Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 6, No. 3, (November 2013), pp. 455–465
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Energy of a fuzzy graph

Anjali Narayanan, Sunil Mathew

Received 25 October 2012; Received 2 December 2012; Accepted 1 March 2013

Abstract. Energy of a simple graph has been defined and many of its properties have been studied. In this paper, the concept of energy is extended to fuzzy graphs. Adjacency matrix of a fuzzy graph is defined and energy of a fuzzy graph is defined as the sum of absolute values of the eigenvalues of the adjacency matrix of the fuzzy graph. Some bounds on energy of fuzzy graphs are also obtained.

2010 AMS Classification: 05C22, 05C90

Keywords: Fuzzy graph, Adjacency matrix, Spectrum, Energy.

Corresponding Author: Sunil Mathew (sm@nitc.ac.in)

1. Introduction

Many real world systems can be modelled using graphs. Graphs represent the connections between the entities in these systems. The connections may be physical as in electrical networks and computer networks or relationships as in ecosystems. Graphs are abstractions of networks.

The foundation for graph theory was laid in 1735 by Euler when he solved the Konigsberg bridges' problem. Conclusions he made on studying the problem were some of the fundamental properties of graphs. Attempts to solve diagram tracing puzzles in which one has to draw the given diagram connecting the given points in fewest number of connected strokes also led to the discovery of principles later recognized as the properties of certain classes of graphs. The connection between diagram tracing puzzles and the Konigsberg bridges' problem was recognized only at the end of nineteenth century by Rouse Ball in *Mathematical Recreations and Problems*. It was the interest and attempts to solve puzzles during 18^{th} and 19^{th} centuries that uncovered majority of concepts and results related to graphs. In 1878, Sylvester wrote a short note in *Nature* about the possible connection between chemical molecules and binary quantics. The term graph was used in the graph theoretic sense for the first time in this note. One can refer [27] to know more about the history of graph theory.

The pictorial representation of a graph consists of a set of points joined by arcs. To make use of computers to solve problems on graphs, they had to be stored in the memory of computers. This is done using matrices. Many kinds of matrices are associated with a graph. The spectrum of one such matrix, adjacency matrix is called the spectrum of the graph. The properties of the spectrum of a graph is related to the properties of the graph. The area of graph theory that deals with this is called spectral graph theory. The spectrum of a graph first appeared in a paper by Collatz and Sinogowitz in 1957. At present, it is widely studied owing to its applications in physics, chemistry, computer science and other branches of mathematics. In chemistry, it has applications in the theory of unsaturated conjugated molecular hydrocarbons called Huckel molecular orbital theory. Graph spectrum appears in problems in statistical physics and in combinatorial optimization problems in mathematics. Spectrum of a graph also plays an important role in pattern recognition, modelling virus propagation in computer networks and in securing personal data in databases. Cvetkovic and I. Gutman have discussed these applications in detail in [5].

A concept related to the spectrum of a graph is that of energy. As its name suggests, it is inspired by energy in chemistry. The study of π-electron energy in chemistry dates back to 1940's but it is in 1978 that I. Gutman defined it mathematically for all graphs. Organic molecules can be represented by graphs called molecular graphs. In case of unsaturated conjugated hydrocarbons, the energy of π electrons of the molecule is approximately the energy of its molecular graph [4]. Energy of graphs has many mathematical properties which are being investigated. The nature of values it takes are discussed in [2] and [21]. Certain bounds on energy are studied in [3], [13] and [8]. Different classes of graphs namely hyperenergetic, hypoenergetic and equienergetic based on their energy are identified and one can know more about them from [3], [5], [1, 22] and [10]. Energy of different graphs including regular [9], non-regular [11], circulant [24] and random graphs [6] is also under study.

The physical meaning and application of energy of a graph may not be known exactly at present but the properties it is found to possess are of interest to a mathematician. Energy is defined for signed graphs in [7] and for weighted graphs by I. Gutman and Shao in 2011. In this paper, we extend the definition of energy to fuzzy graphs. Fuzzy graphs are generalizations of graphs. Our intension is the generalization of energy. Intuition that the results known so far on energy of a graph might be a particular case of more general results on fuzzy graphs is the motivation behind this paper.

Fuzzy graphs are encountered in fuzzy set theory. A fuzzy set was defined by Zadeh in 1965. His purpose was to develop a theory for sets which are ambiguous and imprecise in definition, a characteristic of most of the sets found in the real world. Every element in the universal set is assigned a grade of membership, a value in $[0, 1]$. The elements in the universal set along with their grades of membership form a fuzzy set. It is called a fuzzy subset of the universal set. It is a generalization of usual sets also called crisp sets. A grade of membership of 1 to some elements and

0 to all others in the universal set gives a crisp set. Fuzzy sets are representations of how a human brain perceives the objects in the world. Hence, fuzzy set theory has applications in those areas where machine replacements are sought for humans for instance control engineering, artificial intelligence, expert systems, robotics, pattern recognition and so on. As in case of crisp sets, relations are defined for fuzzy subsets called fuzzy relations. Graphs are representations of binary relations. Similarly, fuzzy binary relations are represented by graphs called fuzzy graphs. Rosenfeld introduced the theory for fuzzy graphs in 1975. Complement [26], fuzzy line graph [17], fuzzy cycle [19], fuzzy tree [25], operations on fuzzy graph [18], connectivity [14, 16, 15] and much more have been defined and studied. The results obtained are applied in clustering analysis and modelling information networks [28]. [20] explains fuzzy subsets, fuzzy relations and fuzzy graphs with applications.

Section 2 consists of preliminaries and definition of energy of a fuzzy graph. In Section 3, we present some results on energy of a fuzzy graph.

2. Preliminaries

An ordered pair of sets $G = (V, E)$ where V is a nonempty finite set and E consisting of two element subsets of elements of V is called a graph. It is denoted by $G = (V, E)$. V is called the vertex set and E, the edge set. The elements in V and E are called vertices and edges respectively. If elements of E are ordered pairs, then G is called a directed graph or digraph [4]. The vertices between which an edge exists are called endpoints of the edge. An edge whose endpoints are the same is called a loop. A graph without loops is called a simple graph.

Let $G = (V, E)$ be a simple graph with n vertices and m edges. Let $v_1, v_2, ..., v_n$ represent the vertices of G. It can be represented by an $n \times n$ matrix giving the adjacency between all vertices. This matrix is called adjacency matrix and let us denote it by A. An element a_{ij} of A gives the number of edges between vertices v_i and v_j for $v_i, v_j \in V$. In a simple graph, there can be atmost 1 edge between two vertices. So, the entries in A are either 0 or 1. The diagonal is zero since there are no loops. A is symmetric and so the spectrum of A is real. The eigenvalues of A are called eigenvalues of G [3] and the spectrum of A is called the spectrum of G. Energy of a simple graph $G = (V, E)$ with adjacency matrix A is defined as the sum of absolute values of eigenvalues of A [1]. It is denoted by $E(G)$.

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

where λ_i is an eigenvalue of A, $i = 1,\ 2, \ldots,\ n$. Suppose k eigenvalues are positive. Then,

$$E(G) = 2\sum_{i=1}^{k} \lambda_i \tag{2.1}$$

Also,

$$\sum_{1 \le i < j \le n} \lambda_i \lambda_j = -2m \tag{2.2}$$

Hence,

$$\sum_{i=1}^{n} \lambda_i^2 = 2m \tag{2.3}$$

The energy of a graph is zero iff it is trivial. Using (2.1) it is proved that the energy of a nontrivial graph is an even number if it is rational [2]. Suppose G is a simple graph with n vertices and m edges. Let A be its adjacency matrix. The following are some of the bounds on energy [3, 8]. Applying Cauchy Schwarz inequality for $(1, \ldots, 1)$ and $(|\lambda_1|, \ldots, |\lambda_n|)$ and using (2.3)

$$E(G) \leq \sqrt{2mn} \tag{2.4}$$

Expanding $[E(G)]^2$ and using the following results

(1) The sum of k real numbers is k times their arithmetic mean.

(2) The geometric mean of non negative numbers cannot exceed their arithmetic mean.

(3) $\{GM|\lambda_i\lambda_j|\} = |A|^{\frac{2}{n}},\ 1 \leq i < j \leq n$

we get,

$$\sqrt{2m + n(n-1)|A|^{\frac{2}{n}}} \leq E(G) \tag{2.5}$$

Combining (2.4) and (2.5)

$$\sqrt{2m + n(n-1)|A|^{\frac{2}{n}}} \leq E(G) \leq \sqrt{2mn}$$

Suppose G is a simple graph. Then applying Cauchy Schwarz inequality for $(1, \ldots, 1)$ and $|\lambda_2|, \ldots, |\lambda_n|$, we get

$$E(G) \leq \lambda_1 + \sqrt{(n-1)(2m - \lambda_1^2)}$$

where λ_1 is the largest eigenvalue of G. If G has no isolated vertices, we have $n \leq 2m$. Then, $2m/n$ and λ_1 lies in an interval in which the function $F(x) = x + \sqrt{(n-1)(2m - x^2)}$ is decreasing. Since $\lambda_1 \geq 2m/n$, we get

$$E(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left\{2m - \left(\frac{2m}{n}\right)^2\right\}}$$

Let V be a nonempty set. A fuzzy subset of V is a function $\sigma : V \to [0,1]$. σ is called the membership function and $\sigma(v)$ is called the membership of v where $v \in V$. Let V_1 and V_2 be nonempty sets. σ_1 and σ_2 be fuzzy subsets of V_1 and V_2 respectively. Define a fuzzy subset μ of $V_1 \times V_2$ as $\mu(v_i, v_j) \leq min\{\sigma_1(v_i), \sigma_2(v_j)\}$. Then, μ is called a fuzzy relation from σ_1 to σ_2 . Suppose $\sigma_1(x) = 1,\ \forall\ x \in V_1$ and $\sigma_2(y) = 1,\ \forall\ y \in V_2$. Then μ is called a fuzzy relation from V_1 into V_2. $\mu(v_i, v_j)$ is interpreted as the strength of relation between v_i and v_j [12]. Suppose $V_1 = V_2 = V$ and $\sigma_1 = \sigma_2 = \sigma$. Then, μ is called a fuzzy relation on σ. Suppose $V_1 = V_2 = V$ and $\sigma_1(x) = 1, \forall\ x \in V_1$ and $\sigma_2(y) = 1,\ \forall\ y \in V_2$. Then, μ is called a fuzzy relation on V. From definition, it follows that binary relations on crisp sets are particular cases of fuzzy relations. Let V be a nonempty set and σ, a fuzzy subset of V. Let μ be a fuzzy relation on σ. μ is said to be symmetric, if $\mu(v_i, v_j) = \mu(v_j, v_i)$ for

$v_i, v_j \in V$. A fuzzy relation can also be expressed by a matrix called fuzzy relation matrix $M = [m_{ij}]$ [12] where

$$m_{ij} = \mu(v_i, v_j)$$

A fuzzy graph $G = (V, \sigma, \mu)$ is a nonempty set V together with a pair of functions (σ, μ) where σ is a fuzzy subset of V and μ is a fuzzy relation on σ. G considered hereafter is undirected and without loops [20].

For a graph G with adjacency matrix $A = [a_{ij}]$, a_{ij} gives the number of vertices between v_i and v_j. a_{ij} can also be interpreted as the strength of relation between v_i and v_j. This interpretation can be extended to fuzzy graphs. Then, the fuzzy relation matrix becomes the adjacency matrix.

Definition 2.1. The *adjacency matrix* A of a fuzzy graph $G = (V, \sigma, \mu)$ is an $n \times n$ matrix defined as $A = [a_{ij}]$ where $a_{ij} = \mu(v_i, v_j)$. Note that A becomes the usual adjacency matrix when all the nonzero membership values are 1. ie; when the fuzzy graph becomes a crisp graph.

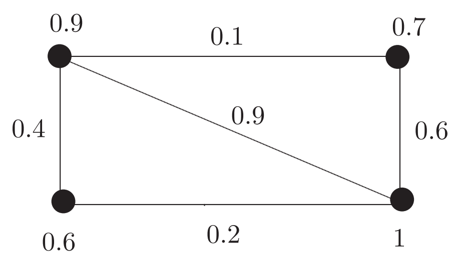

FIGURE 1. G_1, A fuzzy graph

Example 2.2. Adjacency matrix of the fuzzy graph G_1 (Fig. 1) is

$$A=\begin{pmatrix} 0 & 0.1 & 0.9 & 0.4 \\ 0.1 & 0 & 0.6 & 0 \\ 0.9 & 0.6 & 0 & 0.2 \\ 0.4 & 0 & 0.2 & 0 \end{pmatrix}.$$

Definition 2.3. Let $G = (V, \sigma, \mu)$ be a fuzzy graph and A be its adjacency matrix. The eigenvalues of A are called *eigenvalues* of G. The spectrum of A is called the *spectrum* of G. It is denoted by Spec G.

Example 2.4. For the graph in Fig. 1, spec G_1=$\{-1.0464, -0.3164, 0.1174, 1.2454\}$.

Definition 2.5. Let $G = (V, \sigma, \mu)$ be a fuzzy graph and A be its adjacency matrix. *Energy* of G is defined as the sum of absolute values of eigenvalues of G.

Example 2.6. For the graph in Fig. 1, $E(G_1) = 1.0464+0.3164+0.1174+1.2454 = 2.7256$.

3. Results

The energy of a nontrivial simple graph is always greater than 1 [1]. But, this result is not true for a fuzzy graph as seen from the following example.

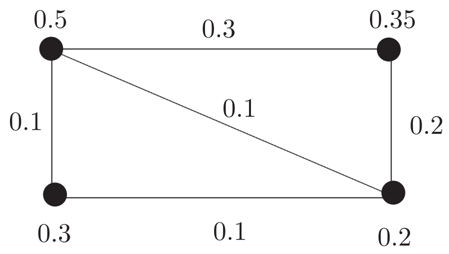

Figure 2. G_2, A fuzzy graph with energy < 1

Example 3.1. For G_2 (Fig. 2), adjacency matrix is B= $\begin{pmatrix} 0 & 0.1 & 0 & 0.1 \\ 0.1 & 0 & 0.2 & 0.1 \\ 0 & 0.2 & 0 & 0.3 \\ 0.1 & 0.1 & 0.3 & 0 \end{pmatrix}$.
Spec G_2=$\{-0.3442, -0.1, 0.0066, 0.4376\}$. $E(G_2) = 0.8884 < 1$.

Some bounds for energy of fuzzy graphs are given below.

Theorem 3.2. ([23]) *Let G be a weighted graph of order n each of whose edges has nonzero weight and $e_1, \ldots, e_m$ be all the edges of G. Then*

$$E(G) \leq 2\sum_{i=1}^{m} |w(e_i)|$$

where equality holds iff each of the connected component of G has atmost two vertices.

The above theorem is for a weighted graph. Here, $w(e_i)$ denotes the weight of an edge $e_i, i = 1, \ldots, m$. Considering a fuzzy graph as a weighted graph with weights in the interval $[0, 1]$, Theorem 3.2 can be restated as follows:

Theorem 3.3. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph with $|V| = n$ and $\mu^* = \{e_1, \ldots, e_m\}$. Then,*

$$E(G) \leq 2\sum_{i=1}^{m} \mu(e_i)$$

Using Theorem 3.3, upper bounds for energy of a fuzzy graph can be obtained in terms of membership values of its vertices.

Proposition 3.4. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph with $|V| = n$ and $\mu^* = \{e_1, \ldots, e_m\}$. Then,*

$$E(G) \leq (n-1)\sum_{i=1}^{n} \sigma(v_i)$$

Proof. From Theorem 3.3,

$$E(G) \leq 2\sum_{i=1}^{m} \mu(e_i) = 2\sum_{i=1}^{\frac{n(n-1)}{2}} \mu(e_i)$$

where $\mu(e_i) = 0, \forall i > m$.

We have $\mu(e_i) = \min\{\sigma(v_i), \sigma(v_j)\}$ for some $v_i, v_j \in V$. Hence,

$$\begin{aligned} E(G) &\leq 2\sum_{i=1}^{\frac{n(n-1)}{2}} \mu(e_i) \\ &= \sum_{i=1}^{\frac{n(n-1)}{2}} \mu(e_i) + \mu(e_i) \\ &\leq \sum_{1\leq i<j\leq n} \sigma(v_i) + \sigma(v_j) \\ &= (n-1)\sum_{i=1}^{n} \sigma(v_i) \end{aligned}$$

□

Proposition 3.5. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph with $|V| = n$, G^* a cycle and $\mu^* = \{e_1, \ldots, e_n\}$. Then,*

$$E(G) \leq 2\sum_{i=1}^{n} \sigma(v_i)$$

where $v_i \in V, i = 1, \ldots, n$.

Proof. From Theorem 3.3,

$$E(G) \leq 2\sum_{i=1}^{n} \mu(e_i) \tag{3.1}$$

$$e_i = \begin{cases} v_i v_{i+1}, & i = 1, \ldots, n-1 \\ v_n v_1, & i = n \end{cases}$$

Each edge in $G = (V, \sigma, \mu)$ can be uniquely mapped to a vertex by a map, defined as

$$f(e_i) = v_i, i = 1, \ldots, n$$

Therefore, (3.1) can be expressed in terms of membership values of vertices of G as

$$E(G) \leq 2\sum_{i=1}^{n} \sigma(v_i)$$

since $\mu(e_i) \leq \sigma(v_i), i = 1, \ldots, n$ by definition of a fuzzy graph. □

Next we have a result giving a lower bound and an improved upper bound for energy of a fuzzy graph.

Theorem 3.6. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph with $|V| = n$ vertices and $\mu^* = \{e_1, \ldots, e_m\}$. If $m_i = \mu(e_i)$ is the strength of the relation associated with the i^{th} edge, then*

$$\sqrt{2\sum_{i=1}^{m} m_i^2 + n(n-1)|A|^{\frac{2}{n}}} \leq E(G) \leq \sqrt{2\left(\sum_{i=1}^{m} m_i^2\right) n}$$

Proof. Upper bound

Applying Cauchy Schwarz inequality to the n numbers $(1,\ldots,1)$ and $(|\lambda_1|,\ldots,|\lambda_n|)$,

$$\sum_{i=1}^{n}|\lambda_i| \leq \sqrt{n}\sqrt{\sum_{i=1}^{n}|\lambda_i|^2} \tag{3.2}$$

$$\left(\sum_{i=1}^{n}\lambda_i\right)^2 = \sum_{i=1}^{n}|\lambda_i|^2 + 2\sum_{1\leq i<j\leq n}\lambda_i\lambda_j \tag{3.3}$$

By comparison of coefficients of

$$\prod_{i=1}^{n}(\lambda-\lambda_i) = |A-\lambda I|$$

We get

$$\sum_{1\leq i<j\leq n}\lambda_i\lambda_j = -\sum_{i=1}^{m}m_i^2$$

Substituting in (3.3),

$$\sum_{i=1}^{n}|\lambda_i|^2 = 2\sum_{i=1}^{m}m_i^2 \tag{3.4}$$

Substituting (3.4) in (3.2),

$$\begin{aligned}\sum_{i=1}^{n}|\lambda_i| \leq \sqrt{n}\sqrt{2\sum_{i=1}^{m}m_i^2} &= \sqrt{2\left(\sum_{i=1}^{m}m_i^2\right)n}\\ E(G) &\leq \sqrt{2\left(\sum_{i=1}^{m}m_i^2\right)n}\end{aligned}$$

Lower bound

$$\begin{aligned}[E(G)]^2 &= \left(\sum_{i=1}^{n}|\lambda_i|\right)^2\\ &= \sum_{i=1}^{n}|\lambda_i|^2 + 2\sum_{1\leq i<j\leq n}|\lambda_i\lambda_j|\\ &= 2\sum_{i=1}^{m}m_i^2 + 2\frac{n(n-1)}{2}AM\{|\lambda_i\lambda_j|\}\\ AM\{|\lambda_i\lambda_j|\} &\geq GM\{|\lambda_i\lambda_j|\},\ 1\leq i<j\leq n\\ E(G) &\geq \sqrt{2\sum_{i=1}^{m}m_i^2 + n(n-1)GM(|\lambda_i\lambda_j|}\end{aligned}$$

$$
\begin{aligned}
GM\{|\lambda_i\lambda_j|\} &= \left(\prod_{1\le i<j\le n}|\lambda_i\lambda_j|\right)^{\frac{2}{n(n-1)}}\\
&= \left(\prod_{i=1}^{n}|\lambda_i|^{n-1}\right)^{\frac{2}{n(n-1)}}\\
&= \left(\prod_{i=1}^{n}|\lambda_i|\right)^{\frac{2}{n}} = |A|^{\frac{2}{n}}\\
E(G) &\ge \sqrt{2\sum_{i=1}^{m}m_i^2+n(n-1)|A|^{\frac{2}{n}}}
\end{aligned}
$$

Therefore, $\sqrt{2\sum_{i=1}^{m}m_i^2+n(n-1)|A|^{\frac{2}{n}}} \le E(G) \le \sqrt{2\left(\sum_{i=1}^{m}m_i^2\right)n}$ □

Example 3.7. (Illustration to Theorem 1) For the graph in Fig. 1, $E(G_1) = 2.7256$

Lower bound $= 2.3238$

Upper bound $= 3.3226$

We have $2.3238 < 2.7256 < 3.322$.

Now, we have another result giving an upper bound for energy of a fuzzy graph which has less number of vertices.

Theorem 3.8. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph $|V| = n$ and $\mu^* = \{e_1, \ldots, e_m\}$. Let A be its adjacency matrix. If $m_i = \mu(e_i)$ and $n \le 2\sum_{i=1}^{m}m_i^2$, then*

$$E(G) \le \frac{2\sum_{i=1}^{m}m_i^2}{n} + \sqrt{(n-1)\left\{2\sum_{i=1}^{m}m_i^2 - \left(\frac{2\sum_{i=1}^{m}m_i^2}{n}\right)^2\right\}}$$

Proof. If $A = [a_{ij}]_{n\times n}$ is a symmetric matrix with zero diagonal, then

$$\lambda_{max} \ge \frac{2\sum_{1\le i<j\le n}a_{ij}}{n}$$

where, λ_{max} is the maximum eigenvalue of A. If A is the adjacency matrix of G, then $\lambda_1 \ge (2\sum_{i=1}^{m}m_i)/n$ where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$.

$$\sum_{i=1}^{n}|\lambda_i|^2 = 2\sum_{i=1}^{m}m_i^2$$

$$\sum_{i=2}^{n}|\lambda_i|^2 = 2\sum_{i=1}^{m}m_i^2 - \lambda_1^2 \tag{3.5}$$

Using Cauchy Schwarz inequality,

$$E(G) - \lambda_1 = \sum_{i=2}^{n}|\lambda_i| \le \sqrt{(n-1)\sum_{i=2}^{n}|\lambda_i|^2} \tag{3.6}$$

Substituting (3.5) in (3.6),

$$E(G) - \lambda_1 \leq \sqrt{(n-1)\left(2\sum_{i=1}^{m} m_i^2 - \lambda_1^2\right)}$$

$$E(G) \leq \lambda_1 + \sqrt{(n-1)\left(2\sum_{i=1}^{m} m_i^2 - \lambda_1^2\right)}$$

$F(x) = x + \sqrt{(n-1)\left(2\sum_{i=1}^{m} m_i^2 - x^2\right)}$ is decreasing in the interval $\left(\sqrt{\frac{2\sum_{i=1}^{m} m_i^2}{n}}, \sqrt{2\sum_{i=1}^{m} m_i^2}\right]$. Since, $n \leq 2\sum_{i=1}^{m} m_i^2, 1 \leq (2\sum_{i=1}^{m} m_i^2)/n$. Therefore,

$$\sqrt{\frac{2\sum_{i=1}^{m} m_i^2}{n}} \leq \frac{2\sum_{i=1}^{m} m_i^2}{n} \leq \frac{2\sum_{i=1}^{m} m_i}{n} \leq \lambda_1 \leq \sqrt{2\sum_{i=1}^{m} m_i^2} \tag{3.7}$$

We have

$$E(G) \leq \lambda_1 + \sqrt{(n-1)\left(2\sum_{i=1}^{m} m_i^2 - \lambda_1^2\right)}$$

Applying (3.7), we get

$$E(G) \leq \frac{2\sum_{i=1}^{m} m_i^2}{n} + \sqrt{(n-1)\left\{2\sum_{i=1}^{m} m_i^2 - \left(\frac{2\sum_{i=1}^{m} m_i^2}{n}\right)^2\right\}}$$

□

Example 3.9. (Illustration to Theorem 2) The energy bound for the graph in Fig. 1 as per the previous theorem is 3.3075 while energy is 2.7256.

4. Conclusion

Adjacency matrix and energy for a fuzzy graph are defined. Some results on energy bounds for weighted and simple graphs are extended to fuzzy graphs. Further study on energy and the spectra of fuzzy graphs may reveal more analogous results of these kind and will be discussed in the forthcoming papers.

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Anjali Narayanan (anjalin0.0@gmail.com)
Department of Mathematics, National Institute of Technology, Calicut-673601, Kerala, India

sunil mathew (sm@nitc.ac.in)
Department of Mathematics, National Institute of Technology, Calicut-673601, Kerala, India