. Consequently, $C_{(t,s)}(A)$ is an interior ideal of S.

Conversely, let $C_{(t,s)}(A) \neq \phi$ is an interior ideal of S where $t \in (\frac{1-k}{2}, 1], s \in [0, \frac{1-k}{2})$. Let $a, b \in S$ with $a \leq b$ and $t \in (0,1]$, $s \in [0,1)$. Consider

$$\mu_A(b) > \max\left\{\mu_A(a), \frac{1-k}{2}\right\} \text{ and } \gamma_A(b) < \min\left\{\gamma_A(a), \frac{1-k}{2}\right\},$$

then there exists some $t \in (0,1]$, $s \in [0,1)$ such that

$$\begin{aligned} \mu_A(b) &\geq t > \max\left\{\mu_A(a), \frac{1-k}{2}\right\} \\ &\text{and} \\ \gamma_A(b) &\leq s < \min\left\{\gamma_A(a), \frac{1-k}{2}\right\}. \end{aligned}$$

This shows that $b \in C_{(t,s)}(A)$ but $a \notin C_{(t,s)}(A)$, a contradiction. Hence

$$\mu_A(y) \leq \max\left\{\mu_A(x), \frac{1-k}{2}\right\} \text{ and } \gamma_A(y) \geq \min\left\{\gamma_A(x), \frac{1-k}{2}\right\},$$

for all $x, y \in S$ with $x \leq y$.

Let $a, b \in S$ and consider

$$\begin{aligned} \max\left\{\mu_A(ab), \frac{1-k}{2}\right\} &< \min\{\mu_A(a), \mu_A(b)\} \\ &\text{and} \\ \min\left\{\gamma_A(ab), \frac{1-k}{2}\right\} &> \max\{\gamma_A(a), \gamma_A(b)\}, \end{aligned}$$

then there exists $t \in (0,1]$ and $s \in [0,1)$ such that

$$\begin{aligned} \max\left\{\mu_A(ab), \frac{1-k}{2}\right\} &< t \leq \min\{\mu_A(a), \mu_A(b)\} \\ &\text{and} \\ \min\left\{\gamma_A(ab), \frac{1-k}{2}\right\} &> s \geq \max\{\gamma_A(a), \gamma_A(b)\}, \end{aligned}$$

it follows that $a, b \in C_{(t,s)}(A)$ but $ab \notin C_{(t,s)}(A)$, contradiction. Hence

$$\begin{aligned} \max\left\{\mu_A(xy), \frac{1-k}{2}\right\} &\geq \min\{\mu_A(x), \mu_A(y)\} \\ &\text{and} \\ \min\left\{\gamma_A(xy), \frac{1-k}{2}\right\} &\leq \max\{\gamma_A(x), \gamma_A(y)\} \text{ for all } x, y \in S. \end{aligned}$$

Let $a, b, c \in S$ and consider

$$\max\left\{\mu_A(abc), \frac{1-k}{2}\right\} < \mu_A(b) \text{ and } \min\left\{\gamma_A(abc), \frac{1-k}{2}\right\} > \gamma_A(b),$$

then there exists $t \in (0,1]$ and $s \in [0,1)$ such that

$$\max\left\{\mu_A(abc), \frac{1-k}{2}\right\} < t \leq \mu_A(b)$$

and

$$\min\left\{\gamma_A(abc), \frac{1-k}{2}\right\} > s \geq \gamma_A(b),$$

it follows that $b \in C_{(t,s)}(A)$ but $abc \notin C_{(t,s)}(A)$, a contradiction. Thus,

$$\max\left\{\mu_A(xyz), \frac{1-k}{2}\right\} \geq \mu_A(y) \text{ and } \min\left\{\gamma_A(xyz), \frac{1-k}{2}\right\} \leq \gamma_A(y),$$

for all $x, y, z \in S$.

Therefore $A = \langle x, \mu_A, \gamma_A \rangle$ is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-intuitionistic fuzzy interior ideal of S. □

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