Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 6, No. 2, (September 2013), pp. 433–446
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

A method to obtain fuzzy Pareto set of fuzzy multi-criteria quadratic programming problems

Debdas Ghosh, Debjani Chakraborty

Received 19 October 2012; Revised 3 December 2012; Accepted 28 January 2013

Abstract. In this paper, an attempt has been made to solve fuzzy multi-criteria quadratic programming problems. A new fuzzy inequality relation, depending on the concept of 'same points' in fuzzy geometry, has been developed. Using proposed fuzzy inequality relation, fuzzy decision feasible region is being formed. Fuzzy criteria feasible region is then obtained through transformation of entire fuzzy decision feasible region by the vector criteria mapping. After defining fuzzy non-dominated points in the criterion space, a method has been proposed to capture entire fuzzy Pareto set of the problem.

2010 AMS Classification: 90C70, 90C20

Keywords: Fuzzy quadratic programming, Multi-criteria optimization, Same points, Fuzzy inequality.

Corresponding Author: D. Chakraborty (debjani@maths.iitkgp.ernet.in)

1. Introduction

In decision making problems, a major concern is that most of them involve multiple criteria which are usually conflicting in nature. Since innovation of Pareto set (1896), study on multiple-criteria optimization problems (MOPs) eventually involves analyzing trade-off between the criteria on a set of efficient solutions or on a set of satisficing solutions to decision maker (DM). In last few decades, there has been many classical methods—like weighted sum, ϵ-constraint, normal boundary intersection, normal constraint, direct search domain, etc.—to obtain Pareto set of MOPs. All these classical methods try to capture entire Pareto set of MOPs. However, classical methods are not enough to tackle all practical problems, since often real-world situations cannot be modeled precisely [15].

To deal with imprecise nature of multiple criteria decision making problems, fuzzy multi-criteria optimization problems (FMOPs) are being studied since 1970, after the seminal work by Bellman and Zadeh [4] on decision-making in a fuzzy environment. A particular area of FMOPs is fuzzy multi-criteria quadratic programming problems (FMQPPs) where fuzzy criteria have fuzzy quadratic expressions with an addition of fuzzy linear term. It may be noticed that there is a vast literature on formulation and solution procedures in fuzzy multi-criteria linear programming problems, but the studies on FMQPPs have been rather scarce until very recently. Though fuzzy quadratic programming problems (FQPPs) is studied by several authors [3, 5, 6, 7, 14, 16, 17, 18, 20, 23, 25], but fuzzy multi-criteria quadratic programming problems have not yet been focused extensively. Very few partial works on FQPPs may be obtained in [1, 2]. In the literature on solving FMQPPs, usually, DM end up with a conventional MOP to get a compromise solution or most preferable solution to DM.

In this paper, an attempt has been made to obtain fuzzy Pareto set of FMQPPs. On solving FMQPPs, here, first the fuzzy decision feasible region has been constructed under the concept of 'same points' [12] in fuzzy geometry. As, irrespective of nature of the problems whether crisp or fuzzy, usually final decision of any decision making problem has to be crisp, we have considered decision variables as crisp in our problem. To simplify our discussion in this introductory paper of our methodology on solving FMQPPs, we have also taken criteria as crisp. Under these considerations, fuzzy decision feasible region is transformed to criterion space through vector criteria mapping. As decision feasible region is fuzzy, criteria feasible region must also be fuzzy. A most promising and simple approach to make a bridge between fuzzy set and crisp set is the use of α-cut. In the proposed methodology, we will obtain entire fuzzy Pareto set of FMQPPs using α-cuts of criteria feasible region. Delineation of the presented work in this paper is as follows.

Preliminaries on multi-criteria quadratic programming problems (MQPPs) and fuzzy set theory are given in the next section. A simple technique to obtain Pareto set of MQPPs is studied in Section 3. Section 4 includes construction procedure of fuzzy decision and criteria feasible region under the concept of 'same points'. The Section 4 also provides definitions of fuzzy Pareto point and generalized fuzzy Pareto point in FMQPPs. A method to obtain fuzzy Pareto set of FMQPPs has also been given in Section 4. One numerical example, to illustrate proposed methodology, is presented in the Section 5. Section 6 includes conclusion and future work of the proposed study.

2. Preliminaries

In this section, necessary definitions and terminologies which are used throughout this paper are studied.

2.1. **Fuzzy set.**

Definition 2.1. (*Fuzzy set* [24]). Let X be a classical set of elements which must be evaluated with regard to a fuzzy statement. Then the set of order pairs

$$\widetilde{A} = \{(x,\, \mu(x|\widetilde{A})) : x \in X\}, \text{ where } \mu : X \to [0, 1],$$

is called a fuzzy set in X. The evaluation function $\mu(x|\widetilde{A})$ is called the membership function or the grade of membership of x in $\widetilde{A}$.

Definition 2.2. (*α-cut of a fuzzy set* [12]). For a fuzzy set $\widetilde{A}$ of $\mathbb{R}$, an α-cut of $\widetilde{A}$ is denoted by $\widetilde{A}(\alpha)$ and is defined by:

$$\widetilde{A}(\alpha) = \begin{cases} \{x : \mu(x|\widetilde{A}) \geq \alpha\} & \text{if } 0 < \alpha \leq 1 \\ closure\{x : \mu(x|\widetilde{A}) > 0\} & \text{if } \alpha = 0. \end{cases}$$

The set $\{x : \mu(x|\widetilde{A}) > 0\}$ is called support of the fuzzy set $\widetilde{A}$.

To represent the construction of membership function of a fuzzy set $\widetilde{A}$, the notation $\bigvee\{x : x \in \widetilde{A}(0)\}$ is frequently used, which means $\mu(x|\widetilde{A}) = \sup\{\alpha : x \in \widetilde{A}(\alpha)\}$.

Definition 2.3. (*Fuzzy number* [12]). A fuzzy set $\widetilde{A}$ of $\mathbb{R}$ is called a fuzzy number if its membership function μ has the following properties:

(i) $\mu(x|\widetilde{A})$ is upper semi-continuous,
(ii) $\mu(x|\widetilde{A}) = 0$, outside some interval $[a, d]$, and
(iii) there exist real numbers b and c, $a \leq b \leq c \leq d$ such that $\mu(x|\widetilde{A})$ is increasing on $[a, b]$, decreasing on $[c, d]$, and $\mu(x|\widetilde{A}) = 1$ for each x in $[b, c]$.

For $b = c$, letting $f(x) = \mu(x|\widetilde{A})$ for all x in $[a, b]$ and $g(x) = \mu(x|\widetilde{A})$ for all x in $[c, d]$, in this paper the notation $(a/c/d)_{fg}$ is used to represent above defined fuzzy number. In particular, if $f(x)$ and $g(x)$ are linear functions, then the fuzzy number is called as triangular fuzzy number and is denoted by $(a/c/d)$.

Definition 2.4. (*Same points* [12]). Let x and y be two numbers belong to supports of two continuous fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$ respectively. The numbers x and y are said to be same points with respect to $\widetilde{a}$ and $\widetilde{b}$ if:

(i) $\mu(x|\widetilde{a}) = \mu(y|\widetilde{b})$, and
(ii) $x \leq a$ and $y \leq b$, or $x \geq a$ and $y \geq b$, where a, b are mid points of $\widetilde{a}(1)$, $\widetilde{b}(1)$ respectively.

Now let us give a brief idea on MQPPs.

2.2. **Classical MQPPs.**

In mathematical notions, MQPPs are defined in the following way

$$\min_{x\in\mathcal{X}} f(x) = \left(\frac{1}{2}x^tQ_1x + c_1^tx, \frac{1}{2}x^tQ_2x + c_2^tx, \ldots, \frac{1}{2}x^tQ_kx + c_k^tx\right)^t, k \geq 2, \tag{2.1}$$

where $\mathcal{X} = \{x \in \mathbb{R}^n : Ax \leq b,\ l \leq x \leq u\}$ is the feasible set, A is an $m \times n$ matrix, b is an m-vector; each Q_i is an $n \times n$ matrix, each c_i is an n-vector ($i = 1, 2, \ldots, k$); the vectors l, $u \in (\mathbb{R} \cup \{\infty\})^n$ are respectively lower and upper bounds of the decision vector $x = (x_1, x_2, \ldots, x_n)^t$.

We denote the image of the decision feasible set $\mathcal{X}$ under the vector mapping f by $\mathcal{Y} := f(\mathcal{X})$. Therefore, $\mathcal{Y}$ is the feasible set in the criterion space. Let $f_i(x) = \frac{1}{2}x^tQ_ix + c_i^tx$, $i = 1, 2, \ldots, k$. If for each individual i in $\{1, 2, \ldots, k\}$, x_i^* is the point of global minima of the function f_i, the point $y_i^* := f(x_i^*) = f_i^*$ say (for

$i = 1, 2, ..., k$) in the criterion space is said to be an *anchor point*. Again, the point $f^* = (f_1^*, f_2^*, \ldots, f_k^*)^t$ is called as *ideal point* or *utopia point*. Without loss of generality, let us consider all the criteria are positive valued and their global minimum value is zero. Usually MQPPs may not qualify this consideration, and thus we will redefine $f(x)$ as $f(x) - f^*$. Therefore, under this consideration, the criteria feasible set $\mathcal{Y}$ must be a subset of non-negative hyper-octant of $\mathbb{R}^k$. Origin of $\mathbb{R}^k$ is the ideal point and anchor points corresponding to k-th criterion must lie on the plane perpendicular to the axis of f_k. As in general ideal point f^* is not attainable by f, thus the notion of Pareto optimality is being introduced as follows. The definitions of weak Pareto optimality is also given subsequently.

Definition of Pareto optimality depends on a dominance structure or componentwise order in the space $\mathbb{R}^k$. To represent dominance structure on $\mathbb{R}^k$, the following subsets are usually used. The non-negative orthant of $\mathbb{R}^k$ is represented by $\mathbb{R}^k_{\geqq} := \{y \in \mathbb{R}^k : y \geqq 0\}$. The notation $y \geqq 0$ implies $y_i \geq 0$ for each $i = 1, 2, ..., k$. The set $\mathbb{R}^k_{\geq}$ is defined by $\{y \in \mathbb{R}^k : y \geq 0\}$ where $y \geq 0$ means $y \geqq 0$ but $y \neq 0$. The notation $\mathbb{R}^k_{>} := \{y \in \mathbb{R}^k : y > 0\}$ indicates the positive orthant of $\mathbb{R}^k$. Here, $y > 0$ stands for $y_i > 0$ for each $i = 1, 2, ..., k$. For $\hat{x}$, $\bar{x}$ in $\mathcal{X}$, the vector $f(\bar{x})$ is said to be dominated by another vector $f(\hat{x})$ if $f(\hat{x}) \leq f(\bar{x})$.

Definition 2.5. (*Pareto optimality* [9]). A feasible point $\hat{x}$ in $\mathcal{X}$ is called efficient or Pareto optimal point, if there is no other x in $\mathcal{X}$ such that $f(x) \leq f(\hat{x})$. If $\hat{x}$ is efficient point, $f(\hat{x})$ is called non-dominated point. The set of all efficient points is denoted by $\mathcal{X}_E$ and the collection of all non-dominated points is denoted by $\mathcal{Y}_N$.

Definition 2.6. (*Weak Pareto optimality* [9]). A feasible point $\hat{x}$ in $\mathcal{X}$ is called weakly Pareto optimal if there is no x in $\mathcal{X}$ such that $f(x) < f(\hat{x})$. The point $\hat{y} = f(\hat{x})$ is then called weakly non-dominated.

In the following section, a classical method [11] to obtain entire Pareto set of a MQPP (2.1) is presented. Presented method may be appeared as a restriction of Pascoletti-Serafini scalarization [19] with $a = 0$ (the ideal point). In the literature, there are some other [8, 10, 13, 21, 22] restriction of Pascoletti-Serafini scalarization. However, we may note that presented method [11], in the next section, attempted to search Pareto points in each and every possible directions from the ideal point, and thus generated Pareto set obviously maintains an approximately uniform diversity throughout the Pareto surface. In contrast, other methods in [8, 10, 13, 21, 22] start from a reference point which is restricted to lie on a plane and then search Pareto points along the normal to the considered plane, i.e., those methods search Pareto points along a particular direction. Thus, unless the Pareto surface is approximately parallel to that plane, generated Pareto set by those methods trivially cannot maintain equal diversity over the entire Pareto set. Moreover, parameter restriction of these methods depends on criteria feasible set and they cannot work efficiently for more than two objectives [10, 22]. More importantly, through solution set of the presented technique [11] we can easily get positions of weak Pareto points, knee regions of the Pareto set and region of unbounded trade-offs of the objectives. This information may facilitate DM's final selection of solution. But no such information can be extracted from solution set of the above mentioned other existing methods in [8, 10, 13, 21, 22].

3. A method to obtain Pareto set in classical MQPPs

In this section we present a technique to obtain Pareto points of MQPP (2.1). The technique is confined under the following three noteworthy observations on Pareto optimality—

- a point $\hat{x}$ in $\mathcal{X}$ is a Pareto optimal point if and only if $f(\mathcal{X}) \cap \Big(f(\hat{x}) - \mathbb{R}^k_{\geqq} \Big) = \{f(\hat{x})\}$,
- a point $\hat{x}$ in $\mathcal{X}$ is weakly Pareto optimal if and only if $f(\mathcal{X}) \cap \Big(f(\hat{x}) - \mathbb{R}^k_{>} \Big) = \varnothing$ and
- sets of non-dominated and weakly non-dominated points must be subsets of the boundary of the criterion feasible region, $bd(\mathcal{Y})$.

Geometrically, first observation means that – if criterion feasible region and translated non-positive orthant $-\mathbb{R}^k_{\geqq}$ whose vertex is being shifted from origin to the point $f(\hat{x})$ have intersection only the single point $f(\hat{x})$, then $\hat{x}$ is a Pereto optimal solution. Thus, to get a Pareto optimal solution, we may translate the cone of non-positive orthant of the criterion space along a particular direction $\hat{\beta}$ in $\mathbb{R}^k_{\geqq}$ until this cone does not touch the criterion feasible region. Translation of the cone $-\mathbb{R}^k_{\geqq}$ along a particular direction $\hat{\beta}$ in $\mathbb{R}^k_{\geqq}$ means that the vertex of the cone is retained on the line $z\hat{\beta}$, for z in $\mathbb{R}$. Now if the cone $-\mathbb{R}^k_{\geqq}$ is being translated along $\hat{\beta}$ in $\mathbb{R}^k_{\geqq}$, then it can touch the boundary of the criterion feasible region $\mathcal{Y}$ in two possible ways: either the vertex of the cone touches first or one boundary plane of the cone touches first. If the first case happens, the point where the cone touches the criterion feasible region is certainly be a globally non-dominated point. If the latter case happens, it is possible in two different ways: touch portion is either a single point or a set of points. In the first subcase, the touch point is a Pareto optimal point. In the second subcase, it can be easily perceived that all the points except the extreme points of the touch portion are weakly Pareto optimal solutions.

Let us illustrate how the above said touch portion of $(z\hat{\beta} - \mathbb{R}^k_{\geqq})$ and $bd(\mathcal{Y})$, for a particular direction $\hat{\beta}$ in $\mathbb{R}^k_{\geqq}$, can be found. To demonstrate, let us take a graphical illustration for a bi-criteria optimization problem. Figure 1 portrays the criterion feasible region $\mathcal{Y} = f(\mathcal{X})$ for a generic bi-criteria problem and the cone $z\hat{\beta} - \mathbb{R}^2_{\geqq}$ for a specific value of $z = \overline{OA}$. Let us now consider the set $\big\{y : z\hat{\beta} \geqq f(x), y = f(x), x \in \mathcal{X}\big\}$, $z \in \mathbb{R}$. For each specific value of $z \in \mathbb{R}$, this set represents the intersecting region of $(z\hat{\beta} - \mathbb{R}^k_{\geqq})$ and $f(\mathcal{X})$. Now for generic z in $\mathbb{R}$ let us try to minimize the intersecting region between $(z\hat{\beta} - \mathbb{R}^k_{\geqq})$ and $f(\mathcal{X})$ by translating the cone $(z\hat{\beta} - \mathbb{R}^k_{\geqq})$ along $\hat{\beta}$ such a way that the cone does not leave $f(\mathcal{X})$. In the optimum situation if the intersection $(z\hat{\beta} - \mathbb{R}^k_{\geqq}) \bigcap f(\mathcal{X})$ contains only one point, then that singleton point indeed be a non-dominated point. We note that minimizing the intersecting region $(z\hat{\beta} - \mathbb{R}^k_{\geqq}) \bigcap f(\mathcal{X})$ eventually involves minimizing the value of z with the constraints

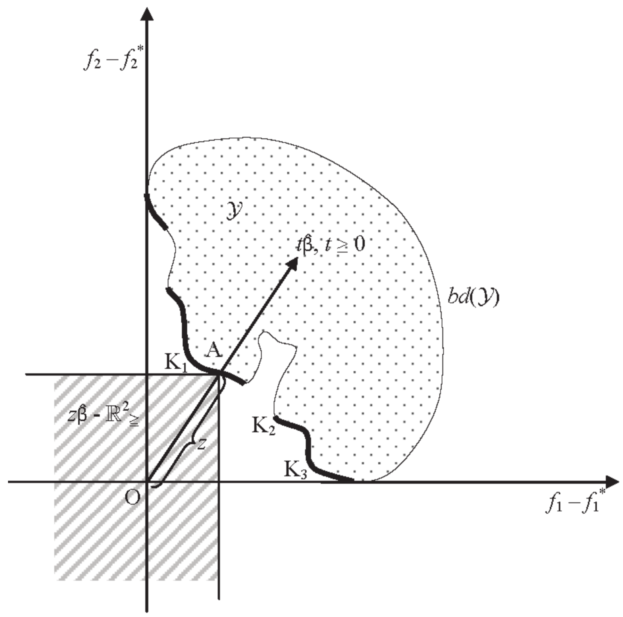

Figure 1. Illustration of CM($\hat{\beta}$) for a bi-criteria problem

$z\hat{\beta} \geqq f(x)$ and $x \in \mathcal{X}$. It is worthy to note that above discussions do not depend on the number of criteria. Therefore, to get a non-dominated solution of the MQPP (2.1) we may solve the following minimization problem:

$$\text{(3.1)} \qquad \text{CM}(\hat{\beta}) \begin{cases} \min & z \\ \text{subject to} & z\hat{\beta} \geqq f(x), \\ & x \in \mathcal{X}. \end{cases}$$

Solving (3.1) for various values of $\hat{\beta}$ in $\mathbb{R}^k_{\geqq} \cap \mathbb{S}^{k-1}$, whole non-dominated set (eventually weakly non-dominated set) of the considered MQPP can be generated; here $\mathbb{S}^{k-1}$ represents unit $(k-1)$-sphere. The abbreviation CM in (3.1) refers to Cone Method, since the closed convex cone $\mathbb{R}^n_{\geqq}$ facilitate us to capture Pareto set of MQPPs. It is to observe that any non-dominated point is attainable by above constructed minimization problem (3.1). For instance, if y_0 belongs to $\mathcal{Y}_N$, then solution of CM($\hat{\beta}$) corresponding to $\hat{\beta} = \frac{y_0}{||y_0||}$ is x_0 for which $y_0 = f(x_0)$. In the Figure 1 we note that solution of CM($\hat{\beta}$) corresponding to $\hat{\beta} = \frac{\overrightarrow{OA}}{||OA||}$ in $\mathbb{R}^2_{\geqq} \cap \mathbb{S}^1$ is the point A which is a Pareto optimal solution of the considered problem. Varying $\hat{\beta}$ for all possible values on $\mathbb{R}^2_{\geqq} \cap \mathbb{S}^1$, all the points in the dark portion of $bd(\mathcal{Y})$ can be obtained. Collection of all the points on the dark portion is the whole Pareto set/non-dominated set of the problem. Now a study on FMQPPs and finding their fuzzy Pareto set has been performed in the next section.

4. Solving FMQPPs

A general model of a fuzzy multi-criteria quadratic programming problem is described by the following system:

(4.1)
$$\min \quad f(x; \widetilde{Q}_1, \ldots, \widetilde{Q}_k; \widetilde{c}_1, \ldots, \widetilde{c}_k) = \Big(\frac{1}{2}x^t\widetilde{Q}_1x + \widetilde{c}_1^t x, \ldots, \frac{1}{2}x^t\widetilde{Q}_kx + \widetilde{c}_k^t x\Big)^t, \; k \geq 2$$

$$\text{subject to} \quad \widetilde{C}_i: \; \widetilde{a}_i^t\, x \;\widetilde{\leq}\; \widetilde{b}_i, \; i = 1, 2, \ldots, m,$$
$$x = (x_1, x_2, \ldots, x_n)^t \in \mathbb{R}^n_{\geqq},$$

where $\widetilde{Q}_j$ is a fuzzy matrix $\big(\widetilde{q}^{j}_{st}\big)_{n\times n}$, $\widetilde{c}_j = \big(\widetilde{c}_{j1}, \widetilde{c}_{j2}, \ldots, \widetilde{c}_{jn}\big)^t$ for each $j = 1, 2, \ldots, k$, $(s,\; t \in \{1, 2, \ldots, n\})$ and $\widetilde{a}_i = \big(\widetilde{a}_{i1}, \widetilde{a}_{i2}, \ldots, \widetilde{a}_{in}\big)^t$ for $i = 1, 2, \ldots, m$. Here, each of $\widetilde{c}_{jr}$, $\widetilde{a}_{il}$ are fuzzy sets $(r,\; l \in \{1, 2, \ldots, n\})$. In this paper, investigation has been made on FMQPPs where all $\widetilde{a}_{il}$, $\widetilde{b}_i$ fuzzy sets are fuzzy numbers and $\widetilde{c}_{jr}$, $\widetilde{q}^{j}_{st}$ are crisp numbers. Under these assumptions all the criteria are continuous crisp functions. It may be noted that fuzzy inequality $\widetilde{\leq}$ in each $\widetilde{C}_i$ eventually depends on ordering of two fuzzy numbers corresponding to each x in $\mathbb{R}^n_{\geqq}$.

Here we will take a new definition of fuzzy inequality $\widetilde{\leq}$ using the concept of 'same points' [12] as follows:

$$\widetilde{C}_i: \; \widetilde{a}_i^t\, x \;\widetilde{\leq}\; \widetilde{b}_i \iff \bigvee_{\alpha\in[0,1]} \Big\{x : a_{i\alpha}^t x \;\leq\; b_{i\alpha}\Big\} \tag{4.2}$$

where $b_{i\alpha}$, $a_{i\alpha} = \big(a_{i1\alpha}, a_{i2\alpha}, \ldots, a_{in\alpha}\big)^t$ are same points with respect to $\widetilde{b}_i$ and $\widetilde{a}_i = \big(\widetilde{a}_{i1}, \widetilde{a}_{i2}, \ldots, \widetilde{a}_{in}\big)^t$ respectively (co-ordinate wise). Therefore, entire fuzzy constraint set of FMQPP (4.1), $\widetilde{\mathcal{X}}$ say, can be represented by collection of crisp points x in $\mathbb{R}^n_{\geqq}$ with varied membership values as follows:

$$\widetilde{\mathcal{X}} = \bigcap_{i=1}^{m} \widetilde{C}_i \bigcap \mathbb{R}^n_{\geqq} = \bigcap_{i=1}^{m} \bigvee_{\alpha\in[0,1]} \Big\{x \in \mathbb{R}^n_{\geqq} : a_{i\alpha}^t x \;\leq\; b_{i\alpha}\Big\}. \tag{4.3}$$

Here on the decision space $\mathbb{R}^n$, the fuzzy set $\widetilde{\mathcal{X}}$ is fuzzy constraint of FMQPP (4.1). If we denote the image of the fuzzy constraint set $\widetilde{\mathcal{X}}$ under the vector mapping f by $\widetilde{\mathcal{Y}} := f(\widetilde{\mathcal{X}})$, then on the criterion space $\mathbb{R}^k$, constraint set of the optimization problem (4.1) will be $\widetilde{\mathcal{Y}}$. Geometrically, $\widetilde{\mathcal{X}}$ is collection of points on $\mathbb{R}^n$ with varied membership value and hence this collection of points will determine a fuzzy region on $\mathbb{R}^n$. This region is known as fuzzy feasible set/region of FMQPP (4.1) and each point on this set/region is known as feasible point. In this paper, since an attempt has been made to capture entire fuzzy Pareto set of FMQPP (4.1) through geometrical aspect of the problem, we may refer the fuzzy constraint $\widetilde{\mathcal{X}}$ as fuzzy feasible region. Similarly, $\widetilde{\mathcal{Y}}$ may be said as fuzzy feasible region on the criterion space $\mathbb{R}^k$.

In formation of fuzzy decision feasible region $\widetilde{\mathcal{X}}$, following result may be useful. We skip the proof which includes a straightforward calculation.

Theorem 4.1. *Let* $\widetilde{C}$: $\widetilde{a}_1 f_1(x) + \widetilde{a}_2 f_2(x) + \cdots + \widetilde{a}_p f_p(x) \lessapprox \widetilde{b}$, $x \in \mathbb{R}^n$ *be a fuzzy inequality, where* $\widetilde{a}_i = (a_i - \gamma_i/a_i/a_i + \delta_i)_{fg}$, $i = 1, 2, \ldots, p$ *and* $\widetilde{b} = (b - \gamma/b/b + \delta)_{fg}$. *If for each* i, $\frac{b-\gamma}{a_i-\gamma_i} < \frac{b}{a_i} < \frac{b+\delta}{a_i+\delta_i}$ *or* $\frac{b-\gamma}{a_i-\gamma_i} > \frac{b}{a_i} > \frac{b+\delta}{a_i+\delta_i}$ *and* $0 \notin \widetilde{a}_i(0), \widetilde{b}(0)$, *then*

(i) *the fuzzy set determined by* $\bigvee_{\alpha\in[0,1]}\{\frac{b_\alpha}{a_{i\alpha}}\}$, *where* $a_{i\alpha}$ *and* b_α *are same points with respect to* $\widetilde{a}_i$ *and* $\widetilde{b}$ *respectively, is a fuzzy number for each* $i = 1, 2, \ldots, p$, *and*

(ii) *the set representing* α*-cut of* $\widetilde{C} \bigcap \mathbb{R}^n_{\geqq}$, *for each* $\alpha \in [0, 1]$, *can be expressed by* $\left\{x \in \mathbb{R}^n_{\geqq} : \frac{f_1(x)}{\frac{b_\alpha}{a_{1\alpha}}} + \frac{f_2(x)}{\frac{b_\alpha}{a_{2\alpha}}} + \cdots + \frac{f_p(x)}{\frac{b_\alpha}{a_{p\alpha}}} \leq 1\right\}$.

Now after the construction of fuzzy decision feasible region $\widetilde{\mathcal{X}}$, let us try to formulate fuzzy criteria feasible region of the FMQPP (4.1). Let us note that $\widetilde{\mathcal{X}}$ is the collection of crisp points x in $\mathbb{R}^n$ with varied membership values and the criteria are considered as crisp functions. Thus, if x belongs to $\mathbb{R}^n$ is a decision feasible point with membership value α on $\widetilde{\mathcal{X}}$, then $f(x)$ must be a criteria feasible point and, by the supremum composition of fuzzy set, membership value of $f(x)$ on the fuzzy criteria feasible region $\widetilde{\mathcal{Y}} = f(\widetilde{\mathcal{X}})$ must be at least α. Here supremum composition is taken, since $\widetilde{\mathcal{Y}}$ is the collection (union) of all the points $y = f(x)$ where x lies in $\widetilde{\mathcal{X}}$. Thus, $\widetilde{\mathcal{Y}} = \bigvee_{\alpha\in[0,1]}\{f(x) : x \in \widetilde{\mathcal{X}}(\alpha)\}$. Membership function of fuzzy criteria feasible region may be obtained by $\mu(y|\widetilde{\mathcal{Y}}) = \sup\{\alpha : y = f(x), \mu(x|\widetilde{\mathcal{X}}) = \alpha\}$.

Here due to continuity of each f_j we obtain $f(\widetilde{\mathcal{X}}(\alpha)) = f(\widetilde{\mathcal{X}})(\alpha) = \widetilde{\mathcal{Y}}(\alpha)$. Thus corresponding to each α in $[0, 1]$, defining a crisp MQPP, FMQPP$_\alpha$ say, as follows:

$$\text{FMQPP}_\alpha \begin{cases} \min \quad \left(\frac{1}{2}x^tQ_1x + c_1^tx, \ldots, \frac{1}{2}x^tQ_kx + c_k^tx\right)^T, \ k \geq 2 \\ \text{subject to } x \in \widetilde{\mathcal{X}}(\alpha), \end{cases} \tag{4.4}$$

we must get that fuzzy decision/criteria feasible region of FMQPP (4.1) is exactly equal to union (by supremum composition) of all decision/criteria feasible region of FMQPP$_\alpha$s.

Let us now give definition fuzzy non-dominated point or fuzzy Pareto optimality.

Definition 4.1. (*Fuzzy Pareto optimal point*). A fuzzy set $\widetilde{P}$ of $\widetilde{\mathcal{Y}}(0)$ is said to be a fuzzy Pareto optimal point of FMQPP (4.1) if:

(i) $\widetilde{P}$ is a normal fuzzy set, i.e., there exists p_0 in $\widetilde{P}$ such that $\mu(p_0|\widetilde{P}) = 1$,

(ii) $\mu(p|\widetilde{P})$ is upper semi-continuous, and

(iii) for any p in $\widetilde{P}$, there exists α in $[0, 1]$ such that p is a Pareto optimal point of FMQPP$_\alpha$.

Note 4.1. *Core of a fuzzy Pareto optimal point is a Pareto optimal point of FMQPP*$_1$.

Example 4.1. *In the Figure 2, a typical fuzzy criteria feasible region* $\widetilde{\mathcal{Y}} = f(\widetilde{\mathcal{X}})$ *is depicted. The fuzzy arc* $\widehat{AB}$, (*rectangle #1*) *is a fuzzy Pareto optimal point. Membership value of any point on the fuzzy arc* $\widehat{AB}$ *is same as that on the fuzzy set* $f(\widetilde{\mathcal{X}})$. *Similarly the fuzzy arc in the rectangle* #3 *is a Pareto optimal point.*

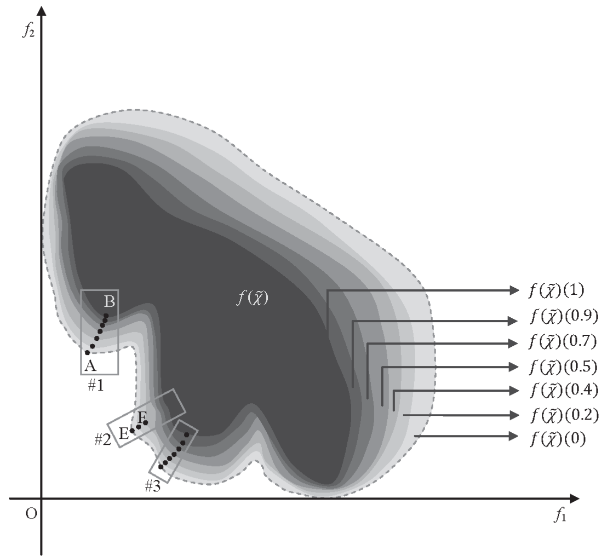

FIGURE 2. Explaining fuzzy Pareto point and generalized fuzzy Pareto point

However, we observe that the fuzzy arc $\widehat{EF}$, though meets (ii) and (iii) conditions of the Definition 4.1, but it does not meet the normality condition (i). Hence, fuzzy arc $\widehat{EF}$ is not a fuzzy Pareto point. We may say this type of fuzzy arc as generalized fuzzy Pareto optimal point. Mathematically generalized fuzzy Pareto point may be defined as follows.

Definition 4.2. (*Generalized fuzzy Pareto optimal point*). A fuzzy set $\widetilde{GP}$ of $\widetilde{\mathcal{Y}}(0)$ is said to be a generalized fuzzy Pareto optimal point of FMQPP (4.1) if:

- (i) $\mu(p|\widetilde{GP})$ is upper semi-continuous, and
- (ii) for any p in $\widetilde{GP}$, there exists α in $[0, 1]$ such that p is a Pareto optimal point of FMQPP$_\alpha$.

Note 4.2. *Here we note that a normal generalized fuzzy Pareto optimal point is a fuzzy Pareto optimal point. This concept is analogous to the concept of generalized fuzzy number and fuzzy number—generalized fuzzy number differs from fuzzy number in the condition of normality.*

Example 4.2. *In the Figure 2, the fuzzy arc $\widehat{EF}$ (rectangle #2) is a generalized fuzzy Pareto optimal point.*

Now to get whole fuzzy Pareto set or non-dominated set of a FMQPP, it is natural to take union of its all possible fuzzy Pareto points and generalized fuzzy Pareto points. However it can be easily perceived that if $\widetilde{\mathcal{Y}}_N$ is the set of all the Pareto points and $\widetilde{\mathcal{Y}}_{GN}$ is the set of all generalized fuzzy Pareto points, then $\widetilde{\mathcal{Y}}_{GN} \subseteq \widetilde{\mathcal{Y}}_N$ (for two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ in X, the relation $\widetilde{A} \subseteq \widetilde{B}$ holds when $\mu(x|\widetilde{A}) \leq \mu(x|\widetilde{B})$

for all x in X). Obviously, $\widetilde{\mathcal{X}}_{GE} = f^{-1}(\widetilde{\mathcal{Y}}_{GN}) \subseteq f^{-1}(\widetilde{\mathcal{Y}}_N) = \widetilde{\mathcal{X}}_E$. Thus, to find whole non-dominated points in a FMQPP, we only have to obtain $\widetilde{\mathcal{X}}_E$.

Again if $\mathcal{X}_{E\alpha}$ is the set of Pareto points of FMQPP$_\alpha$, then according to the mathematical formulation of FMQPP$_\alpha$, following result holds true

$$\widetilde{\mathcal{X}}_E = \bigvee_{\alpha \in [0,1]} \mathcal{X}_{E\alpha}. \tag{4.5}$$

Therefore, evaluating each $\mathcal{X}_{E\alpha}$, by solving FMQPP$_\alpha$ (using the classical method given in the Section 3) and then applying relation (4.5), compete fuzzy Pareto set of the FMQPP (4.1) can be obtained.

Note 4.3. *We note that some methods are already available to solve fuzzy multi-criteria decision making problems. Now question may arise: how the proposed technique is different from these available methods? In what way this is an improved version? To answer these questions, we note that all the existing methods in the literature give only a single solution of the problem with a measure of satisfaction of the DM* [3] *and do not attempt to capture complete set of fuzzy efficient solutions. On the other hand, proposed method intended to generate entire set of fuzzy efficient points and, as solutions of the existing methods are appeared to be a single Pareto optimum point, solutions of existing methods are subset of the complete fuzzy Pareto set obtained by our method.*

Note 4.4. *One another question may also arise that any quadratic programming problem can be made linear and thereafter it can be solved by the linear programming methods, then why do we need a new method to solve FMQPP? To address this question, we note that a single criteria optimization problem whose objective function is quadratic can be solved by linear programming method, but if we apply this method for multi-criteria quadratic optimization problem, then only a single Pareto optimum solution of the problem will be obtained. Owing to this fact, the technique studied in the Section 3 is applied in the proposed method to obtain complete Pareto set of MQPPs and the same technique is applied to obtain fuzzy Pareto set of FMQPPs.*

To illustrate the proposed method, one numerical example is given in the next section.

5. Numerical illustration

Example 5.1. *Let us consider the following fuzzy bi-criteria quadratic programming problem:*

$$\begin{aligned} &\min \begin{pmatrix} \frac{1}{4}(-3x_1^2 - 4x_2^2 + x_1 + 18x_2 + 14) \\ \frac{1}{5}(-3x_1^2 - 12x_2^2 - \frac{1}{2}x_1 - 4x_2 + 40) \end{pmatrix} \\ &\textit{subject to } \widetilde{C}_1 : (0.5/1/1.5)x_1 + (1/3/4)x_2 \precsim (1/3/6), \\ &\qquad \widetilde{C}_2 : (2/2.5/3)x_1 + (0.5/1/2)x_2 \precsim (2/2.5/6), \\ &\qquad x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

First let us determine the fuzzy decision feasible set $\widetilde{\mathcal{X}} = \widetilde{C}_1 \bigcap \widetilde{C}_2 \bigcap \mathbb{R}^2_{\geqq}$ using (4.3).

Fuzzy constraint set $\widetilde{C}_1$ is determined by (according to Equation 4.2)

$$\widetilde{C}_1 \equiv \bigvee_{\alpha\in[0,1]} \left\{x \in \mathbb{R}^2 : \frac{1+\alpha}{2}x_1+(1+2\alpha)x_2 \leq (1+2\alpha) \text{ or } \frac{3-\alpha}{2}x_1+(4-\alpha)x_2 \leq 3(2-\alpha)\right\}.$$

Supports of x_1 and x_2-intercept of $\widetilde{C}_1$ are $\bigcup_{\alpha\in[0,1]} \left\{\frac{2(1+2\alpha)}{1+\alpha}, \frac{6(2-\alpha)}{3-\alpha}\right\} = [2,4]$ and $\bigcup_{\alpha\in[0,1]} \left\{1, \frac{3(2-\alpha)}{4-\alpha}\right\} = [1, \frac{3}{2}]$ respectively.

Similarly, fuzzy constraint sets $\widetilde{C}_2$ is determined by (according to Equation 4.2)

$$\widetilde{C}_2 \equiv \bigvee_{\alpha\in[0,1]} \left\{x \in \mathbb{R}^2 : \frac{4+\alpha}{2}x_1+\frac{1+\alpha}{2}x_2 \leq \frac{4+\alpha}{2} \text{ or } \frac{6-\alpha}{2}x_1+(2-\alpha)x_2 \leq \frac{12-7\alpha}{2}\right\}.$$

Supports of x_1 and x_2-intercept of $\widetilde{C}_2$ are $\bigcup_{\alpha\in[0,1]} \left\{1, \frac{12-7\alpha}{6-\alpha}\right\} = [1,2]$ and $\bigcup_{\alpha\in[0,1]} \left\{\frac{4+\alpha}{1+\alpha}, \frac{12-7\alpha}{2(2-\alpha)}\right\} = [\frac{5}{2}, 3]$ respectively.

Entire decision feasible region $\widetilde{\mathcal{X}} = \widetilde{C}_1 \bigcap \widetilde{C}_2 \bigcap \mathbb{R}^2_{\geqq}$ has been displayed in the Figure 3. For each α in $[0,1]$, the α-cut of the decision constraint set, i.e., $\widetilde{\mathcal{X}}(\alpha)$ is the set

$$\left\{x \in \mathbb{R}^2_{\geqq} : \frac{3-\alpha}{2}x_1+(4-\alpha)x_2 \leq 3(2-\alpha)\right\} \bigcap \left\{x \in \mathbb{R}^2_{\geqq} : \frac{4+\alpha}{2}x_1+\frac{1+\alpha}{2}x_2 \leq \frac{4+\alpha}{2}\right\}.$$

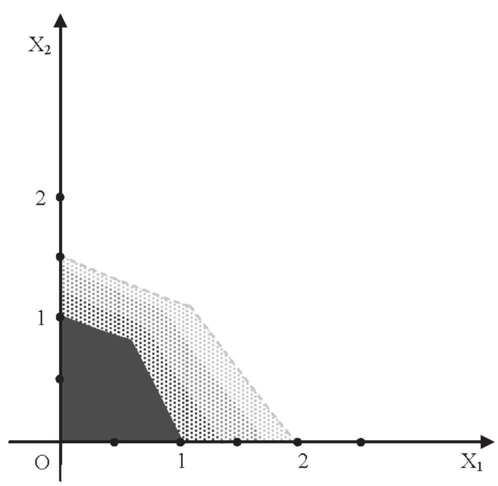

FIGURE 3. Fuzzy decision feasible region $\widetilde{\mathcal{X}}$ of the Example 5.1

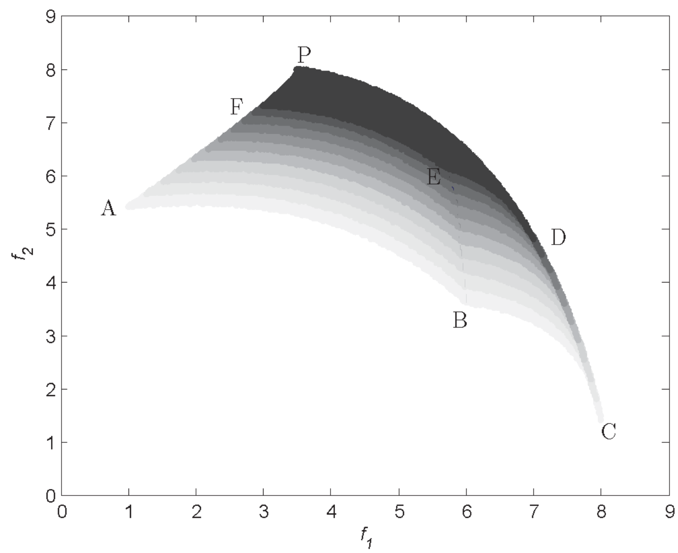

FIGURE 4. Fuzzy criteria feasible region $\widetilde{\mathcal{Y}}$ of the Example 5.1

Therefore, according to the formulation of FMQPP_α (Problem 4.4), we get

$$\text{FMQPP}_\alpha \quad \begin{cases} \qquad \min \begin{pmatrix} \frac{1}{4}(-3x_1^2 - 4x_2^2 + x_1 + 18x_2 + 14) \\ \frac{1}{5}(-3x_1^2 - 12x_2^2 - \frac{1}{2}x_1 - 4x_2 + 40) \end{pmatrix} \\ \text{subject to } \dfrac{3-\alpha}{2}x_1 + (4-\alpha)x_2 \le 3(2-\alpha), \\ \qquad \dfrac{4+\alpha}{2}x_1 + \dfrac{1+\alpha}{2}x_2 \le \dfrac{4+\alpha}{2}, \\ \qquad x_1 \ge 0, x_2 \ge 0. \end{cases}$$

Applying presented classical method in the Section 3, Pareto set $\mathcal{X}_{E\alpha}$ of FMQPP_α will be obtained. Taking union, by supremum composition, of all the Pareto set $\mathcal{X}_{E\alpha}$ (for all α in $[0,1]$) we obtain the fuzzy Pareto set $\widetilde{\mathcal{X}}_E$. Image of the set $\widetilde{\mathcal{X}}_E$ by the vector mapping f is the fuzzy non-dominated set $\widetilde{\mathcal{Y}}_N$. The fuzzy criteria feasible region $\widetilde{\mathcal{Y}}$ and the fuzzy non-dominated set $\widetilde{\mathcal{Y}}_N$ are shown in the Figure 4. $\widetilde{\mathcal{Y}}_N$ is the interior and boundary of the region bounded by $ABCDEFA$ on the Figure 4. Its core is the arc $DE \bigcup EF$. Coordinates of the points A, B, C, D, E and F are $(1, 5.4)$, $(6.006, 3.606)$, $(8, 1.4)$, $(7, 4.8)$, $(5.762, 6.122)$ and $(3.012, 7.313)$ respectively.

6. Conclusion

In this paper, we have studied a method to obtain fuzzy Pareto set in FMQPPs. The technique essentially depends on a classical method to capture Pareto set of MQPPs. By the classical method, first we obtain Pareto solutions of each FMQPP_α, i.e., the set $\mathcal{X}_{E\alpha}$, for each α in $[0,1]$. Then taking union, by the supremum composition, of all $\mathcal{X}_{E\alpha}$ sets, the method has obtained entire fuzzy Pareto set of FMOQPPs.

As number of points in the fuzzy Pareto set may be huge, it may be very difficult to DM to pick best solution(s) out of this large set of alternatives in the support of fuzzy Pareto set. The selection would become more difficult for large number of fuzzy criteria. A proper mathematical construction of DM's preferences while dealing with large number of imprecise criteria and a huge set of imprecise alternatives seems to be really complex. In this situation, the fuzzy knees of the fuzzy Pareto optimal set are likely to be more relevant to the DM. Thus, finding fuzzy knees may reduce the final selection procedure on a smaller number of potentially more relevant solutions on fuzzy Pareto set. Future research work may be focused on this topic.

In this introductory work on our methodology to solve FMQPPs, the proposed study has been made on the FMQPPs where decision variables and criteria are crisp. Investigation on more generalized FMOPs may be obtained in our future research.

Acknowledgements. Authors are grateful to editors-in-chief, managing editors and anonymous reviewers for their constructive comments and valuable suggestions. First author gratefully acknowledges a research scholarship awarded by the Council of Scientific & Industrial Research, Government of India (award no. 09/081(1054) /2010-EMR-I). Second author acknowledges financial support given by the Department of Science & Technology, Government of India (SR/S4/M:497/07).

References

[1] E. E. Ammar, On solutions of fuzzy random multiobjective quadratic programming with applications in portfolio problem, Inform. Sci. 178(2) (2008) 468–484.

[2] E. E. Ammar, On fuzzy random multiobjective quadratic programming, European J. Oper. Res. 193(2) (2009) 329–341.

[3] C. R. Bector and S. Chandra, Fuzzy mathematical programming and fuzzy matrix games, STUDFUZZ, Vol. 169, Springer-Verlag, Berlin, Heidelberg, 2005.

[4] R. E. Bellman and L. A. Zadeh, Decision-making in a fuzzy environment, Management Sci. 17(4) (1970) B141–B164.

[5] C. Cruz, R. C. Silva, J. L. Verdegay and A. Yamakami, A survey of fuzzy quadratic programming, Recent Patents on Comp. Sci. 1(3) (2008) 182–1930.

[6] C. Cruz, R. C. Silva and J. L. Verdegay, Extending and relating different approaches for solving fuzzy quadratic problems, Fuzzy Optim. Decis. Mak. 10 (2011) 193–210.

[7] C. Cruz, R. C. Silva and J. L. Verdegay, Solving real-world fuzzy quadratic programming problems by a parametric method, in: S. Greco et al. (Eds.): IPMU 2012, Part III, CCIS 299, pp. 102–111, Springer-Verlag, Berlin, Heidelberg, 2012.

[8] I. Das and J. E. Dennis, Normal-boundary intersection: A new method for generating the pareto surface in nonlinear multicriteria optimization problems, SIAM J. Optim. 8(3) (1998) 631–657.

[9] M. Ehrgott, Multicriteria optimization, Vol. 491, Springer-Verlag, Berlin, Heidelberg, 2005.

[10] G. Eichfelder, Adaptive scalarization methods in multiobjective optimization, Springer-Verlag, Berlin, Heidelberg, 2008.

[11] D. Ghosh and D. Chakraborty, Modified normal boundary interesection method for multiobjective optimization problems to obtain knee of the pareto frontier, in: 44th Annual Convention of ORSI, Operational Research Society of India, 2011.

[12] D. Ghosh and D. Chakraborty, Analytical fuzzy plane geometry I, Fuzzy Sets and Systems 209 (2012) 66–83.

[13] S. Helbig, An interactive algorithm for nonlinear vector optimization, Appl. Math. Optim. 22(2) (1990) 147–151.

[14] B. Kheirfam, A method for solving fully fuzzy quadratic programming problems, Acta Univ. Apulensis Math. Inform. 27 (2011) 69–76.

[15] Y. J. Lai and C. L. Hwang, Fuzzy multiple objective decision making: methods and applications, Lecture Notes in Economics and Mathematical Systems, Vol. 404, Springer, New York, 1994.

[16] S. T. Liu, Solving quadratic programming with fuzzy parameters based on extension principle, in: Fuzzy Systems Conference, FUZZ-IEEE 2007, IEEE International, pp. 1–5, IEEE, 2007.

[17] S. T. Liu, Quadratic programming with fuzzy parameters: A membership function approach, Chaos Solitons Fractals 40(1) (2009) 237–245.

[18] S. T. Liu, A revisit to quadratic programming with fuzzy parameters, Chaos Solitons Fractals 41(3) (2009) 1401–1407.

[19] A. Pascoletti and P. Serafini, Scalarizing vector optimization problems, J. Optim. Theory Appl. 42(4) (1984) 499–524.

[20] R. C. Silva, J. L. Verdegay and A. Yamakami, Two-phase method to solve fuzzy quadratic programming problems, in: Fuzzy Systems Conference, FUZZ-IEEE 2007, IEEE International, pp. 1–6. IEEE, 2007.

[21] P. K. Shukla, On the normal boundary intersection method for generation of efficient front, in: Y. Shi et al. (Eds.): ICCS 2007, Part I, LNCS 4487, pp. 310–17, Springer, Berlin, Heidelberg, 2007.

[22] S. Siddiqui, S. Azarm and S. A. Gabriel, On improving normal boundary intersection method for generation of Pareto frontier, Struct. Multidisc. Optim. 46 (2012) 839–852.

[23] J. Tang and D. Wang, An interactive approach based on a genetic algorithm for a type of quadratic programming problems with fuzzy objective and resources, Comput. Oper. Res. 24(5) (1997) 413–422.

[24] L. A. Zadeh, Fuzzy sets, Information and Control 8(3) (1965) 338–353.

[25] C. Zhang, X. Yuan and E. S. Lee, Duality theory in fuzzy mathematical programming problems with fuzzy coefficients, Comput. Math. Appl. 49 (2005) 1709–1730.

Debdas Ghosh (debdas.email@gmail.com)
Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur, West Bengal, 721302, India.

Debjani Chakraborty (debjani@maths.iitkgp.ernet.in)
Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur, West Bengal, 721302, India.