Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 6, No. 3, (November 2013), pp. 467–477
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

Lattice structures of intuitionistic fuzzy soft sets

Zhaowen Li, Shijie Li

Received 29 September 2012; Revised 30 January 2013; Accepted 13 March 2013

Abstract. In this paper, the concept of extended intersection and restricted union of intuitionistic fuzzy soft sets are introduced. Some operations on intuitionistic fuzzy soft sets are investigated, and we prove that De Morgan's laws hold in intuitionistic fuzzy soft sets theory. Based on these properties, we discuss the algebraic structures of intuitionistic fuzzy soft sets, which is lattice structures.

2010 AMS Classification: 06D72, 03E72, 49J53

Keywords: Lattices, Lattice structures, Intuitionistic fuzzy soft sets.

Corresponding Author: Zhaowen Li (lizhaowen8846@126.com)

1. Introduction

Most of traditional methods for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. However, many practical problems within the fields of economics, engineering, environmental science, medical science and social sciences involve data that contain uncertainties. We can not use traditional methods because of various types of uncertainties present in these problems.

There are several theories: probability theory, theory of fuzzy sets [26], theory of interval mathematics, and theory of rough sets [20], which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties (see [14]). To overcome these difficulties, Molodtsov [14] proposed a completely new approach, which is called theory of soft sets, for modeling uncertainty.

Presently, works on soft set theory are progressing rapidly. Maji et al. [18, 17] further studied soft set theory and used this theory to solve some decision making problems. Aktas and Cağman [3] defined soft groups. Jiang et al. [12] extended soft sets with description logics. Feng et al. [7, 8] investigated the relationship among soft sets, rough sets and fuzzy sets. Ge et al. [10] discussed the relationship between soft sets and topological spaces. Babitha et al. [4] proposed relations on soft sets.

Shabir et al. [23] introduced soft topological spaces over the universe with a fixed set of parameters. Cağman et al. [5] defined topologies on soft sets.

Maji et al. [15] devoted the concept of fuzzy soft sets by combining soft sets with fuzzy sets. Roy et al. [22] presented a fuzzy soft set theoretic approach towards decision making problems. Jiang et al. [13] generalized the adjustable approach to fuzzy soft sets based decision making. Feng et al. [9] proposed an adjustable approach to (weighted) fuzzy soft set based decision making. Yang et al. [25] introduced the concept of interval-valued fuzzy soft set. Jun et al. [11] discussed the applications of fuzzy soft sets to study BCK/BCI-algebras. Tanay et al. [24] investigated the topological structure of fuzzy soft sets.

K. Atanassov [1, 2] introduced the concept of intuitionistic fuzzy sets. Maji et al. [16, 19] introduced the theory of intuitionistic fuzzy soft set. P. Rajarajeswari et al. [21] gave the concepts of intuitionistic fuzzy contra weakly generalized continuous map-pings in intuitionistic fuzzy topological space.

In this paper, we deal with the algebraic structures based on lattice of intuitionistic fuzzy soft sets. Some operations on intuitionistic fuzzy soft sets are introduced. Some lattice structures of intuitionistic fuzzy soft sets are established. The De Morgan's laws in intuitionistic fuzzy soft set theory are verified.

2. Preliminaries

In this paper, U denotes initial universe, E denotes parameter set, I denotes $[0,1]$ and 2^U denotes the family of all subsets of U.

Throughout this paper, we only consider the case where U and E are both nonempty finite sets.

We briefly recall some basic concepts of soft sets, intuitionistic fuzzy sets, intuitionistic fuzzy soft sets and lattices.

2.1. Soft sets and intuitionistic fuzzy sets.

Definition 2.1 ([14])**.** A pair (f, E) is called a soft set over U, if f is a mapping given by $f : E \to 2^U$.

Definition 2.2 ([1])**.** An intuitionistic fuzzy (briefly IF) set A in U is an object the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) : x \in U\},$$

where $\mu_A : U \to [0,1]$ and $\gamma_A : U \to [0,1]$ satisfying $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in U$, and $\mu_A(x)$ and $\gamma_A(x)$ are, respectively, used to define the degree of membership and the degree of non-membership of the element x to A.

Obviously, every fuzzy set has the form $\{(x, \mu_A(x), 1 - \mu_A(x)) : x \in U\}$ and is thus an IF set. Every crisp set A is still regarded as an IF set with the form $\{(x, 1, 0) : x \in A\}$, and if $y \notin A$, then $\mu_A(y) = 0$ and $\gamma_A(y) = 1$.

In this paper, $\mathcal{IF}(U)$ denotes the family of all IF sets in U, $\widetilde{U}$ represents the IF set which satisfies $\widetilde{U}(x) = \{(x, 1, 0) : x \in U\}$ and $\widetilde{\varnothing}$ represents the IF set which satisfies $\widetilde{\varnothing}(x) = \{(x, 0, 1) : x \in U\}$.

If $A, B \in \mathcal{IF}(U)$, then some IF set relations and operations are given componentwise proposed by Atanassov [1] as follows:

(1) $A = B$ iff $\mu_A(x) = \mu_B(x)$ and $\gamma_A(x) = \gamma_B(x)$ for all $x \in U$.
(2) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_A(x)$ for all $x \in U$.
(3) $A \cap B = \{(x, min(\mu_A(x), \mu_B(x)), max(\gamma_A(x), \gamma_B(x))) : x \in U\}$.
(4) $A \cup B = \{(x, max(\mu_A(x), \mu_B(x)), min(\gamma_A(x), \gamma_B(x))) : x \in U\}$.
(5) $A^c(x) = \{(x, \gamma_A(x), \mu_A(x)) : x \in U\}$.

Moreover,

$$\bigcap_{\alpha\in\Gamma} A_\alpha = \{(x, \bigwedge_{\alpha\in\Gamma} \mu_{A_\alpha}(x), \bigvee_{\alpha\in\Gamma} \gamma_{A_\alpha}(x)) : x \in U\}.$$

for any $x \in U$ and

$$\bigcup_{\alpha\in\Gamma} A_\alpha = \{(x, \bigvee_{\alpha\in\Gamma} \mu_{A_\alpha}(x), \bigwedge_{\alpha\in\Gamma} \gamma_{A_\alpha}(x)) : x \in U\}$$

for any $x \in U$, where $\{A_\alpha : \alpha \in \Gamma\} \subseteq \mathcal{IF}(U)$ and Γ is an index set.

Obviously, $A = B \iff A \subseteq B$ *and* $B \subseteq A$.

2.2. **Intuitionistic fuzzy soft sets.**

Definition 2.3 ([16]). Let $A \subseteq E$. A pair (f, A) is called an intuitionistic fuzzy soft (briefly IFS) set over U, if f is a mapping given by $f : A \to \mathcal{IF}(U)$. We also denote (f, A) by f_A.

In other words, an IFS set f_E over U is a parameterized family of IF sets in the universe U.

Let $A \subseteq E$. Denote

$$S(U)_A = \{f_A : f_A \text{ is an IFS set over } U\},$$

$$S(U) = \{f_A : f_A \text{ is an IFS set over } U \text{ and } A \subseteq E\}.$$

Obviously,

$$S(U)_A \subseteq S(U).$$

Example 2.4. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $E = \{e_1, e_2, e_3\}$. Let f_E be an IFS set over U, defined as follows

$f(e_1) = \frac{(0,1)}{x_1} + \frac{(0.3,0.6)}{x_2} + \frac{(0.8,0.1)}{x_3} + \frac{(0.5,0.2)}{x_4} + \frac{(0.7,0.1)}{x_5} + \frac{(0.3,0.6)}{x_6}$,
$f(e_2) = \frac{(0.7,0.2)}{x_1} + \frac{(0.5,0.5)}{x_2} + \frac{(0.1,0.8)}{x_3} + \frac{(0.2,0.6)}{x_4} + \frac{(0.2,0.7)}{x_5} + \frac{(0.6,0.3)}{x_6}$,
$f(e_3) = \frac{(0.1,0.8)}{x_1} + \frac{(0.9,0.1)}{x_2} + \frac{(1,0)}{x_3} + \frac{(0.5,0.4)}{x_4} + \frac{(0.1,0.7)}{x_5} + \frac{(0.7,0.3)}{x_6}$.

Then f_E is described by the following Table 1.

TABLE 1. Tabular representation of the intuitionistic fuzzy soft sets f_E

	x_1	x_2	x_3	x_4	x_5	x_6
e_1	$(0, 1)$	$(0.3, 0.6)$	$(0.8, 0.1)$	$(0.5, 0.2)$	$(0.7, 0.1)$	$(0.3, 0.6)$
e_2	$(0.7, 0.2)$	$(0.5, 0.5)$	$(0.1, 0.8)$	$(0.2, 0.6)$	$(0.2, 0.7)$	$(0.6, 0.3)$
e_3	$(0.1, 0.8)$	$(0.9, 0.1)$	$(1, 0)$	$(0.5, 0.4)$	$(0.1, 0.7)$	$(0.7, 0.3)$

Definition 2.5 ([16]). Let $A, B \subseteq E$ and let $f_A, g_B \in S(U)$. f_A is called a IFS subset of g_B, if
(i) $A \subseteq B$,
(ii) $f(e) \subseteq g(e)$ for any $e \in E$.
We denote it by $f_A \widetilde{\subset} g_B$, where
$f(e) = \{(x, \mu_{f(e)}(x), \gamma_{f(e)}(x)) : x \in U\}$ and $g(e) = \{(x, \mu_{g(e)}(x), \gamma_{g(e)}(x)) : x \in U\}$.

Definition 2.6 ([16]). Let $A, B \subseteq E$ and let $f_A, g_B \in S(U)$. f_A and g_B are called IFS equal, if $f_A \widetilde{\subset} g_B$ and $g_B \widetilde{\subset} f_A$. We denote it by $f_A = g_B$.

Definition 2.7 ([16]). Let $A, B \subseteq E$ and let $f_A, g_B \in S(U)$. The intersection of them is the IFS set h_C where $C = A \cap B$, and $h(e) = f(e) \cap g(e)$ for any $e \in C$. We denote it by $f_A \widetilde{\cap} g_B = h_C$.

Definition 2.8 ([16]). Let $A, B \subseteq E$ and let $f_A, g_B \in S(U)$. The union of them is the IFS set h_C where $C = A \cup B$, and for any $e \in C$,

$$h(e) = \begin{cases} f(e), & e \in A - B, \\ g(e), & e \in B - A, \\ f(e) \cup g(e), & e \in A \cap B. \end{cases}$$

We denote it by $f_A \widetilde{\cup} g_B = h_C$.

Definition 2.9 ([16]). Let $A \subseteq E$ and let $f_A \in S(U)$. The relative complement of f_A is denoted f'_A, where $f' : A \to \mathcal{IF}(U)$ is a mapping given by $f'(e) = f^c(e)$ for any $e \in A$.

2.3. **Lattices.**

Definition 2.10 ([6]). Let $(L, \leq)$ be a poset and $a, b \in L$.
(1) a is called a top (or maximal) element of L, if $x \leq a$ for any $x \in L$.
(2) b is called a bottom (or minimal) element of L, if $b \leq x$ for any $x \in L$.

If a poset L has top elements a_1, a_2 (resp. bottom elements b_1, b_2), then $a_1 = a_2$ (resp. $b_1 = b_2$). We denote this sole top element (resp. this sole bottom element) by 1_L (resp. 0_L).

Definition 2.11 ([6]). Let $(L, \leq)$ be a poset, $S \subseteq L$ and $a, b \in L$.
(1) a is called a above boundary in S, if $x \leq a$ for any $x \in S$.
(2) b is called a under boundary in S, if $b \leq x$ for any $x \in S$.
(3) $a = sup\ S\ or \vee S$, if a is a minimal above boundary in S.
(4) $b = inf\ S\ or \wedge S$, if b is a maximal under boundary in S.

Let $(L, \leq)$ be a poset, $S \subseteq L$. If S has $\vee S$ (resp. $\wedge S$), then $\vee S$ (resp. $\wedge S$) is sole. But we can not claim that $\vee S \in S$ (resp. $\wedge S \in S$) although $\vee S \in L$ (resp. $\wedge S \in L$).

We stipulate that $\vee \varnothing = 0_L$ and $\wedge \varnothing = 1_L$. If $S = \{a, b\}$, then we denote $\vee S = a \vee b$ and $\wedge S = a \wedge b$.

Remark 2.12 ([6]). *Let $(L, \leq)$ be a poset and $a, b, c \in L$. Then*
(1) $a = a \wedge b \Longleftrightarrow a \leq b \Longleftrightarrow b = a \vee b$.
(2) $a \leq b \Longrightarrow a \wedge c \leq b \wedge c$; $\quad a \leq b \Longrightarrow a \vee c \leq b \vee c$.
(3) *" $a \leq b$ and $b < a$ " can not be simultaneously true.*

Definition 2.13 ([6]). Let $(L, \leq)$ be a poset.
(1) L is called a lattice, if $a \vee b \in L,\ a \wedge b \in L$ for any $a, b \in L$.
(2) L is called a complete lattice, if $\vee\ S \in L,\ \wedge\ S \in L$ for any $S \subseteq L$.
(3) L is called a distributive lattice, if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for any $a, b, c \in L$.

Definition 2.14 ([6]). Let L be a lattice with 1_L and 0_L and $a, b \in L$. b is called a complement element of a, if $a \vee b = 1_L, a \wedge b = 0_L$.

If L is a distributive lattice and $a \in L$ has complement elements b_1, b_2, then $b_1 = b_2$. We denote the complement element of a by a'.

Example 2.15. Let $L = [0, 1]$. For any $a, b \in L$, we define $a \leq b$ by $b - a \geq 0$. Obviously, $1_L = 1,\ 0_L = 0$. It is easily proved that L is a complete distributive lattice.

3. The properties of IFS sets

In this section, we introduce some operations on IFS sets and investigate their related properties.

Theorem 3.1. *Let $f_A, g_B \in S(U)$. Then*
(1) $(f_A\ \widetilde{\cup}\ g_B)\ \widetilde{\cap}\ f_A = f_A$,
(2) $(f_A\ \widetilde{\cap}\ g_B)\ \widetilde{\cup}\ f_A = f_A$.

Proof. Put $(f_A\ \widetilde{\cup}\ g_B)\ \widetilde{\cap}\ f_A = m_{(A\cup B)\cap B}$. For any $e \in A \cup B$.
i) If $e \in B - A$, then $m(e) = g(e) \cap f(e) = \widetilde{\varnothing} = f(e)$.
ii) If $e \in A - B$, then $m(e) = f(e) \cap f(e) = f(e)$.
iii) If $e \in A \cup B$, then $m(e) = (f(e) \cup g(e)) \cap f(e) = f(e)$.
Thus $(f_A\ \widetilde{\cup}\ g_B)\ \widetilde{\cap}\ f_A = f_A$.
(2) This is similar to the proof of (1). □

Proposition 3.2. *Let $f_A, g_B, h_C \in S(U)$. Then*
(1) $f_A\ \widetilde{\cup}\ f_A = f_A$,
(2) $f_A\ \widetilde{\cup}\ g_B\ =\ g_B\ \widetilde{\cup}\ f_A$,
(3) $(f_A\ \widetilde{\cup}\ g_B)\ \widetilde{\cup}\ h_C\ =\ f_A\ \widetilde{\cup}\ (g_B\ \widetilde{\cup}\ h_C)$.

Proof. (1) and (2) are trivial. We only prove (3). Put

$$(f_A\ \widetilde{\cup}\ g_B)\ \widetilde{\cup}\ h_C = k_{A\cup B\cup C},\ \ f_A\ \widetilde{\cup}\ (g_B\ \widetilde{\cup}\ h_C) = l_{A\cup B\cup C}.$$

For any $e \in A \cup B \cup C$, it follows that $e \in A$, or $e \in B$, or $e \in C$. Without losing of generality, we can suppose that $e \in C$.
i) If $e \notin A \cup B$, then $k(e) = h(e) = l(e)$.
ii) If $e \in B - A$, then $k(e) = g(e) \cup h(e) = l(e)$.
iii) If $e \in A - B$, then $k(e) = f(e) \cup h(e) = l(e)$.
iv) If $e \in A \cap B$, then $k(e) = (f(e) \cup g(e)) \cup h(e) = f(e) \cup (g(e) \cup h(e)) = l(e)$.
Thus $(f_A\ \widetilde{\cup}\ g_B)\ \widetilde{\cup}\ h_C\ =\ f_A\ \widetilde{\cup}\ (g_B\ \widetilde{\cup}\ h_C)$. □

Proposition 3.3. *Let $f_A, g_B, h_C \in S(U)$. Then*
(1) $f_A\ \widetilde{\cap}\ f_A = f_A$,
(2) $f_A\ \widetilde{\cap}\ g_B\ =\ g_B\ \widetilde{\cap}\ f_A$,
(3) $(f_A\ \widetilde{\cap}\ g_B)\ \widetilde{\cap}\ h_C\ =\ f_A\ \widetilde{\cap}\ (g_B\ \widetilde{\cap}\ h_C)$.

Proof. (1) and (2) are trivial. We only prove (3). Put

$$(f_A \mathbin{\widetilde{\cap}} g_B) \mathbin{\widetilde{\cap}} h_C = k_{A\cap B\cap C},\ f_A \mathbin{\widetilde{\cap}} (g_B \mathbin{\widetilde{\cap}} h_C) = l_{A\cap B\cap C}.$$

For any $e \in A\cap B\cap C$, it follows that $e \in A$, $e \in B$ and $e \in C$. Since

$$k(e) = (f(e)\cap g(e))\cap h(e) = f(e)\cap (g(e)\cap h(e)) = l(e),$$

then $(f_A \mathbin{\widetilde{\cap}} g_B) \mathbin{\widetilde{\cap}} h_C \ = \ f_A \mathbin{\widetilde{\cap}} (g_B \mathbin{\widetilde{\cap}} h_C)$. □

Proposition 3.4. *Let $f_A, g_B, h_C \in S(U)$. Then*
(1) $(f_A \mathbin{\widetilde{\cup}} g_B) \mathbin{\widetilde{\cap}} h_C \ = \ (f_A \mathbin{\widetilde{\cap}} h_C) \mathbin{\widetilde{\cup}} (g_B \mathbin{\widetilde{\cap}} h_C)$,
(2) $(f_A \mathbin{\widetilde{\cap}} g_B) \mathbin{\widetilde{\cup}} h_C \ = \ (f_A \mathbin{\widetilde{\cup}} h_C) \mathbin{\widetilde{\cap}} (g_B \mathbin{\widetilde{\cup}} h_C)$.

Proof. (1) Put

$$(f_A \mathbin{\widetilde{\cup}} g_B) \mathbin{\widetilde{\cap}} h_C \ = k_{(A\cup B)\cap C}\ ,$$
$$(f_A \mathbin{\widetilde{\cap}} h_C)\cup (g_B \mathbin{\widetilde{\cap}} h_C) = l_{(A\cap C)\cup(B\cap C)}.$$

Obviously, $(A\cup B)\cap C = (A\cap C)\cup(B\cap C)$. For any $e \in (A\cup B)\cap C$, it follows that $e \in A\cap C$, or $e \in B\cap C$.

i) If $e \notin A\cap C$ and $e \in B\cap C$, then $e \notin A, e \in B$ and $e \in C$. So $k(e) = g(e)\cap h(e) = l(e)$.

ii) If $e \in A\cap C$ and $e \notin B\cap C$, then $e \in A, e \notin B$ and $e \in C$. So $k(e) = f(e)\cap h(e) = l(e)$.

iii) If $e \in A\cap C$ and $e \in B\cap C$, then $e \in A, e \in B$ and $e \in C$. So $k(e) = (f(e)\cup g(e))\cap h(e) = (f(e)\cap h(e))\cup (g(e)\cap h(e)) = l(e)$.

Thus

$$(f_A \mathbin{\widetilde{\cup}} g_B) \mathbin{\widetilde{\cap}} h_C \ = \ (f_A \mathbin{\widetilde{\cap}} g_B)\cup (g_B \mathbin{\widetilde{\cap}} h_C).$$

(2) This is similar to the proof of (1). □

Definition 3.5. Let $A, B \subseteq E$ and let $f_A, g_B \in S(U)$. The extended intersection of them is the IFS set h_C where $C = A\cup B$, and for any $e \in C$,

$$h(e) = \begin{cases} f(e), & e \in A - B, \\ g(e), & e \in B - A, \\ f(e)\cap g(e), & e \in A\cap B. \end{cases}$$

We denote it by $f_A \sqcap g_B = h_C$.

Definition 3.6. Let $A, B \subseteq E$ and let $f_A, g_B \in S(U)$. The restricted union of them is the IFS set h_C where $C = A\cap B$, and $h(e) = f(e)\cup g(e)$ for any $e \in C$. We denote it by $f_A \sqcup g_B = h_C$.

Theorem 3.7. *Let $f_A, g_B, h_C \in S(U)$. Then*
(1) $f_A \sqcup f_A = f_A$,
(2) $f_A \sqcup g_B = g_B \sqcup f_A$,
(3) $(f_A \sqcup g_B) \sqcup h_C = f_A \sqcup (g_B \sqcup h_C)$.

Proof. This is obvious. □

Theorem 3.8. *Let $f_A, g_B, h_C \in S(U)$. Then*
(1) $f_A \sqcap f_A = f_A$,
(2) $f_A \sqcap g_B = g_B \sqcap f_A$,
(3) $(f_A \sqcap g_B) \sqcap h_C = f_A \sqcap (g_B \sqcap h_C)$.

Proof. This is obvious. □

Theorem 3.9. *Let $f_A, g_B \in S(U)$. Then*
(1) $(f_A \sqcup g_B) \sqcap f_A = f_A$,
(2) $(f_A \sqcap g_B) \sqcup f_A = f_A$.

Proof. Let $(f_A \sqcup g_B) = h_{A\cap B}$. For any $e \in E$,
i) If $e \in A - B$, then $e \notin A \cap B$ and $h(e) = \widetilde{\varnothing}$. Thus $(h_{A\cap B} \sqcap f_A)(e) = f(e)$.
ii) If $e \in B - A$, then $e \notin A \cap B$ and $h(e) = \widetilde{\varnothing}$. Thus $(h_{A\cap B} \sqcap f_A)(e) = \widetilde{\varnothing} = f(e)$.
iii) If $e \in A \cap B$, then $h(e) = f(e) \cup g(e)$. Thus $(h_{A\cap B} \sqcap f_A)(e) = (f(e) \cup g(e)) \cap f(e) = f(e)$.
Hence $(f_A \sqcup g_B) \sqcap f_A = f_A$.
(2) This is similar to (1). □

Theorem 3.10. *Let $f_A, g_B, h_C \in S(U)$. Then*
(1) $f_A \sqcup (g_B \sqcap h_C) = (f_A \sqcup g_B) \sqcap (f_A \sqcup h_C)$,
(2) $f_A \sqcap (g_B \sqcup h_C) = (f_A \sqcap g_B) \sqcup (f_A \sqcap h_C)$.

Proof. (1) Let $f_A \sqcup (g_B \sqcap h_C) = m_{A\cap(B\cup C)}$ and $(f_A \sqcup g_B) \sqcap (f_A \sqcup h_C) = n_{(A\cap B)\cup(A\cap C)} = n_{A\cap(B\cup C)}$.
For any $e \in A \cap (B \cup C)$, this implies $e \in A$ and $e \in B \cup C$.
i) If $e \in B - C$, then $m(e) = f(e) \cup g(e) = n(e)$.
ii) If $e \in C - B$, then $m(e) = f(e) \cup h(e) = n(e)$.
iii) If $e \in B \cap C$, then $m(e) = f(e) \cup (g(e) \cap h(e)) = (f(e) \cup g(e)) \cap (f(e) \cup h(e)) = n(e)$.
Thus $f_A \sqcup (g_B \sqcap h_C) = (f_A \sqcup g_B) \sqcap (f_A \sqcup h_C)$.
(2) This proof is similar to (1). □

Theorem 3.11. *Let $f_A, g_B \in S(U)$. Then*
(1) $(f_A \widetilde{\cup} g_B)' = f'_A \sqcap g'_B$,
(2) $(f_A \sqcap g_B)' = f'_A \widetilde{\cup} g'_B$.

Proof. Let $f_A \widetilde{\cup} g_B = h_{A\cup B}$ and $f'_A \sqcap g'_B = l_{A\cup B}$. For any $e \in E$,
i) If $e \notin A \cup B$, then $h'(e) = \widetilde{\varnothing} = l(e)$.
ii) If $e \in B - A$, then $h'(e) = g'(e) = l(e)$.
iii) If $e \in A - B$, then $h'(e) = f(e) = l(e)$.
iv) If $e \in A \cap B$, then $h'(e) = (f(e) \cup g(e))' = f(e)' \cap g(e)' = l(e)$.
Thus $(f_A \widetilde{\cup} g_B)' = f'_A \sqcap g'_B$.
(2) This is similar to the proof of (1). □

Theorem 3.12. *Let $f_A, g_B, h_C \in S(U)$. Then*
(1) $f_A \widetilde{\cap} (g_B \sqcup h_C) = (f_A \widetilde{\cap} g_B) \sqcup (f_A \widetilde{\cap} h_C)$,
(2) $f_A \sqcup (g_B \widetilde{\cap} h_C) = (f_A \sqcup g_B) \widetilde{\cap} (f_A \sqcup h_C)$.

Proof. (1) Let $f_A \widetilde{\cap} (g_B \sqcup h_C) = h_{A\cap(B\cap C)}$ and $(f_A \widetilde{\cap} g_B) \sqcup (f_A \widetilde{\cap} h_C) = l_{(A\cap B)\cap(A\cap C)}$. Then for any $e \in A \cap (B \cap C)$, we have $e \in A$, $e \in B$ and $e \in C$.
Thus $h(e) = f(e) \cap (g(e) \cup h(e)) = (f(e) \cap g(e)) \cup (f(e) \cap h(e)) = l(e)$.
(2) This is similar to the proof of (1). □

Theorem 3.13. *Let $f_A, g_B \in S(U)$. Then*
(1) $(f_A \mathbin{\widetilde{\cup}} g_B) \sqcap f_A = (f_A \sqcap g_B) \mathbin{\widetilde{\cup}} f_A$,
(2) $(f_A \sqcup g_B) \mathbin{\widetilde{\cap}} f_A = (f_A \mathbin{\widetilde{\cap}} g_B) \sqcup f_A$.

Proof. (1) Let $(f_A \mathbin{\widetilde{\cup}} g_B) \sqcap f_A = m_{A\cup B}$ and $(f_A \sqcap g_B) \mathbin{\widetilde{\cup}} f_A = n_{A\cup B}$. For any $e \in A \cup B$,
(i) If $e \in A - B$, then $m(e) = f(e) = n(e)$.
(ii) If $e \in B - A$, then $m(e) = g(e) = n(e)$.
(iii) If $e \in A \cap B$, then $m(e) = (f(e) \cup g(e)) \cap f(e) = f(e) = (f(e) \cap g(e)) \cup f(e) = n(e)$.
Thus $(f_A \mathbin{\widetilde{\cup}} g_B) \sqcap f_A = (f_A \sqcap g_B) \mathbin{\widetilde{\cup}} f_A$.
(2) Let $(f_A \sqcup g_B) \mathbin{\widetilde{\cap}} f_A = m_{A\cap B}$ and $(f_A \mathbin{\widetilde{\cap}} g_B) \sqcup f_A = n_{A\cap B}$. For any $e \in A \cap B$, $m(e) = (f(e) \cup g(e)) \cap f(e) = f(e) = (f(e) \cap g(e)) \cup f(e) = n(e)$. □

Example 3.14. Let $U = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2, e_3\}$. Let $f_A, g_B \in S(U)$, where $A = \{e_1, e_2\}$ and $B = \{e_2, e_3\}$.
Let $f_A \mathbin{\widetilde{\cup}} g_B = h_{A\cup B}$ and $h_{A\cup B} \sqcap f_A = m_{A\cup B}$. Then

$$h(e_1) = f(e_1),\ h(e_2) = f(e_2) \cup g(e_2),\ h(e_3) = g(e_3)$$

and

$$m(e_1) = h(e_1) \cap f(e_1) = f(e_1),\ m(e_2) = h(e_2) \cap f(e_2) = (f(e_2) \cup g(e_2)) \cap f(e_2).$$

Note that

$$m(e_3) = g(e_3) \neq f(e_3) = \widetilde{\varnothing}.$$

Thus $(f_A \mathbin{\widetilde{\cup}} g_B) \sqcap f_A \neq f_A$.
Let $f_A \sqcup g_B = l_{A\cap B}$ and $l_{A\cap B} \mathbin{\widetilde{\cap}} f_A = n_{A\cap B}$. Then

$$l(e_1) = \widetilde{\varnothing},\ l(e_2) = f(e_2) \cup g(e_2),\ l(e_3) = \widetilde{\varnothing}$$

and

$$n(e_1) = \widetilde{\varnothing} \neq f(e_1),\ n(e_3) = \widetilde{\varnothing} \neq f(e_3).$$

Thus $(f_A \sqcup g_B) \mathbin{\widetilde{\cap}} f_A \neq f_A$.

This example shows that

$$(f_A \mathbin{\widetilde{\cup}} g_B) \sqcap f_A = f_A \text{ and } (f_A \sqcup g_B) \mathbin{\widetilde{\cap}} f_A = f_A$$

are do not hold in general.

4. Lattice structures of IFS sets

Theorem 4.1. *For any $f_A, g_B \in S(U)$, define*

$$f_A \leq g_B \ \Leftrightarrow\ f_A \mathbin{\widetilde{\subset}} g_B,$$

$$f_A \vee g_B = f_A \mathbin{\widetilde{\cup}} g_B,$$

$$f_A \wedge g_B = f_A \mathbin{\widetilde{\cap}} g_B.$$

Then $(S(U), \widetilde{\cup}, \widetilde{\cap})$ is a complete distributive lattice with top element and bottom element.

Proof. Denote $\sum = S(U)$. It is easily proved that

$$0_{\sum} = \widetilde{\varnothing} \text{ and } 1_{\sum} = \widetilde{U}.$$

By Proposition 3.4, $S(U)$ is a distributive lattice with $1_{\sum}$ and $0_{\sum}$. □

Theorem 4.2. *For any $f_A, g_A \in S(U)_A$, define*

$$f_A \leq g_A \ \Leftrightarrow \ f_A \mathrel{\widetilde{\subset}} g_A,$$

$$f_A \vee g_A = \ f_A \mathbin{\widetilde{\cup}} g_A,$$

$$f_A \wedge g_A = \ f_A \mathbin{\widetilde{\cap}} g_A.$$

Then $(S(U)_A, \widetilde{\cup}, \widetilde{\cap})$ *is a distributive lattice.*

Proof. This is obvious. □

Corollary 4.3. $(S(U)_A, \widetilde{\cup}, \widetilde{\cap})$ *is a sublattice of* $(S(U), \widetilde{\cup}, \widetilde{\cap})$.

Theorem 4.4. *Let $f_A, g_B \in S(U)$, define*

$$f_A \leq g_B \ \Leftrightarrow \ f_A \mathrel{\widetilde{\subset}} g_B,$$

$$f_A \vee g_B = \ f_A \sqcup g_B,$$

$$f_A \wedge g_B = \ f_A \sqcap g_B.$$

Then $(S(U), \sqcup, \sqcap)$ *is a complete distributive lattice with top element and bottom element.*

Proof. By Theorem 3.10, this is straightforward. □

Theorem 4.5. *Let $f_A, g_A \in S(U)_A$, define*

$$f_A \leq g_A \ \Leftrightarrow \ f_A \mathrel{\widetilde{\subset}} g_A,$$

$$f_A \vee g_A = \ f_A \sqcup g_A,$$

$$f_A \wedge g_A = \ f_A \sqcap g_A.$$

Then $(S(U)_A, \sqcup, \sqcap)$ *is a distributive lattice.*

Proof. This is obvious. □

Corollary 4.6. $(S(U)_A, \sqcup, \sqcap)$ *is a sublattice of* $(S(U), \sqcup, \sqcap)$.

Theorem 4.7. *Let $f_A, g_B \in S(U)$, define*

$$f_A \leq g_B \;\Leftrightarrow\; f_A \,\widetilde{\subset}\, g_B,$$

$$f_A \vee g_B = \; f_A \sqcup g_B,$$

$$f_A \wedge g_B = \; f_A \,\widetilde{\cap}\, g_B.$$

Then $(S(U), \sqcup, \widetilde{\cap})$ is a distributive lattice with top element and bottom element.

Proof. By Theorem 3.12, this is obvious. □

Similarly, by the definition of Theorem 4.7, we can have that $(S(U)_A, \sqcup, \widetilde{\cap})$ is a distributive lattice.

5. Conclusions

In this paper, we study the lattice structures of intuitionistic fuzzy soft sets. Based on intersection (extended intersection) and union (restricted union) of intuitionistic fuzzy sets, five lattice structures of intuitionistic fuzzy soft sets are established and several related properties are investigated. We prove that De Morgan's laws hold in intuitionistic fuzzy soft set theory.

Acknowledgements. This work is supported by the Grants of China-Dong Meng Research Center (Guangxi Scientific Experiment Center) and Guangxi University Science and Technology Research Project (No. 2013ZD020, 2013ZD061).

References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and systems 20 (1986) 87–96.
[2] K. Atanassov, Operators over interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 64 (1994) 159–174.
[3] H. Aktas and N. Cağman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726–2735.
[4] K. V. Babitha and J. J. Sunil, Soft set relations and functions, Comput. Math. Appl. 60 (2010) 1840–1849.
[5] N. Cağman, S. Karatas and S. Enginoglu, Soft topology, Comput. Math. Appl. 62 (2011) 351–358.
[6] B. A. Davey and H. A. Priestley, Introduction to lattices and order, Cambridge University Press, Cambridge, 1990.
[7] F. Feng, C. Li, B. Davvaz and M. Irfan Ali, Soft sets combined with fuzzy sets and rough sets: a tentative approach, Soft Comput. 14 (2010) 899–911.
[8] F. Feng, X. Liu, V. Leoreanu-Fotea and Y. B. Jun, Soft sets and soft rough sets, Inform. Sci. 181 (2011) 1125–1137.
[9] F. Feng, Y. B. Jun, X. Liu and L. Li, An adjustable approach to fuzzy soft set based decision making, J. Comput. Appl. Math. 234 (2010) 10–20.
[10] X. Ge, Z. Li and Y. Ge, Topological spaces and soft sets, J. Comput. Anal. Appl. 13 (2011) 881–885.
[11] Y. B. Jun, K. J. Lee and C. H. Park, Fuzzy soft set theory applied to BCK/BCI-algebras, Comput. Math. Appl. 59 (2010) 3180–3192.
[12] Y. Jiang, Y. Tang, Q. Chen, J. Wang and S. Tang, Extending soft sets with description logics, Comput. Math. Appl. 59 (2010) 2087–2096.
[13] Y. Jiang, Y. Tang and Q. Chen, An adjustable approach to intuitionistic fuzzy soft sets based decision making, Appl. Math. Model. 35 (2011) 824–836.

[14] D. Molodtsov, Soft set theory-first result, Comput. Math. Appl. 37 (1999) 19–31.
[15] P. K. Maji, R. Biswas and A. R. Roy, Fuzzy soft sets, J. Fuzzy Math. 9 (2001) 589–602.
[16] P. K. Maji, R. Biswas and A. R. Roy, Intuitionistic fuzzy soft sets, J. Fuzzy Math. 9 (2001) 677–692.
[17] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[18] P. K. Maji and A. R. Roy, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077–1083.
[19] P. K. Maji, A. R. Roy and R. Biswas, On intuitionistic fuzzy soft sets, J. Fuzzy Math. 12 (2004) 669–682.
[20] Z. Pawlak, Rough Sets: Theoretical aspects of reasoning about data, Kluwer Academic Publishers, Boston, 1991.
[21] P. Rajarajeswari and R. Krishna Moorthy, Intuitionistic fuzzy contra weakly generalized continuous mappings, Ann. Fuzzy Math. Inform. 5 (2013) 361–369.
[22] A. R. Roy and P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, J. Comput. Appl. Math. 203 (2007) 412–418.
[23] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (2011) 1786–1799.
[24] B. Tanay and M. B. Kandemir, Topological structure of fuzzy soft sets, Comput. Math. Appl. 61 (2011) 2952–2957.
[25] X. Yang, T. Lin, J. Yang, Y. Li and D. Yu, Combination of interval-valued fuzzy set and soft set, Comput. Math. Appl. 58 (2009) 521–527.
[26] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

Zhaowen Li (lizhaowen8846@126.com)
College of Science, Guangxi University for Nationalities, Nanning-530006, Guangxi, P. R. China

Shijie Li (lishijie1026@126.com)
College of Science, Guangxi University for Nationalities, Nanning-530006, Guangxi, P. R. China