Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 6, No. 2, (September 2013), pp. 391–400
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

General fuzzy soft groups and fuzzy normal soft groups

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Received 3 December 2012; Accepted 7 February 2013

Abstract. Based on the analysis of soft sets and fuzzy sets, we introduce the concepts of $(\in, \in \vee q_k)$ fuzzy soft groups and $(\in, \in \vee q_k)$ fuzzy normal soft groups, and study their properties. Then we investigate the properties of their images under fuzzy soft homomorphism. Moreover, we introduce the concepts of $(\in, \in \vee q_k)$ fuzzy left(right) soft cosets and $(\in, \in \vee q_k)$ fuzzy quotient soft group.

2010 AMS Classification: 06D72, 20N25

Keywords: $(\in, \in \vee q_k)$ fuzzy soft group, $(\in, \in \vee q_k)$ fuzzy normal soft group, $(\in, \in \vee q_k)$ fuzzy quotient soft group.

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1. Introduction

To solve uncertainties in many areas such as economics, engineering, environment, sociology and medical science, we cannot successfully use classical methods. In 1965, Zadeh [15] proposed the theory of fuzzy set as a new mathematical tool for dealing with uncertainty. In 1971, Rosenfeld introduced the concept of fuzzy subgroups [12]. Next, Bhakat and Das [4] introduced a new type of fuzzy subgroups, that is, the $(\in, \in \vee q)$ fuzzy subgroups. In 2011, Young Baejun et al. [6] introduced the notion of $(\in, \in \vee q_k)$ fuzzy subgroup. In fact, the $(\in, \in \vee q_k)$ fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. Although other useful approaches such as probability theory, rough set theory and vague set theory are dedicated to modeling uncertain data, each of these theories has its own difficulties. To deal with fuzzy information and uncertainties more effectively, Molodtsov [11] introduced the concept of soft sets in 1999. Later, Maji et al.[9] further studied the theory of soft sets. The application of fuzzy soft set theory to a decision making problem were described [8, 13]. They also studied several operations on the theory of soft sets. In 2001, Maji et al. [10] introduced the concept of fuzzy soft sets. In 2007, Aktaş and Çağman [1] introduced the notion of soft groups. Next, A. Aygünoğlu and H. Aygün [3] introduced the concept of fuzzy soft groups. K. Kaygisiz [7] studied soft int-groups. In 2009, Ali et al. [2] gave some new notions of soft sets. In 2008, F. Feng et al. [5] introduced soft semirings.

In this paper, we first study some preliminary results of fuzzy sets and soft sets. Then we introduce the notions of $(\in, \in \vee q_k)$ fuzzy soft groups and $(\in, \in \vee q_k)$ fuzzy normal soft groups, and discuss their related properties. We also study the relation between $(\in, \in \vee q_k)$ fuzzy soft group and $(\in, \in \vee q_k)$ fuzzy normal soft group. Moreover, we introduce the concept of $(\in, \in \vee q_k)$ fuzzy left (right) soft cosets, then we introduce the notion of $(\in, \in \vee q_k)$ fuzzy quotient soft groups.

2. Preliminaries

Definition 2.1 ([15]). Let X be a nonempty set and I be the unit interval $[0, 1]$. The map $\lambda : X \to I$ is called a fuzzy subset of X.

Definition 2.2 ([15]). A fuzzy subset λ of X of the form

$$\lambda(y) = \begin{cases} t, & y = x, \\ 0 & y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

Definition 2.3 ([4, 6]). A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fuzzy subset λ, written as $x_t \in \lambda$ (resp. or $x_t q \lambda$) if $\lambda(x) \geq t$ (resp. $\lambda(x) + t \geq 1$). If $\lambda(x) + t + k > 1$, we write $x_t q_k \lambda$. If $x_t \in \lambda$ or $x_t q \lambda$, then we write $x_t \in \vee q \lambda$. If $x_t \in \lambda$ or $x_t q_k \lambda$, we write $x_t \in \vee q_k \lambda$.

Definition 2.4 ([15]). A fuzzy subset λ of group G is said to be a fuzzy subgroup of G if for any $x, y \in G$,

(1) $\lambda(xy) \geq \lambda(x) \wedge \lambda$ (y);
(2) $\lambda(x^{-1}) \geq \lambda(x)$.

Definition 2.5 ([4]). A fuzzy subset λ of group G is said to be an $(\in, \in \vee q)$ fuzzy subgroup of G if for any $x, y \in G$ and $t, r \in (0, 1]$,

(1) $x_t \in \lambda$, $y_r \in \lambda \Rightarrow (xy)_{t \wedge r} \in \vee q \lambda$;
(2) $x_t \in \lambda \Rightarrow (x^{-1})_t \in \vee q \lambda$.

Definition 2.6 ([4]). An $(\in, \in \vee q)$ fuzzy subgroup λ of G is said to be an $(\in, \in \vee q)$ fuzzy normal subgroup if for any $x, y \in G$ and $t \in (0, 1]$, $x_t \in \lambda \Rightarrow (y^{-1}xy)_t \in \vee q \lambda$.

Definition 2.7 ([6]). A fuzzy subset λ of group G is said to be an $(\in, \in \vee q_k)$ fuzzy subgroup of G if for any $x, y \in G$ and $t, r \in (0, 1]$, $k \in [0, 1)$,

(1) $x_t \in \lambda$, $y_r \in \lambda \Rightarrow (xy)_{t \wedge r} \in \vee q_k \lambda$;
(2) $x_t \in \lambda \Rightarrow (x^{-1})_t \in \vee q_k \lambda$.

Definition 2.8 ([6]). An $(\in, \in \vee q_k)$ fuzzy subgroup λ of G is said to be an $(\in, \in \vee q_k)$ fuzzy normal subgroup if for any $x, y \in G$ and $t \in (0, 1]$, $x_t \in \lambda \Rightarrow (y^{-1}xy)_t \in \vee q_k \lambda$.

In the following, we recall some basic notions in soft set theory. Let U be an initial universe set, E be a set of parameters, $P(U)$ be the power set of U, and $\varnothing \neq A \subseteq E$.

Definition 2.9 ([11])**.** A pair (F, A) is called a soft set over U where F is a mapping given by $F : A \to P(U)$.

Definition 2.10 ([1])**.** Let (F, A) be a soft set over G. Then (F, A) is called a soft group over G if $F(\alpha)$ is a subgroup of G for all $\alpha \in A$.

Definition 2.11 ([14])**.** Let (F, A) be a soft set over G. Then (F, A) is called a normal soft group over G if $F(\alpha)$ is a normal subgroup of G for all $\alpha \in A$.

Definition 2.12 ([10])**.** A pair (F, A) is called a fuzzy soft set over U, where $F : A \to I^U$ is a mapping, $I = [0, 1]$, $F(\alpha)$ is a fuzzy subset of U for all $\alpha \in A$.

Definition 2.13 ([3])**.** Let (F, A) be a fuzzy soft set over G. Then (F, A) is called a fuzzy soft group if $F(\alpha)$ is a fuzzy subgroup of G for all $\alpha \in A$.

Definition 2.14 ([10])**.** Let (F, A) and (G, B) be two fuzzy soft sets over U. Then (F, A) is called a fuzzy soft subset of (G, B), denoted by $(F, A) \subseteq (G, B)$. if (1) $A \subseteq B$, (2) $F(\alpha)$ is a fuzzy subset of $G(\alpha)$ for each $\alpha \in A$.

Definition 2.15 ([2])**.** Let (F, A) and (H, B) be two fuzzy soft sets over a common universe U. The intersection of (F, A) and (H, B) is defined to be the fuzzy soft set (K, C), where $C = A \cap B$, and for each $c \in C$, $K(c) = F(c) \cap H(c)$. We write $(F, A) \cap (H, B) = (K, C)$

Definition 2.16 ([2])**.** Let (F, A) and (H, B) be two fuzzy soft sets over a common universe U. The union of (F, A) and (H, B) is defined to be the fuzzy soft set (K, C), where $C = A \cup B$, and for each $c \in C$,

$$K(c) = \begin{cases} F(c), & if\ c \in A \setminus B \\ H(c), & if\ c \in B \setminus A \\ F(c) \cup H(\varepsilon), & if\ c \in A \cap B \end{cases}$$

We write $(F, A) \cup (H, B) = (K, C)$

Definition 2.17 ([2])**.** Let (F, A) and (H, B) be two fuzzy soft sets over a common universe U. Then (F, A) AND (H, B) denoted by $(F, A) \vee (H, B)$ is defined as the fuzzy soft set (K, C), where $C = A \times B$, and for each $(\alpha, \beta) \in C$, $K(\alpha, \beta) = F(\alpha) \cap H(\beta)$.

Definition 2.18 ([3])**.** Let (F, A) be a fuzzy soft set over U. For each $t \in [0, 1]$, the soft set $(F, A)^t = (F^t, A)$ is called a t-level set of (F, A), where $F^t(\alpha) = \{x | F(\alpha)(x) \geq t, x \in G\}$ for each $\alpha \in A$.

3. $(\in, \in \vee q_k)$ fuzzy soft group

In this section, G denotes a group, E is a set of parameters, $A \subseteq E, A \neq \varnothing$, $k \in [0, 1)$, unless otherwise specified.

Definition 3.1. Let (F, A) be a fuzzy soft set over G. Then (F, A) is called an $(\in, \in \vee q_k)$ fuzzy soft group if $F(\alpha)$ is an $(\in, \in \vee q_k)$ fuzzy subgroup of G for all $\alpha \in A$.

Example 3.2. Let $Z/(3) = \{\bar{0}, \bar{1}, \bar{2}\}$ be a modulo 3 residue class group, $A = \{\alpha_1, \alpha_2\}$. Define a fuzzy soft set (F, A) over $< Z/(3), + >$ as : $F(\alpha_1)(\bar{0}) = 0.6$, $F(\alpha_1)(\bar{1}) = 0.8$, $F(\alpha_1)(\bar{2}) = 0.9$, $F(\alpha_2)(\bar{0}) = 0.4$, $F(\alpha_2)(\bar{1}) = 0.7$, $F(\alpha_2)(\bar{2}) = 0.9$. It is easy to verify that $F(\alpha_1), F(\alpha_2)$ are $(\in, \in \vee q_k)$ fuzzy subgroups of $< Z/(3), + >$. Therefore, (F, A) is an $(\in, \in \vee q_k)$ fuzzy soft group over $< Z/(3), + >$, where $t \in [0.2, 1)$.

Remark 3.3. If λ is a fuzzy subgroup of G, then λ is an $(\in, \in \vee q_k)$ fuzzy subgroup of G. Therefore, a fuzzy soft group over G must be an $(\in, \in \vee q_k)$ fuzzy soft group over G.

Example 3.4. In Example 3.2, since $\bar{1}_{0.5} \in F(\alpha_2)$, $\bar{2}_{0.5} \in F(\alpha_2)$, $F(\alpha_2)(\bar{1} + \bar{2}) = F(\alpha_2)(\bar{0}) = 0.4$. Thus $(\in, \in \vee q_k)$ fuzzy soft group (F, A) is not an $(\in, \in \vee q)$ fuzzy soft group over group $< Z/(3), + >$. Actually, $(\in, \in \vee q_k)$ fuzzy soft group is the generalization of $(\in, \in \vee q)$ fuzzy soft group.

Remark 3.5 ([6])**.** Let (F, A) be an $(\in, \in \vee q_k)$ fuzzy soft group and e be the identity element of G. If $\forall \alpha \in A$, $F(\alpha)(e) < \frac{1-k}{2}$, then (F, A) is a fuzzy soft group over G.

Lemma 3.6 ([6])**.** *Let λ be a fuzzy set of G. Then λ is an $(\in, \in \vee q_k)$ fuzzy group if and only if $\lambda(x^{-1}y) \geq (\lambda(x) \wedge \lambda(y)) \wedge \frac{1-k}{2}$, $\forall x, y \in G$.*

Theorem 3.7. *Let (F, A) be a fuzzy soft set over G. Then (F, A) is an $(\in, \in \vee q_k)$ fuzzy soft group if and only if $(F, A)^t$ is a soft group over G, $\forall t \in (0, \frac{1-k}{2}]$.*

Proof. Let (F, A) be an $(\in, \in \vee q_k)$ fuzzy soft group over G. Let $\forall t \in (0, \frac{1-k}{2}]$, $\alpha \in A$ and $x_1, x_2 \in F^t(\alpha)$, that is $F(\alpha)(x_1) \geq t$, $F(\alpha)(x_2) \geq t$. Now by Definition 3.1, $F(\alpha)$ is an $(\in, \in \vee q_k)$ fuzzy subgroup. Hence $F(\alpha)(x_1^{-1}x_2) \geq F(\alpha)(x_1) \wedge F(\alpha)(x_2) \wedge \frac{1-k}{2} \geq t \wedge \frac{1-k}{2} = t$, then $x_1^{-1}x_2 \in F^t(\alpha)$, that is $F^t(\alpha)$ is a subgroup of G. So $\forall t \in (0, \frac{1-k}{2}]$, $(F, A)^t$ is a soft group over G.

Conversely, assume that $\forall t \in (0, \frac{1-k}{2}]$, $(F, A)^t$ is a soft group over G. $\forall \alpha \in A$, $x_1, x_2 \in G$, if $F(\alpha)(x_1) \wedge F(\alpha)(x_2) = 0$, then $F(\alpha)(x_1^{-1}x_2) \geq F(\alpha)(x_1) \wedge F(\alpha)(x_2) \wedge \frac{1-k}{2}$; if $F(\alpha)(x_1) \wedge F(\alpha)(x_2) > \frac{1-k}{2}$, then $x_1 \in F^{\frac{1-k}{2}}(\alpha)$, $x_2 \in F^{\frac{1-k}{2}}(\alpha)$, hence $x_1^{-1}x_2 \in F^{\frac{1-k}{2}}(\alpha)$, therefore $F(\alpha)(x_1^{-1}x_2) \geq \frac{1-k}{2} = F(\alpha)(x_1) \wedge F(\alpha)(x_2) \wedge \frac{1-k}{2}$; if $F(\alpha)(x_1) \wedge F(\alpha)(x_2) \in (0, \frac{1-k}{2}]$, let $t = F(\alpha)(x_1) \wedge F(\alpha)(x_2)$, then $x_1, x_2 \in F^t(\alpha)$, that is $F(\alpha)(x_1^{-1}x_2) \geq t = t \wedge \frac{1-k}{2} = F(\alpha)(x_1) \wedge F(\alpha)(x_2) \wedge \frac{1-k}{2}$. Hence $\forall \alpha \in A$, $F(\alpha)$ is an $(\in, \in \vee q_k)$ fuzzy subgroup of G. So (F, A) is an $(\in, \in \vee q_k)$ fuzzy soft group over G. □

Theorem 3.8. *Let (F, A) and (G, B) be two $(\in, \in \vee q_k)$ fuzzy soft groups over G. Then the following hold.*

(1) *$(F, A) \cap (H, B)$ is an $(\in, \in \vee q_k)$ fuzzy soft group over G if $A \cap B \neq \varnothing$.*
(2) *$(F, A) \cup (G, B)$ is an $(\in, \in \vee q_k)$ fuzzy soft group over G if $A \cap B = \varnothing$.*
(3) *$(F, A) \wedge (H, B)$ is an $(\in, \in \vee q_k)$ fuzzy soft group over G.*

Proof. (1) Let $(F, A) \cap (H, B) = (K, C)$. Now by Definition 2.15, $\forall c \in C$, $\forall x, y \in G$, $K(c)(x^{-1}y) = (F(c) \cap H(c))(x^{-1}y) = F(c)(x^{-1}y) \wedge H(c)(x^{-1}y) \geq F(c)(x) \wedge F(c)(y) \wedge \frac{1-k}{2} \wedge H(c)(x) \wedge H(c)(y) \wedge \frac{1-k}{2} = (F(c) \cap H(c))(x) \wedge (F(c) \cap H(c))(y) \wedge \frac{1-k}{2} =$

$K(c)(x) \wedge K(c)(y) \wedge \frac{1-k}{2}$. $\forall c \in C$, $K(c)$ is an $(\in, \in \vee q_k)$ fuzzy subgroup of G. So $(F, A) \cap (H, B)$ is an $(\in, \in \vee q_k)$ fuzzy soft group over G.

(2) Let $(F, A) \cup (H, B) = (K, C)$, and $A \cap B = \varnothing$. Now by Definition 2.16, $\forall c \in C$, $K(c) = F(c)$ or $K(c) = H(c)$. Since $F(c), H(c)$ are $(\in, \in \vee q_k)$ fuzzy subgroups of G, then $\forall c \in C$, $K(c)$ is an $(\in, \in \vee q_k)$ fuzzy subgroups of G. So $(F, A) \cup (G, B)$ is an $(\in, \in \vee q_k)$ fuzzy soft group over G.

(3) Let $(F, A) \wedge (H, B) = (K, C)$, where $C = A \times B$. Now by Definition 2.17, $\forall (\alpha, \beta) \in C$, $F(\alpha), H(\beta)$ are $(\in, \in \vee q_k)$ fuzzy subgroups of G. According to (1), $K(\alpha, \beta) = F(\alpha) \cap H(\beta)$ is an $(\in, \in \vee q_k)$ fuzzy subgroup of G. So $(F, A) \wedge (H, B)$ is an $(\in, \in \vee q_k)$ fuzzy soft group over G. □

Definition 3.9 ([5]). Let $(F_i, A_i)_{i \in I}$ be a nonempty family of soft sets over a common universe set U. The intersection of these soft sets is defined to be the soft set (H, B) such that $B = \bigcap_{i \in I} A_i \neq \varnothing$, and for all $\beta \in B$, $H(\beta) = \bigcap_{i \in I} F_i(\beta)$. We write $\bigcap_{i \in I} (F_i, A_i) = (H, B)$.

Similarly, the definitions of union and AND of $(F_i, A_i)_{i \in I}$ are given by [5].

Theorem 3.10. *Let $(F_i, A_i)_{i \in I}$ be a nonempty family of $(\in, \in \vee q_k)$ fuzzy soft groups over G. Then the following hold.*

(1) *$\bigcap_{i \in I} (F_i, A_i)$ is an $(\in, \in \vee q_k)$ fuzzy soft group over G if $\bigcap_{i \in I} A_i \neq \varnothing$.*
(2) *$\bigcup_{i \in I} (F_i, A_i)$ is an $(\in, \in \vee q_k)$ fuzzy soft group over G if $i, j \in I, i \neq j$, $A_i \cap A_j = \varnothing$.*
(3) *$\bigwedge_{i \in I} (F_i, A_i)$ is an $(\in, \in \vee q_k)$ fuzzy soft group over G.*

Proof. (1) Let $\bigcap_{i \in I} (F_i, A_i) = (H, B)$, and $B = \bigcap_{i \in I} A_i \neq \varnothing$. Now by Definition 3.9 and Theorem 3.8(1), $\forall \beta \in B$, $H(\beta) = \bigcap_{i \in I} F_i(\beta)$ is an $(\in, \in \vee q_k)$ fuzzy subgroup of G. So $\bigcap_{i \in I} (F_i, A_i)$ is an $(\in, \in \vee q_k)$ fuzzy soft group over G.

Similarly, one can easily prove (2) (3) by Theorem 3.8 and Definition 3.9. □

Definition 3.11. Let (F, A) and (H, B) be two $(\in, \in \vee q_k)$ fuzzy soft groups over G. Then (F, A) is called an $(\in, \in \vee q_k)$ fuzzy soft subgroup of (G, B), if (F, A) is a fuzzy soft subset of (G, B), denoted by $(F, A) \tilde{\subset} (G, B)$.

Example 3.12. Let $Z/(3) = \{\bar{0}, \bar{1}, \bar{2}\}$ be a modulo 3 residue class group, $A = \{\alpha_1, \alpha_2\}$. In Example 3.2, (F, A) is an $(\in, \in \vee q_k)$ fuzzy soft group over group $< Z/(3), + >$. Let $B = \{\alpha_1\}$. Define a fuzzy soft set (H, B) over $< Z/(3), + >$ as : $H(\alpha_1)(\bar{0}) = 0.35$, $H(\alpha_1)(\bar{1}) = 0.7$, $H(\alpha_1)(\bar{2}) = 0.2$. It is easy to verify that (H, B) is an $(\in, \in \vee q_k)$ fuzzy soft group over $< Z/(3), + >$, where $k = 0.3$. Since $(H, B) \subset (F, A)$, so $(H, B) \tilde{\subset} (F, A)$.

Theorem 3.13. *Let (F, A) be an $(\in, \in \vee q_k)$ fuzzy soft groups over G and $(F_i, A_i)_{i \in I}$ be a nonempty family of $(\in, \in \vee q_k)$ fuzzy soft subgroups of (F, A). Then the following hold.*

(1) *$\bigcap_{i \in I} (F_i, A_i)$ is an $(\in, \in \vee q_k)$ fuzzy soft subgroup of (F, A) if $\bigcap_{i \in I} A_i \neq \varnothing$.*
(2) *$\bigcup_{i \in I} (F_i, A_i)$ is an $(\in, \in \vee q_k)$ fuzzy soft subgroup of (F, A) if $i, j \in I, i \neq j$, $A_i \cap A_j = \varnothing$.*
(3) *$\bigwedge_{i \in I} (F_i, A_i)$ is an $(\in, \in \vee q_k)$ fuzzy soft subgroup of (F, A).*

Proof. The proof is obvious by Definition 3.11 and Theorem 3.10. □

Lemma 3.14. *Let $f : G_1 \to G_2$ be a group epimorphism. If λ is an $(\in, \in \vee q_k)$ fuzzy subgroup of G_1, then $f(\lambda)$ is an $(\in, \in \vee q_k)$ fuzzy subgroup of G_2, where $f(\lambda)(y) = \sup\{\lambda(x)|f(x) = y, x \in G_1\}$.*

Proof. Let $y_1, y_2 \in G_2$, then $f(\lambda)(y_1^{-1}y_2) = \sup\{\lambda(x_1^{-1}x_2)|f(x_1^{-1}x_2) = y_1^{-1}y_2\} \geq \sup\{\lambda(x_1) \wedge \lambda(x_2) \wedge \frac{1-k}{2}|f(x_1) = y_1, f(x_2) = y_2\} = \sup\{\lambda(x_1)|f(x_1) = y_1\} \wedge \sup\{\lambda(x_2)|f(x_2) = y_2\} \wedge \frac{1-k}{2} = f(\lambda)(y_1) \wedge f(\lambda)(y_2) \wedge \frac{1-k}{2}$. So $f(\lambda)$ is an $(\in, \in \vee q_k)$ fuzzy subgroup of G_2. □

Definition 3.15. Let (F, A) and (H, B) be fuzzy soft sets over G_1 and G_2, respectively. Then a pair (f, g) is called a fuzzy soft homomorphism if

(1) f is a group homomorphism from G_1 to G_2;
(2) g is surjective from A to B;
(3) $f(F(\alpha)) = H(g(\alpha))$ for all $\alpha \in A$.

Then we say (F, A) is fuzzy soft homomorphic to (H, B) under the fuzzy soft homomorphism (f, g).

Theorem 3.16. *Let (F, A) and (G, B) be fuzzy soft sets over G_1 and G_2, respectively. Let (F, A) be fuzzy soft homomorphic to (H, B) under the fuzzy soft homomorphism (f, g). If (F, A) is an $(\in, \in \vee q_k)$ fuzzy soft group over G_1, then (H, B) is an $(\in, \in \vee q_k)$ fuzzy soft group over G_2.*

Proof. According to Definition 3.15, (F, A) is fuzzy soft homomorphic to (H, B) under the fuzzy soft homomorphism (f, g), that is $\forall \alpha \in A$, $f(F(\alpha)) = H(g(\alpha))$. Since g is surjective, then $\forall \beta \in B$, $\exists \alpha \in A$, satisfied $g(\alpha) = \beta$. Therefore $H(\beta) = H(g(\alpha)) = f(F(\alpha))$. By Lemma 3.14, $f(F(\alpha))$ is an $(\in, \in \vee q_k)$ fuzzy subgroup of G_2, that is $H(\beta)$ is an $(\in, \in \vee q_k)$ fuzzy subgroup of G_2. According to Definition 3.1, (H, B) is an $(\in, \in \vee q_k)$ fuzzy soft group over G_2. □

4. $(\in, \in \vee q_k)$ FUZZY NORMAL SOFT GROUP

Definition 4.1. Let (F, A) be an $(\in, \in \vee q_k)$ fuzzy soft group over G. Then (F, A) is called an $(\in, \in \vee q_k)$ fuzzy normal soft group if $F(\alpha)$ is an $(\in, \in \vee q_k)$ fuzzy normal subgroup of G for all $\alpha \in A$.

Example 4.2. Let $A_3 = \{1, (123), (132)\}$ be an alternating group of S_3, $A = \{\alpha_1, \alpha_2\}$. Define a fuzzy soft set (F, A) over A_3 as: $F(\alpha_1)(1) = 0.4$, $F(\alpha_1)((123)) = 0.5$, $F(\alpha_1)((132)) = 0.8$, $F(\alpha_2)(1) = 0.8$, $F(\alpha_2)((123)) = 0.6$, $F(\alpha_2)((132)) = 0.7$. It is easy to verify that $F(\alpha_1), F(\alpha_2)$ are $(\in, \in \vee q_k)$ fuzzy normal subgroups of A_3. Therefore, (F, A) is an $(\in, \in \vee q_k)$ fuzzy normal soft group over A_3, where $k \in [0.2, 1)$.

Lemma 4.3 ([6])**.** *Let λ be an $(\in, \in \vee q_k)$ fuzzy subgroup of G. Then λ is an $(\in, \in \vee q_k)$ fuzzy normal subgroup if and only if $\lambda(y^{-1}xy) \geq \lambda(x) \wedge \frac{1-k}{2}$, $\forall x, y \in G$.*

Theorem 4.4. *Let (F, A) be a fuzzy soft set over G. Then (F, A) is an $(\in, \in \vee q_k)$ fuzzy normal soft group if and only if $(F, A)^t$ is a normal soft group over G, $\forall t \in (0, \frac{1-k}{2}]$.*

Proof. Let (F, A) be an $(\in, \in \vee q_k)$ fuzzy normal soft group over G. Now by Theorem 3.7, $(F, A)^t$ is a soft group over G. Since $\forall x \in F^t(\alpha), y \in G$, $F(\alpha)(y^{-1}xy) \geq$

$F(\alpha)(x) \wedge \frac{1-k}{2} \geq t \wedge \frac{1-k}{2} = t$, then $y^{-1}xy \in F^t(\alpha)$. So $\forall t \in (0, \frac{1-k}{2}], (F, A)^t$ is a normal soft group over G.

Conversely, Let $\forall t \in (0, \frac{1-k}{2}]$, $(F, A)^t$ is a normal soft group over G. Now by Theorem 3.7, (F, A) is an $(\in, \in \vee q_k)$ fuzzy soft group over G. $\forall x, y \in G$, let $F(\alpha)(x) = t$, $\forall \alpha \in A$. If $t = 0$, then $F(\alpha)(y^{-1}xy) \geq 0 = F(\alpha)(x) \wedge \frac{1-k}{2}$. If $t \geq \frac{1-k}{2}$, then $x \in F^{\frac{1-k}{2}}(\alpha)$, therefore $y^{-1}xy \in F^{\frac{1-k}{2}}(\alpha)$, that is $F(\alpha)(y^{-1}xy) \geq \frac{1-k}{2} = F(\alpha)(x) \wedge \frac{1-k}{2}$. If $t \in (0, \frac{1-k}{2}]$, since $x \in F^t(\alpha)$, then $y^{-1}xy \in F^t(\alpha)$, that is $F(\alpha)(y^{-1}xy) \geq t = t \wedge \frac{1-k}{2} = F(\alpha)(x) \wedge \frac{1-k}{2}$. So (F, A) is an $(\in, \in \vee q_k)$ fuzzy normal soft group over G. □

Theorem 4.5. *Let $(F_i, A_i)_{i \in I}$ be a nonempty family of $(\in, \in \vee q_k)$ fuzzy normal soft groups over G. Then the following hold.*

(1) *$\bigcap_{i \in I}(F_i, A_i)$ is an $(\in, \in \vee q_k)$ fuzzy normal soft group over G if $\bigcap_{i \in I} A_i \neq \varnothing$.*
(2) *$\bigcup_{i \in I}(F_i, A_i)$ is an $(\in, \in \vee q_k)$ fuzzy normal soft group over G if $i, j \in I, i \neq j$, $A_i \cap A_j = \varnothing$.*
(3) *$\bigwedge_{i \in I}(F_i, A_i)$ is an $(\in, \in \vee q_k)$ fuzzy normal soft group over G.*

Proof. The proof is obvious by Theorem 3.10 and Lemma 4.3. □

Definition 4.6. Let (F, A) and (H, B) be two $(\in, \in \vee q_k)$ fuzzy normal soft groups over G. Then (F, A) is called an $(\in, \in \vee q_k)$ fuzzy normal soft subgroup of (G, B), if (F, A) is a fuzzy soft subset of (G, B), denoted by $(F, A)\tilde{\lhd}(G, B)$.

Example 4.7. Let $A_3 = \{1, (123), (132)\}$ be alternating group of S_3, $A = \{\alpha_1, \alpha_2\}$. In Example 4.2, (F, A) is an $(\in, \in \vee q_k)$ fuzzy soft group over A_3. Let $B = \{\alpha_1\}$. Define a fuzzy soft set (H, B) over A_3 as: $H(\alpha_1)(1) = 0.3$, $H(\alpha_1)((123)) = 0.5$, $H(\alpha_1)((132)) = 0.6$. It is easy to verify that (H, B) is an $(\in, \in \vee q_k)$ fuzzy soft group over A_3. Since $(H, B) \subset (F, A)$, so $(H, B)\tilde{\lhd}(F, A)$, where $k = 0.4$.

Theorem 4.8. *Let (F, A) be an $(\in, \in \vee q_k)$ fuzzy normal soft group over G and $(F_i, A_i)_{i \in I}$ be a nonempty family of $(\in, \in \vee q_k)$ fuzzy normal soft subgroups of (F, A). Then the following hold.*

(1) *$\bigcap_{i \in I}(F_i, A_i)$ is an $(\in, \in \vee q_k)$ fuzzy normal soft subgroup of (F, A) if $\bigcap_{i \in I} A_i \neq \varnothing$.*
(2) *$\bigcup_{i \in I}(F_i, A_i)$ is an $(\in, \in \vee q_k)$ fuzzy normal soft subgroup of (F, A) if $i, j \in I, i \neq j$, $A_i \cap A_j = \varnothing$.*
(3) *$\bigwedge_{i \in I}(F_i, A_i)$ is an $(\in, \in \vee q_k)$ fuzzy normal soft set of (F, A).*

Proof. (1) Now by Theorem 3.13, $\bigcap_{i \in I}(F_i, A_i)$ is an $(\in, \in \vee q_k)$ fuzzy soft subgroup of (F, A) if $\bigcap_{i \in I} A_i \neq \varnothing$. Then we can easily obtain the proof from Theorem 4.5 and Definition 4.6.

Similarly, one can easily prove (2) (3). □

Lemma 4.9. *Let $f : G_1 \to G_2$ be a group epimorphism. If λ is an $(\in, \in \vee q_k)$ fuzzy normal subgroup of G_1, then $f(\lambda)$ is an $(\in, \in \vee q_k)$ fuzzy normal subgroup of G_2, where $f(\lambda)(y) = \sup\{\lambda(x) | f(x) = y, x \in G_1\}$.*

Proof. Straightforward. □

Theorem 4.10. *Let (F, A) and (G, B) be fuzzy soft sets over G_1 and G_2, respectively. Let (F, A) be fuzzy soft homomorphic to (H, B) under the fuzzy soft homomorphism (f, g). If (F, A) is an $(\in, \in \vee q_k)$ fuzzy normal soft group over G_1, then (H, B) is an $(\in, \in \vee q_k)$ fuzzy normal soft group over G_2.*

Proof. By Theorem 3.16, (H, B) is an $(\in, \in \vee q_k)$ fuzzy soft group over G_2. Since g is surjective, then $\forall \beta \in B$, $\exists \alpha \in A$, satisfied $g(\alpha) = \beta$. Therefore $H(\beta) = H(g(\alpha)) = f(F(\alpha))$. By Lemma 4.9, $f(F(\alpha))$ is an $(\in, \in \vee q_k)$ fuzzy normal subgroup of G_2, that is $H(\beta)$ is an $(\in, \in \vee q_k)$ fuzzy normal subgroup of G_2. So (H, B) is an $(\in, \in \vee q_k)$ fuzzy normal soft group over G_2. □

Definition 4.11. Let (F, A) be an $(\in, \in \vee q_k)$ fuzzy soft group over G. Define fuzzy soft set $g \cdot (F, A)$ and $(F, A) \cdot g$ as: $(g \cdot F(\alpha))(x) = F(\alpha)(g^{-1}x) \wedge \frac{1-k}{2}$, $(F(\alpha) \cdot g)(x) = F(\alpha)(xg^{-1}) \wedge 0.5$, $\forall \alpha \in A, x \in G, g \in G$. Then $g \cdot (F, A)$ is called an $(\in, \in \vee q_k)$ fuzzy left soft coset and $(F, A) \cdot g$ is called an $(\in, \in \vee q_k)$ fuzzy right soft coset.

Lemma 4.12 ([6]). *Let λ be an $(\in, \in \vee q_k)$ fuzzy subgroup. Then λ is an $(\in, \in \vee q_k)$ fuzzy normal subgroup if and only if $g \cdot \lambda = \lambda \cdot g$, $\forall g \in G$.*

Theorem 4.13. *Let (F, A) be a fuzzy soft group over G. Then (F, A) is an $(\in, \in \vee q_k)$ fuzzy normal soft group if and only if $g \cdot (F, A) = (F, A) \cdot g$.*

Proof. By Definition 4.1, $\forall \alpha \in A$, $F(\alpha)$ is an $(\in, \in \vee q_k)$ fuzzy normal subgroup of G. By Lemma 4.12, that is $\forall \alpha \in A, g \in G$, $g \cdot F(\alpha) = F(\alpha) \cdot g$. So (F, A) is an $(\in, \in \vee q_k)$ fuzzy normal soft group if and only if $g \cdot (F, A) = (F, A) \cdot g$. □

Theorem 4.14. *Let (F, A) be an $(\in, \in \vee q)$ fuzzy normal soft group over G. Then $(G/(F, A), \otimes)$ is a group, where $G/(F, A) = \{g \cdot (F, A) | g \in G\}$, $(x \cdot (F, A)) \otimes (y \cdot (F, A)) = (xy) \cdot (F, A)$, $\forall x, y \in G$.*

Proof. Let $x \cdot (F, A) = a \cdot (F, A)$, $y \cdot (F, A) = b \cdot (F, A)$. By Definition 4.11, $\forall g \in G, \alpha \in A$, $F(\alpha)(x^{-1}g) \wedge \frac{1-k}{2} = F(\alpha)(a^{-1}g) \wedge \frac{1-k}{2}$, $F(\alpha)(y^{-1}g) \wedge \frac{1-k}{2} = F(\alpha)(b^{-1}g) \wedge \frac{1-k}{2}$. Then $((xy) \cdot F(\alpha))(g) = F(\alpha)(y^{-1}x^{-1}g) \wedge \frac{1-k}{2} = F(\alpha)(b^{-1}x^{-1}g) \wedge \frac{1-k}{2} \geq F(\alpha)(x^{-1}gb^{-1}) \wedge \frac{1-k}{2} = F(\alpha)(a^{-1}gb^{-1}) \wedge \frac{1-k}{2} \geq F(\alpha)(b^{-1}a^{-1}g) \wedge \frac{1-k}{2} = ((ab) \cdot F(\alpha))(g)$. So $(xy) \cdot F(\alpha) \supset (ab) \cdot F(\alpha)$. Similarly, $(ab) \cdot F(\alpha) \supset (xy) \cdot F(\alpha)$. So $\otimes$ is a operation and is associative. Since $e \cdot (F, A)$ is identical element, $x^{-1} \cdot (F, A)$ is the inverse element of $x \cdot (F, A)$. So $(G/(F, A), \otimes)$ is a group. □

Theorem 4.15. *Let (F, A) be $(\in, \in \vee q_k)$ fuzzy soft group over G and $(\tilde{F}, A)$ be a fuzzy soft set over group $(G/(F, A), \otimes)$. Define $\tilde{F}(\alpha)(x \cdot (F, A)) = F(\alpha)(x)$, $\forall \alpha \in A$. Then $(\tilde{F}, A)$ is an $(\in, \in \vee q_k)$ fuzzy soft group over $(G/(F, A), \otimes)$.*

Proof. $\forall \alpha \in A$, $x \cdot (F, A) \in G/(F, A)$, $y \cdot (F, A) \in G/(F, A)$, $\tilde{F}(\alpha)((x^{-1} \cdot (F, A)) \otimes (y \cdot (F, A))) = \tilde{F}(\alpha)((x^{-1}y) \cdot (F, A)) = F(\alpha)(x^{-1}y) \geq F(\alpha)(x) \wedge F(\alpha)(y) \wedge \frac{1-k}{2} = \tilde{F}(\alpha)(x \cdot (F, A)) \wedge \tilde{F}(\alpha)(y \cdot (F, A)) \wedge \frac{1-k}{2}$. So by Definition 4.1, $(\tilde{F}, A)$ is an $(\in, \in \vee q_k)$ fuzzy soft group over $(G/(F, A), \otimes)$. □

We call $(\tilde{F}, A)$ to be an $(\in, \in \vee q_k)$ fuzzy quotient soft group of (F, A) over G.

Theorem 4.16. *Let (F, A) be an $(\in, \in \vee q_k)$ fuzzy normal soft group over G and B be an $(\in, \in \vee q_k)$ fuzzy subgroup of G. Then $B/(F, A)$ is an $(\in, \in \vee q_k)$ fuzzy subgroup of $G/(F, A)$, where $B/(F, A)(g \cdot (F, A)) = \sup\{B(x) | x \cdot (F, A) = g \cdot (F, A), x \in G\}$.*

Proof. Since $B/(F,A)((g_1 \cdot (F,A))^{-1} \otimes (g_2 \cdot (F,A))) = B/(F,A)((g_1^{-1}g_2) \cdot (F,A)) = sup\{B(x_1^{-1}x_2)|(x_1^{-1}x_2) \cdot (F,A) = (g_1^{-1}g_2) \cdot (F,A), x_1, x_2 \in G\} \geq sup\{B(x_1|x_1 \cdot (F,A) = g_1 \cdot (F,A))\} \wedge sup\{B(x_2)|x_2 \cdot (F,A) = g_2 \cdot (F,A)\} \wedge \frac{1-k}{2} = B/(F,A)(g_1 \cdot (F,A)) \wedge B/(F,A)(g_2 \cdot (F,A)) \wedge \frac{1-k}{2}$. So $B/(F,A)$ is an $(\in, \in \vee q_k)$ fuzzy subgroup of $G/(F,A)$. □

Theorem 4.17. *Let (F,A) be $(\in, \in \vee q_k)$ fuzzy normal soft group over G, B be an $(\in, \in \vee q_k)$ fuzzy normal subgroup of G. Then $B/(F,A)$ is an $(\in, \in \vee q_k)$ fuzzy normal subgroup of $G/(F,A)$, where $B/(F,A)(g \cdot (F,A)) = \sup\{B(x)|x \cdot (F,A) = g \cdot (F,A), x \in G\}$.*

Proof. By Theorem 4.16, $B/(F,A)$ is an $(\in, \in \vee q_k)$ fuzzy subgroup of $G/(F,A)$. Since $B/(F,A)(g \cdot (F,A)) = \sup\{B(x)|x \cdot (F,A) = g \cdot (F,A), x \in G\}$.

$$\begin{aligned}
&B/(F,A)((g_1 \cdot (F,A))^{-1} \otimes (g_2 \cdot (F,A)) \otimes (g_1 \cdot (F,A))) \\
&= B/(F,A)((g_1^{-1}g_2g_1) \cdot (F,A)) \\
&= \sup\left\{B(x_1^{-1}x_2x_1) \mid (x_1^{-1}x_2x_1) \cdot (F,A) = (g_1^{-1}g_2g_1) \cdot (F,A), x_1, x_2 \in G\right\} \\
&\geq \sup\{B(x_2) \mid x_2 \cdot (F,A) = g_2 \cdot (F,A)\} \wedge \frac{1-k}{2} \\
&= B/(F,A)(g_2 \cdot (F,A)) \wedge \frac{1-k}{2}.
\end{aligned}$$

So $B/(F,A)$ is an $(\in, \in \vee q_k)$ fuzzy normal subgroup of $G/(F,A)$. □

5. Conclusions

Soft sets and fuzzy sets are new mathematical tools to deal with uncertainties. They have rich potential for applications in several directions. In this paper, we applied the notion of fuzzy sets to soft sets, and introduced the concepts of $(\in, \in \vee q_k)$ fuzzy soft groups and $(\in, \in \vee q_k)$ fuzzy normal soft groups, then studied some properties of them. These obtained results can be applied to other algebraic structures.

Acknowledgements. The authors are highly grateful to the anonymous referees and Editor-in-Chief for their valuable comments and suggestions which greatly improve the quality of this paper.

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