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Fuzzy real inner product space and its properties

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Abstract. Defininition of fuzzy real inner product space is introduced. A decomposition theorem from a fuzzy real inner product into an ascending family of crisp inner products and other decomposition theorem from an ascending family of crisp inner products into a fuzzy real inner product are established.

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1. Introduction

It was Katsaras[17], who while studying fuzzy topological vector spaces, was the first to introduced in 1984, the idea of fuzzy norm on a linear space. Later on many other mathematicians like Felbin[12], Cheng & Mordeson[10], Bag & Samanta[1] etc. introduced definition of fuzzy normed linear spaces in different approach. A large number of paper have been published in fuzzy normed linear spaces, for reference please see [2, 3, 4, 5, 6, 7, 13, 14, 23]. On the other hand studies on fuzzy inner product spaces are relatively recent and few work have been done in fuzzy inner product spaces. R. Biswas[9], A.M.El-Abyed & H.M.Hamouly[11] were among the first who gave a meaningful definition of fuzzy inner product space and associated fuzzy norm function. Later on J.K.Kohli & R.Kumar[18] modified the definition of inner product space introduced by R.Biswas. The definition introduced by Pinaki Mazumder & S.K.Samanta [19], A.Hasankhani, A.Nazari, M.Saheli [16], M.Goudarzi & S.M.Vaezpour [15], S. Vijayabalaji [21, 22] are very recent. In this context it is worth mentioning the work done by Thangaraj et al.[8]. But still now no remarkable development in the study of fuzzy inner product spaces is found. Following the definition of A.Hasankhani, A.Nazari, M.Saheli, we [20] study some properties of fuzzy Hilbert spaces and fixed point theory in such spaces.

In this paper, we redefine the definition of fuzzy inner product introduced by M.Goudarzi & S.M.Vaezpour and choose 't'-norm $*$ as 'min' to develop more results of fuzzy inner product spaces. We are able to establish a decomposition theorem from a fuzzy real inner product into an ascending family of crisp inner product. The novelty of this decomposition theorem is far fetching. We think that with the help of this decomposition theorem it will be possible to establish many fundamental results of functional analysis viz. operator theory, spectral theory in fuzzy setting. The organization of the paper is as follows:

Section 2 provides some preliminary results.

In section 3, the definition of fuzzy real inner product is introduced and some examples are given.

Section 4 is devoted to establish the decomposition theorem of a fuzzy real inner product into a family of crisp inner products and vice-versa.

2. Preliminaries

In this section some definitions and preliminary results are given which are used in this paper.

Definition 2.1 ([1]). Let U be a linear space over a field F(field of real complex numbers). A fuzzy subset N of $U \times R$ (R is the set of real numbers) is called a fuzzy norm on U if $\forall x, u \in U$ and $c \in F$, following conditions are satisfied:

(N1) $\forall t \in R$ with $t \leq 0,\ N(x\ ,\ t) = 0$;

(N2) ($\forall t \in R,\ t > 0,\ N(x\ ,\ t) = 1$) iff $x = \underline{0}$;

(N3) $\forall t \in R,\ t > 0,\ N(cx\ ,\ t) = N(x\ ,\ \frac{t}{|c|})$ if $c \neq 0$;

(N4) $\forall s, t \in R,\ x, u \in U$;

$$N(x + u\ ,\ s + t) \geq min\{N(x\ ,\ s)\ ,\ N(u\ ,\ t)\}$$

(N5) $N(x\ ,\ .)$ is a non-decreasing function of R and $\lim_{t\to\infty} N(x\ ,\ t) = 1$.

The pair $(U\ ,\ N)$ will be referred to as a fuzzy normed linear space.

Theorem 2.2 ([1]). *Let $(U\ ,\ N)$ be a fuzzy normed linear space.*
Assume further that,
(N6) $\forall t > 0,\ N(x\ ,\ t) > 0$ *implies* $x = \underline{0}$.
Define $||x||_\alpha = \wedge\{t > 0 :\ N(x\ ,\ t) \geq \alpha\},\ \alpha \in (0\ ,\ 1)$.
Then $\{||\ ||_\alpha :\ \alpha \in (0\ ,\ 1)\}$ *is an ascending family of norms on U and they are called α-norms on U corresponding to the fuzzy norm N on U.*

Theorem 2.3 ([1]). *Let $(U\ ,\ N)$ be a fuzzy normed linear space satisfying (N6).*
Assume further that,

(N7) for $x \neq \underline{0},\ N(x, .)$ *is a continuous function of* $\mathbf{R}$.

Let $||x||_\alpha = \wedge\{t > 0 :\ N(x\ ,\ t) \geq \alpha\},\ \alpha \in (0\ ,\ 1)$ *and* $N' : U \times R \to [0,\ 1]$ *be a function defined by*

$$N'(x,\ t) = \begin{cases} \vee\{\alpha \in (0,\ 1) : ||x||_\alpha \leq t\} & \textit{if } (x,\ t) \neq (\underline{0},\ 0) \\ 0 & \textit{if } (x,\ t) \neq (\underline{0},\ 0). \end{cases}$$

Then

(i) $\{||\ ||_\alpha : \alpha \in (0\ ,\ 1)\}$ *is an ascending family of norms on* U.

(ii) N' *is a fuzzy norm on* U.

(iii) $N' = N$.

Definition 2.4 ([15]). A fuzzy inner product space (FIP-space) is a triplet $(X,\ F,\ *)$, where X is a real vector space, $*$ is a continuous t-norm, F is a fuzzy set on $X^2 \times R$ and the following conditions hold for every $x,\ y,\ z \in X$ and $s,\ t,\ r \in R$.
(FI-1) $F(x,\ x, 0) = 0$ and $F(x,\ x,\ t) > 0$, for each $t > 0$;
(FI-2) $F(x,\ x,\ t) \neq H(t)$ for some $t \in R$ if and only if $x \neq 0$;
(FI-3) $F(x,\ y,\ t) = F(y,\ x,\ t)$;
(FI-4) For any real number α,

$$F(\alpha x,\ y,\ t) = \begin{cases} F(x,\ y,\ \frac{t}{\alpha}) & \text{if } \alpha > 0 \\ H(t) & \text{if } \alpha = 0 \\ 1 - F(x,\ y,\ \frac{t}{-\alpha}) & \text{if } \alpha < 0 \end{cases}$$

Where

$$H(t) = \begin{cases} 1 & \text{if } \ t > 0 \\ 0 & \text{if } \ t \leq 0. \end{cases}$$

(FI-5) $sup_{s+r=t}(F(x,\ z,\ s) * F(y,\ z,\ r)) = F(x + y,\ z,\ t)$;
(FI-6) $F(x,\ y,\ .) : R \to [0,\ 1]$ is continuous on $R\ \{0\}$;
(FI-7) $\lim_{t\to+\infty} F(x,\ y,\ t) = 1$.

3. Fuzzy real inner product spaces

Following the definition of fuzzy inner product space introduced by Goudarzi & Vaezpour we redefine the fuzzy real inner product space and some examples of such space are given.

Definition 3.1. Let X be a linear space over R (the set of real numbers). Then a fuzzy subset $\mathfrak{F} : X \times X \times R \to [0,\ 1]$ is called fuzzy real inner product on X if $\forall\ x,\ y,\ z \in X$ and $t \in R$, the following conditions hold.
(FI-1) $\mathfrak{F}(x,\ x,\ t) = 0 \ \ \forall\ t < 0$
(FI-2) $(\mathfrak{F}(x,\ x,\ t) = 1 \ \ \forall\ t > 0)$ iff $x = \underline{0}$
(FI-3) $\mathfrak{F}(x,\ y,\ t) = \mathfrak{F}(y,\ x,\ t)$
(FI-4)

$$\mathfrak{F}(cx,\ y,\ t) = \begin{cases} \mathfrak{F}(x,\ y,\ \frac{t}{c}) & \text{if } \ c > 0 \\ H(t) & \text{if } \ c = 0 \\ 1 - \mathfrak{F}(x,\ y,\ \frac{t}{c}) & \text{if } \ c < 0 \end{cases}$$

(FI-5) $\mathfrak{F}(x + y,\ z,\ t + s) \geq \mathfrak{F}(x,\ z,\ t) \wedge \mathfrak{F}(y,\ z,\ s)$
(FI-6) $\lim_{t\to+\infty} \mathfrak{F}(x,\ y,\ t) = 1$.
The pair $(X,\ \mathfrak{F})$ is said to be a fuzzy real inner product space.

Remark 3.2. $\mathfrak{F}(x,\ y, .)$ is a non-decreasing function of R.

Proof. Let $t_1 > t_2$. Therefore $t_1 - t_2 > 0$
$\mathfrak{F}(0 + x,\ y,\ t_1 - t_2 + t_2) \geq \min\ \{\mathfrak{F}(0,\ y,\ t_1 - t_2),\ \mathfrak{F}(x,\ y,\ t_2)\}$
$\Rightarrow \mathfrak{F}(x,\ y,\ t_1) \geq \min\ \{1,\ \mathfrak{F}(x,\ y,\ t_2)\}$ [by (FI 4) and since $H(t_1 - t_2) = 1$]
$\Rightarrow \mathfrak{F}(x,\ y,\ t_1) \geq\ \mathfrak{F}(x,\ y,\ t_2)$. □

Example 3.3. Let $(X, \langle\ ,\ \rangle)$ be an ordinary inner product space over R. Define $\mathfrak{F} : X \times X \times R \to [0,\ 1]$ by $\mathfrak{F}(cx,\ y,\ t) = H(t)$ for $c = 0$ and for $c \neq 0$,

$$\mathfrak{F}(cx,\ y,\ t) = \begin{cases} 1 & \text{if } t > c|\langle x,\ y\rangle| \\ \frac{1}{2} & \text{if } t = c|\langle x,\ y\rangle| \\ 0 & \text{if } t < c|\langle x,\ y\rangle| \end{cases}$$

Then $(X,\ \mathfrak{F})$ is a fuzzy real inner product space.

Proof. **(FI-1)** $\mathfrak{F}(x,\ x,\ t) = 0$ for $t < 0$ since $|\langle x,\ x\rangle| \geq 0$.
(FI-2) $\mathfrak{F}(x,\ x,\ t) = 1\ \ \forall t > 0$
$\Leftrightarrow t > |\langle x,\ x\rangle|\ \ \forall t > 0$
$\Leftrightarrow |\langle x,\ x\rangle| = 0$
$\Leftrightarrow x = 0$.
(FI-3) Follows directly from the definition.
(FI-4) **(Case-I)** Let $c > 0$.
(Subcase-i) Let $t > 0$ then $\frac{t}{c} > 0$.
If $\mathfrak{F}(cx,\ y,\ t) = 1$, then $t > c|\langle x,\ y\rangle|$
$\Rightarrow \frac{t}{c} > |\langle x,\ y\rangle|$
$\Rightarrow \mathfrak{F}(x,\ y,\ \frac{t}{c}) = 1$.
If $\mathfrak{F}(cx,\ y,\ t) = \frac{1}{2}$, then $t = c|\langle x,\ y\rangle|$
$\Rightarrow \frac{t}{c} = |\langle x,\ y\rangle|$
$\Rightarrow \mathfrak{F}(x,\ y,\ \frac{t}{c}) = \frac{1}{2}$.
Similarly if $\mathfrak{F}(cx,\ y,\ t) = 0\ \Rightarrow \mathfrak{F}(x,\ y,\ \frac{t}{c}) = 0$.
(Subcase-ii) Let $t = 0$ then similarly we get $\mathfrak{F}(cx,\ y,\ t) = \mathfrak{F}(x,\ y,\ \frac{t}{c})$.
(Case-II) Let $c = 0$, then by definition $\mathfrak{F}(cx,\ y,\ t) = H(t)$.
(Case-III) Let $c < 0$, then $c = -m$ for some $m > 0$.
(Subcase-i) Let $t > 0$ then $\frac{t}{c} < 0$.
If $\mathfrak{F}(cx,\ y,\ t) = 1$, then $t > c|\langle x,\ y\rangle|$
$\Rightarrow t > -m|\langle x,\ y\rangle|$
$\Rightarrow \frac{t}{m} > -|\langle x,\ y\rangle|$
$\Rightarrow \frac{t}{-m} < |\langle x,\ y\rangle|$
$\Rightarrow \frac{t}{c} < |\langle x,\ y\rangle|$
$\Rightarrow \mathfrak{F}(x,\ y,\ \frac{t}{c}) = 0$.
Therefore $\mathfrak{F}(cx,\ y,\ t) = 1 - \mathfrak{F}(x,\ y,\ \frac{t}{c})$.
(Subcase-ii) Let $t = 0$ then $\frac{t}{c} = 0$.
If $\mathfrak{F}(cx,\ y,\ t) = 1$, then $o > c|\langle x,\ y\rangle|$
$\Rightarrow 0 < |\langle x,\ y\rangle|$
$\Rightarrow \mathfrak{F}(x,\ y,\ \frac{t}{c}) = 0$
$\Rightarrow \mathfrak{F}(cx,\ y,\ t) = 1 - \mathfrak{F}(x,\ y,\ \frac{t}{c})$.
If $\mathfrak{F}(cx,\ y,\ t) = \frac{1}{2}$ then $t = c|\langle x,\ y\rangle|$
$\Rightarrow \frac{t}{c} = |\langle x,\ y\rangle|$
$\Rightarrow \mathfrak{F}(x,\ y,\ \frac{t}{c}) = \frac{1}{2}$
$\Rightarrow \mathfrak{F}(cx,\ y,\ t) = 1 - \mathfrak{F}(x,\ y,\ \frac{t}{c})$.
(Subcase-iii) Let $t < 0$ then $\frac{t}{c} > 0$.
If $\mathfrak{F}(cx,\ y,\ t) = 1$, then $t > c|\langle x,\ y\rangle|$

$\Rightarrow \frac{t}{c} < |\langle x,\ y\rangle|$
$\Rightarrow \mathfrak{F}(x,\ y,\ \frac{t}{c}) = 0.$
Therefore $\mathfrak{F}(cx,\ y,\ t) = 1 - \mathfrak{F}(x,\ y,\ \frac{t}{c})$.
If $\mathfrak{F}(cx,\ y,\ t) = \frac{1}{2}$ then $t = c|\langle x,\ y\rangle|$
$\Rightarrow \frac{t}{c} = |\langle x,\ y\rangle|$
$\Rightarrow \mathfrak{F}(x,\ y,\ \frac{t}{c}) = \frac{1}{2}$
$\Rightarrow \mathfrak{F}(cx,\ y,\ t) = 1 - \mathfrak{F}(x,\ y,\ \frac{t}{c})$.
If $\mathfrak{F}(cx,\ y,\ t) = 0$, then $t < c|\langle x,\ y\rangle|$
$\Rightarrow \frac{t}{c} > |\langle x,\ y\rangle|$
$\Rightarrow \mathfrak{F}(x,\ y,\ \frac{t}{c}) = 1$
$\Rightarrow \mathfrak{F}(cx,\ y,\ t) = 1 - \mathfrak{F}(x,\ y,\ \frac{t}{c})$.
(FI-5) **(Case-I)** Let $t > |\langle x,\ z\rangle|$ and $s > |\langle y,\ z\rangle|$.
So $\mathfrak{F}(x,\ z,\ t) = 1,\ \mathfrak{F}(y,\ z,\ s) = 1$
Then $t + s > |\langle x,\ z\rangle| + |\langle y,\ z\rangle| \geq |\langle x+y,\ z\rangle|$
$\Rightarrow \mathfrak{F}(x+y,\ z,\ t+s) = 1 \geq \mathfrak{F}(x,\ z,\ t) \wedge \mathfrak{F}(y,\ z,\ s)$.
(Case-II) Let $t = |\langle x,\ z\rangle|$ and $s > |\langle y,\ z\rangle$— or $t > |\langle x,\ z\rangle|$ and $s = |\langle y,\ z\rangle|$holds.
So $\mathfrak{F}(x,\ z,\ t) = \frac{1}{2},\ \mathfrak{F}(y,\ z,\ s) = 1$ or $\mathfrak{F}(x,\ z,\ t) = 1,\ \mathfrak{F}(y,\ z,\ s) = \frac{1}{2}$
Then $t + s > |\langle x,\ z\rangle| + |\langle y,\ z\rangle| \geq |\langle x+y,\ z\rangle|$
$\Rightarrow \mathfrak{F}(x+y,\ z,\ t+s) = 1 > \frac{1}{2} \geq \mathfrak{F}(x,\ z,\ t) \wedge \mathfrak{F}(y,\ z,\ s)$.
(Case-III) If any one of $t < |\langle x,\ z\rangle|$ or $s < |\langle y,\ z\rangle|$ holds.
Then $\mathfrak{F}(x,\ z,\ t) = 0$ or $\mathfrak{F}(y,\ z,\ s) = 0$
So $\mathfrak{F}(x,\ z,\ t) \wedge \mathfrak{F}(y,\ z,\ s) = 0$ and obviously
$\mathfrak{F}(x+y,\ z,\ t+s) \geq \mathfrak{F}(x,\ z,\ t) \wedge \mathfrak{F}(y,\ z,\ s)$.
(FI-6) Clearly $\lim_{t \to +\infty} \mathfrak{F}(x,\ y,\ t) = 1$.
Thus $(X,\ \mathfrak{F})$ is a fuzzy real inner product space. □

Theorem 3.4. *Let $(X,\ \mathfrak{F})$ be a fuzzy real inner product space. Assume further that* **(FI-7)** $\mathfrak{F}(x,\ y,\ st) \geq \mathfrak{F}(x,\ x,\ s^2) \wedge \mathfrak{F}(y,\ y,\ t^2)$, $\quad \forall\ s,\ t \in R$ *and* $\forall\ x,\ y \in X$. *Define a function $N : X \times R \to [0,\ 1]$ by*

$$N(x,\ t) = \begin{cases} \mathfrak{F}(x,\ x,\ t^2) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

Then N is a B-S[1] fuzzy norm on X. We call this norm as induced norm of $\mathfrak{F}$.

Proof. (N1) $\forall\ t \leq 0,\ \ N(x,\ t) = 0$ (by definition).
(N2) $N(x,\ t) = 1,\ \ \forall\ t > 0$
$\Leftrightarrow \mathfrak{F}(x,\ x,\ t^2) = 1,\ \ \forall\ t > 0$
$\Leftrightarrow x = \underline{0}$.
(N3) If $c > 0$, then for $t \leq 0,\ N(cx,\ t) = 0 = N(x,\ \frac{t}{c})$ and for $t > 0$,
$N(cx,\ t) = \mathfrak{F}(cx,\ cx,\ t^2) = \mathfrak{F}(x,\ x,\ \frac{t^2}{c^2}) = N(x,\ \frac{t}{c})$
If $c < 0$, then for $t \leq 0,\ N(cx,\ t) = 0 = N(x,\ \frac{t}{-c})$ and for $t > 0$,
$N(cx,\ t) = \mathfrak{F}(cx,\ cx,\ t^2) = 1 - \mathfrak{F}(x,\ cx,\ \frac{t^2}{c}) = 1 - [1 - \mathfrak{F}(x,\ x,\ \frac{t^2}{(c)^2})] = \mathfrak{F}(x,\ x,\ \frac{t^2}{(c)^2}) = N(x,\ \frac{t}{-c})$.
Thus $N(cx,\ t) = N(x,\ \frac{t}{|c|})$.
(N4) $N(x+y,\ s+t) = \mathfrak{F}(x+y,\ x+y,\ (s+t)^2)$

$\geq \mathfrak{F}(x,\ x,\ s^2) \wedge \mathfrak{F}(x,\ y,\ st) \wedge \mathfrak{F}(y,\ x,\ ts) \wedge \mathfrak{F}(y,\ y,\ t^2)$
$= \mathfrak{F}(x,\ x,\ s^2) \wedge \mathfrak{F}(x,\ y,\ st) \wedge \mathfrak{F}(y,\ y,\ t^2)$.
By (FI-7) we have
$N(x+y,\ s+t) \geq \mathfrak{F}(x,\ x,\ s^2) \wedge \mathfrak{F}(y,\ y,\ t^2) = N(x,\ s) \wedge N(y,\ t)$.
(N5) This follows from the nondecreasing property of $\mathfrak{F}(x,\ y,\ .)$ and $\lim_{t \to +\infty} \mathfrak{F}(x,\ y,\ t) = 1$ and $N(x,\ t) = 0$ for $t \leq 0$. □

4. Decomposition theorems

In this section we establish two decomposition theorems from a fuzzy real inner product space into a ascending family of crisp inner product spaces.

Theorem 4.1. *Let* $(X,\ \mathfrak{F})$ *be a fuzzy real inner product space. Further assume that*
(FI-8) $\wedge\{t \in R : \mathfrak{F}(x,\ y,\ t) \geq \alpha\} < \infty,\ \forall\ \alpha \in (0,\ 1)$ *and*
$\mathfrak{F}(x,\ x,\ t) > 0 \quad \forall\ t > 0\ \Rightarrow\ x = \underline{0}$.
Define $\langle x,\ y \rangle_\alpha = \wedge\{t \in R : \mathfrak{F}(x,\ y,\ t) \geq \alpha\},\quad \alpha \in (0,\ 1)$.
Then $\{\langle\ ,\ \rangle_\alpha : \alpha \in (0,\ 1)\}$ *is an ascending family of inner products on* X. *We call these inner products as* α*-inner products of* $\mathfrak{F}$.

Proof. (I) $\langle x,\ x \rangle_\alpha = \wedge\{t \in R : \mathfrak{F}(x,\ x,\ t) \geq \alpha\}$. Since $\mathfrak{F}(x,\ x,\ t) = 0\quad \forall\ t \leq 0$, $\langle x,\ x \rangle_\alpha \geq 0\ \ \forall\ \alpha \in (0,\ 1)\}$.
Now let for $\alpha \in (0,\ 1)$, $\langle x,\ x \rangle_\alpha = 0$
$\Rightarrow \wedge\{t \in R : \mathfrak{F}(x,\ x,\ t) \geq \alpha\} = 0$
$\Rightarrow \forall\ t > 0\ \ \mathfrak{F}(x,\ x,\ t) \geq \alpha > 0$.
By (FI-8) we have $x = \underline{0}$.
Conversely assume that $x = \underline{0}$. Then $\mathfrak{F}(x,\ x,\ t) = H(t)$. Therefore $\langle x,\ x \rangle_\alpha = \wedge\{t \in R : H(t) \geq \alpha\},\ \ \forall\ \alpha \in (0,\ 1) = \wedge\{t \in R : t > 0\} = 0,\ \ \forall\ \alpha \in (0,\ 1)$.
(II) $\langle x,\ y \rangle_\alpha = \wedge\{t \in R : \mathfrak{F}(x,\ y,\ t) \geq \alpha\},\ \ \alpha \in (0,\ 1)$
$= \wedge\{t \in R : \mathfrak{F}(y,\ x,\ t) \geq \alpha\},\ \ \alpha \in (0,\ 1)$ [by (FI-3)]
$= \langle y,\ x \rangle_\alpha$.
(III) Let $\alpha \in (0,\ 1)$. Now $\mathfrak{F}(x+y,\ z,\ \langle x,\ z \rangle_\alpha + \langle y,\ z \rangle_\alpha + \epsilon)$
$\geq \mathfrak{F}(x,\ z,\ \langle x,\ z \rangle_\alpha + \frac{\epsilon}{2}) \wedge \mathfrak{F}(y,\ z,\ \langle y,\ z \rangle_\alpha + \frac{\epsilon}{2})$
$\geq \alpha \wedge \alpha = \alpha$.
Therefore $\wedge\{t \in R : \mathfrak{F}(x+y,\ z,\ t) \geq \alpha\} \leq \langle x,\ z \rangle_\alpha + \langle y,\ z \rangle_\alpha + \epsilon$.
Since ϵ is arbitrary,
$\wedge\{t \in R : \mathfrak{F}(x+y,\ z,\ t) \geq \alpha\} \leq \langle x,\ z \rangle_\alpha + \langle y,\ z \rangle_\alpha$
$\Rightarrow \langle x+y,\ z \rangle_\alpha \leq \langle x,\ z \rangle_\alpha + \langle y,\ z \rangle_\alpha$. (4.1.1)
Again if $A = 1 - [(1 - \mathfrak{F}(x,\ z,\ \langle x,\ z \rangle_\alpha - \frac{\epsilon}{2})) \wedge (1 - \mathfrak{F}(y,\ z,\ \langle y,\ z \rangle_\alpha - \frac{\epsilon}{2}))]$
$= 1 - [\mathfrak{F}(-x,\ z,\ -\langle x,\ z \rangle_\alpha + \frac{\epsilon}{2}) \wedge \mathfrak{F}(-y,\ z,\ -\langle y,\ z \rangle_\alpha + \frac{\epsilon}{2})$
$\geq 1 - \mathfrak{F}(-x-y,\ z,\ -\langle x,\ z \rangle_\alpha - \langle y,\ z \rangle_\alpha + \epsilon)$
$= \mathfrak{F}(x+y,\ z,\ \langle x,\ z \rangle_\alpha + \langle y,\ z \rangle_\alpha - \epsilon)$
Therefore $A \geq \mathfrak{F}(x+y,\ z,\ \langle x,\ z \rangle_\alpha + \langle y,\ z \rangle_\alpha - \epsilon)$. (4.1.2)
Now $\mathfrak{F}(x,\ z,\ \langle x,\ z \rangle_\alpha - \frac{\epsilon}{2}) < \alpha,\ \mathfrak{F}(y,\ z,\ \langle y,\ z \rangle_\alpha - \frac{\epsilon}{2}) < \alpha$
$\Rightarrow 1 - \mathfrak{F}(x,\ z,\ \langle x,\ z \rangle_\alpha - \frac{\epsilon}{2}) > 1 - \alpha,\ 1 - \mathfrak{F}(y,\ z,\ \langle y,\ z \rangle_\alpha - \frac{\epsilon}{2}) > 1 - \alpha$
$\Rightarrow [1 - \mathfrak{F}(x,\ z,\ \langle x,\ z \rangle_\alpha - \frac{\epsilon}{2})] \wedge [1 - \mathfrak{F}(y,\ z,\ \langle y,\ z \rangle_\alpha - \frac{\epsilon}{2}) > 1 - \alpha$
$\Rightarrow A = 1 - [(1 - \mathfrak{F}(x,\ z,\ \langle x,\ z \rangle_\alpha - \frac{\epsilon}{2})) \wedge (1 - \mathfrak{F}(y,\ z,\ \langle y,\ z \rangle_\alpha - \frac{\epsilon}{2}))] < 1 - (1-\alpha) = \alpha$.

From (4.1.2) we have $\mathfrak{F}(x+y,\ z,\ \langle x,\ z\rangle_\alpha + \langle y,\ z\rangle_\alpha - \epsilon) \leq A < \alpha$
$\Rightarrow \langle x+y,\ z\rangle_\alpha \geq \langle x,\ z\rangle_\alpha + \langle y,\ z\rangle_\alpha - \epsilon$.
Since ϵ is arbitrary, we have
$$\langle x+y,\ z\rangle_\alpha \geq \langle x,\ z\rangle_\alpha + \langle y,\ z\rangle_\alpha. \tag{4.1.3}$$
From (4.1.1) and (4.1.3) we have
$\langle x+y,\ z\rangle_\alpha = \langle x,\ z\rangle_\alpha + \langle y,\ z\rangle_\alpha$.
(IV) (a) Let $c > 0$. Then
$\langle cx,\ y\rangle_\alpha = \wedge\{t \in R : \mathfrak{F}(cx,\ y,\ t) \geq \alpha\}$
$= \wedge\{t \in R : \mathfrak{F}(x,\ y,\ \frac{t}{c}) \geq \alpha\}$
$= c \wedge \{s \in R : \mathfrak{F}(x,\ y,\ s) \geq \alpha\}$ where $\frac{t}{c} = s$
$= c\langle x,\ y\rangle_\alpha$.
(b) Let $c = 0$. Then
$\langle cx,\ y\rangle_\alpha = \langle 0.x,\ y\rangle_\alpha$
$= \wedge\{t \in R : \mathfrak{F}(0.x,\ y,\ t) \geq \alpha\}$
$= \wedge\{t \in R : H(t) \geq \alpha\}$
$= \wedge\{t \in R : t > 0\} = 0 = c\langle x,\ y\rangle_\alpha$.
(c) Let $c < 0$. Then
$c = -m$ for some $m > 0$.
Then $\langle -mx + mx,\ y\rangle_\alpha = \langle -mx,\ y\rangle_\alpha + \langle mx,\ y\rangle_\alpha$
$$\Rightarrow \langle 0,\ y\rangle_\alpha = \langle -mx,\ y\rangle_\alpha + \langle mx,\ y\rangle_\alpha. \tag{4.1.4}$$
Now by (FI-4) $\mathfrak{F}(0,\ y,\ t) = H(t)$ and thus by definition of $\langle x,\ y\rangle_\alpha$, it follows that $\langle 0,\ y\rangle_\alpha = 0$.
Hence from (4.1.4) we have
$0 = \langle -mx,\ y\rangle_\alpha + \langle mx,\ y\rangle_\alpha \Rightarrow \langle -mx,\ y\rangle_\alpha = -\langle mx,\ y\rangle_\alpha$
$\Rightarrow \langle -mx,\ y\rangle_\alpha = -m\langle x,\ y\rangle_\alpha$
$\Rightarrow \langle cx,\ y\rangle_\alpha = c\langle x,\ y\rangle_\alpha$.
Thus in any case $\langle cx,\ y\rangle_\alpha = c\langle x,\ y\rangle_\alpha$.
(V) Let $\alpha_1 > \alpha_2$.
Therefore $\{t \in R : \mathfrak{F}(x,\ y,\ t) \geq \alpha_1\} \subset \{t \in R : \mathfrak{F}(x,\ y,\ t) \geq \alpha_2\}$
$\Rightarrow \wedge\{t \in R : \mathfrak{F}(x,\ y,\ t) \geq \alpha_1\} \geq \wedge\{t \in R : \mathfrak{F}(x,\ y,\ t) \geq \alpha_2\}$
$\Rightarrow \langle x,\ y\rangle_{\alpha_1} \geq \langle x,\ y\rangle_{\alpha_2}$.
Thus $\{\langle\ ,\ \rangle_\alpha : \alpha \in (0,\ 1)\}$ is an ascending family of inner products on X. □

Remark 4.2. Let $(X,\ \mathfrak{F})$ is a fuzzy real inner product space satisfying (FI-7) and (FI-8) and N be its induced fuzzy norm. The α-norms derived from induced fuzzy norm N and from α-inner product are same.

Proof. Since $\mathfrak{F}$ satisfies (FI-8) we have $\mathfrak{F}(x,\ x,\ t) > 0\ \ \forall\, t > 0 \Rightarrow x = \underline{0}$.
Now $N(x,\ t) > 0 \Leftrightarrow \mathfrak{F}(x,\ x,\ t) > 0\ \ \forall\, t > 0$. So $N(x,\ t) > 0 \Rightarrow x = \underline{0}$ i.e N satisfies (N6). Therefore for $\alpha \in (0,\ 1)$
$||x||_\alpha = \wedge\{t > 0 : N(x,\ t) \geq \alpha\} = \wedge\{t > 0 : \mathfrak{F}(x,\ x,\ t^2) \geq \alpha\}$
So $\{||x||_\alpha\}^2 = \wedge\{t^2 > 0 : \mathfrak{F}(x,\ x,\ t^2) \geq \alpha\}$
$$= \wedge\{s > 0 : \mathfrak{F}(x,\ x,\ s) \geq \alpha\}. \tag{4.2.1}$$
On the other hand for $\alpha \in (0,\ 1)$,
$\langle x,\ y\rangle_\alpha = \wedge\{t \in R : \mathfrak{F}(x,\ y,\ t) \geq \alpha\}$

$\Rightarrow \langle x,\ x\rangle_\alpha = \wedge\{t \in R : \mathfrak{F}(x,\ x,\ t) \geq \alpha\}$
Now $\mathfrak{F}(x,\ x,\ t) = 0\ \ \forall\, t \leq 0$
$\Rightarrow \langle x,\ x\rangle_\alpha = \wedge\{t > 0 : \mathfrak{F}(x,\ x,\ t) \geq \alpha\}$
$\Rightarrow \{||x||'_\alpha\}^2 = \wedge\{t > 0 : \mathfrak{F}(x,\ x,\ t) \geq \alpha\}$ (4.2.2)
From (4.2.1) and (4.2.2) we get $\{||x||_\alpha\}^2 = \{||x||'_\alpha\}^2 \Rightarrow ||x||_\alpha = ||x||'_\alpha \quad \forall\ \alpha \in (0,\ 1)$. □

Theorem 4.3. *Let $\{\langle\ ,\ \rangle_\alpha : \alpha \in (0,\ 1)\}$ be an ascending family of inner products on a real linear space X. Now define a function $\mathfrak{F}' : X \times X \times R \to [0,\ 1]$ as for $c \geq 0$*

$$\mathfrak{F}'(cx,\ y,\ t) = \begin{cases} 0 & t = 0 \text{ and either } x = 0 \text{ or } y = 0. \\ \vee\{\alpha \in (0,\ 1) : \langle cx,\ y\rangle_\alpha \leq t\} & \text{otherwise} \end{cases}$$

and for $c < 0$

$$\mathfrak{F}'(cx,\ y,\ t) = \begin{cases} 1 & t = 0 \text{ and either } x = 0 \text{ or } y = 0. \\ 1 - \vee\{\alpha \in (0,\ 1) : \langle cx,\ y\rangle_\alpha \geq t\} & \text{otherwise.} \end{cases}$$

Then $\mathfrak{F}'$ is a fuzzy real inner product on X.

Proof. **(FI-1)** For $t < 0$, $\{\alpha \in (0,\ 1) : \langle x,\ x\rangle_\alpha \leq t\} = \phi$ (empty set). Therefore $\mathfrak{F}'(x,\ x,\ t) = \vee\{\alpha \in (0,\ 1) : \langle x,\ x\rangle_\alpha \leq t\} = 0$.

(FI-2) Let $x = \underline{0}$. If $t > 0$, then $\mathfrak{F}'(x,\ x,\ t) = \vee\{\alpha \in (0,\ 1) : \langle x,\ x\rangle_\alpha \leq t\} = \vee\{\alpha \in (0,\ 1) : 0 \leq t\} = \vee\{\alpha \in (0,\ 1) : \alpha \in (0,\ 1)\} = 1$.

Conversely assume that $\mathfrak{F}'(x,\ x,\ t) = 1\ \ \forall\, t > 0 \Rightarrow \vee\{\alpha \in (0,\ 1) : \langle x,\ x\rangle_\alpha \leq t\} = 1$. Choose any $\epsilon \in (0,\ 1)$, then for any $t > 0$, $\exists\ \alpha_t \in (\epsilon,\ 1]$ such that $\langle x,\ x\rangle_{\alpha_t} \leq t$ and hence $\langle x,\ x\rangle_\epsilon \leq t$. Since $t > 0$ is arbitrary, this implies $\langle x,\ x\rangle_\epsilon = 0$ and hence $x = \underline{0}$.

(FI-3) It follows from the definition.

(FI-4) If $c > 0$ and either of x or y is $\underline{0}$ and $t = 0$ then $\mathfrak{F}'(cx,\ y,\ t) = \mathfrak{F}'(x,\ y,\ \frac{t}{c}) = 0$. If $c > 0$ and $x \neq \underline{0} \neq y$ then $\mathfrak{F}'(cx,\ y,\ t) = \vee\{\alpha \in (0,\ 1) : \langle cx,\ y\rangle_\alpha \leq t\}$
$= \vee\{\alpha \in (0,\ 1) : \langle x,\ y\rangle_\alpha \leq \frac{t}{c}\}$
$= \mathfrak{F}'(x,\ y,\ \frac{t}{c})$. If $c = 0$ and $t > 0$ then $\mathfrak{F}'(cx,\ y,\ t) = \vee\{\alpha \in (0,\ 1) : \langle cx,\ y\rangle_\alpha \leq t\}$
$= \vee\{\alpha \in (0,\ 1); 0 \leq t\}$
$= 1$. If $c = 0$ and $t < 0$ then $\{\alpha \in (0,\ 1) : \langle cx,\ y\rangle_\alpha \leq t\} = \phi$
$\vee\ \{\alpha \in (0,\ 1) : \langle cx,\ y\rangle_\alpha \leq t\} = 0$.
If $c = 0$ and $t = 0$ then $0.x = \underline{0}$ and hence by definition $\mathfrak{F}'(cx,\ y,\ t) = 0$.
Thus $\mathfrak{F}'(cx,\ y,\ t) = H(t)$ if $c = 0$.
If $c < 0$ and either x or y is $\underline{0}$ and $t = 0$ then $\mathfrak{F}'(cx,\ y,\ t) = 1$
Now $1 - \mathfrak{F}'(x,\ y,\ \frac{t}{c}) = 1 - 0 = 1 = \mathfrak{F}'(cx,\ y,\ t)$
If $c < 0$ and $x \neq \underline{0} \neq y$ then
$\mathfrak{F}'(cx,\ y,\ t) = 1 - \vee\{\alpha \in (0,\ 1) : \langle cx,\ y\rangle_\alpha \geq t\}$
$= 1 - \vee\{\alpha \in (0,\ 1) : \langle x,\ y\rangle_\alpha \leq \frac{t}{c}\} \qquad = 1 - \mathfrak{F}'(x,\ y,\ \frac{t}{c})$.
(FI-5) Let $p = \mathfrak{F}'(x,\ z,\ t),\ q = \mathfrak{F}'(y,\ z,\ s)$ and $p \leq q$.
If $p = 0$ or $q = 0$, then obviously the relation holds. Let $0 < r < p \leq q$. Then $\exists\ \alpha > r$ such that $\langle x,\ z\rangle_\alpha \leq t$ and $\exists\ \beta > r$ such that $\langle y,\ z\rangle_\beta \leq s$. Take $\gamma = \alpha \wedge \beta > r$. Therefore $\langle x,\ z\rangle_\gamma \leq \langle x,\ z\rangle_\alpha \leq t$ and $\langle y,\ z\rangle_\gamma \leq \langle y,\ z\rangle_\beta \leq s$.

$\Rightarrow \langle x+y,\ z\rangle_\gamma \le \langle x,\ z\rangle_\gamma + \langle y,\ z\rangle_\gamma \le t+s$
$\Rightarrow \mathfrak{F}'(x+y,\ z,\ t+s) \ge \gamma > r$.
Since $0 < r < \gamma$ is arbitrary, thus
$\mathfrak{F}'(x+y,\ z,\ t+s) \ge p = \mathfrak{F}'(x,\ z,\ t) \wedge \mathfrak{F}'(y,\ z,\ s)$
Similarly if $p \ge q$, then the relation also holds.
(FI-6) Clearly $\lim\limits_{t\to+\infty} \mathfrak{F}(x,\ y,\ t) = 1$.
Therefore $\mathfrak{F}'$ is a fuzzy real inner product on X. □

Definition 4.4. Let X be a linear space over R and $\mathfrak{F}$ be a fuzzy real inner product on X. For $x,\ y \in X$, we define
$\mathfrak{F}(x,\ y,\ t+) = \mathfrak{F}_+(x,\ y,\ t) = \lim\limits_{s\to t+} \mathfrak{F}(x,\ y,\ s)$ and
$\mathfrak{F}(x,\ y,\ t-) = \mathfrak{F}_-(x,\ y,\ t) = \lim\limits_{s\to t-} \mathfrak{F}(x,\ y,\ s)$ for $x,\ y \in X$.

Theorem 4.5. *Let X be a linear space over R and $\mathfrak{F}_1,\ \mathfrak{F}_2$ be two fuzzy real inner products on X satisfying (FI-8). Then $\forall\ x,\ y \in X,\ \ \forall\ t \in R$*
$\mathfrak{F}_1(x,\ y,\ t+) = \mathfrak{F}_2(x,\ y,\ t+)$ and $\mathfrak{F}_1(x,\ y,\ t-) = \mathfrak{F}_2(x,\ y,\ t-)$ iff
$\langle x,\ y\rangle^1_\alpha = \langle x,\ y\rangle^2_\alpha,\ \ \forall\ \alpha \in (0,\ 1)$
where $\langle\ ,\ \rangle^1_\alpha$ and $\langle\ ,\ \rangle^2_\alpha$ denote the corresponding α-inner products of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ respectively.

Proof. First suppose $\langle x,\ y\rangle^1_\alpha = \langle x,\ y\rangle^2_\alpha,\ \ \forall\ \alpha \in (0,\ 1)$.
If possible suppose that for some $t = t_0,\ \mathfrak{F}_1(x,\ y,\ t_0+) \ne \mathfrak{F}_2(x,\ y,\ t_0+)$.
Without loss of generality we may assume $\mathfrak{F}_1(x,\ y,\ t_0+) < \mathfrak{F}_2(x,\ y,\ t_0+)$.
Then for $t_0 < t < t_0 + \epsilon\ (\epsilon > 0),\ \mathfrak{F}_1(x,\ y,\ t) < \mathfrak{F}_2(x,\ y,\ t)$.
Choose β such that $\mathfrak{F}_1(x,\ y,\ t) < \beta < \mathfrak{F}_2(x,\ y,\ t)$.
Note that $\langle x,\ y\rangle^1_\alpha = \wedge\{t \in R : \mathfrak{F}_1(x,\ y,\ t) \ge \alpha\}$ and
$\langle x,\ y\rangle^2_\alpha = \wedge\{t \in R : \mathfrak{F}_2(x,\ y,\ t) \ge \alpha\}\ \ \forall \alpha \in (0,\ 1]$. Therefore $\langle x,\ y\rangle^2_\beta \le t_0$ and $\langle x,\ y\rangle^1_\beta \ge t_0 + \epsilon$ -a contradiction.
Hence $\mathfrak{F}_1(x,\ y,\ t+) = \mathfrak{F}_2(x,\ y,\ t+)\ \ \forall\ t \in R$.
Similarly we can prove $\mathfrak{F}_1(x,\ y,\ t-) = \mathfrak{F}_2(x,\ y,\ t-)\ \ \forall\ t \in R$.
Conversely suppose $\mathfrak{F}_1(x,\ y,\ t+) = \mathfrak{F}_2(x,\ y,\ t+)$ and $\mathfrak{F}_1(x,\ y,\ t-) = \mathfrak{F}_2(x,\ y,\ t-)$ holds $\forall\ t \in R$. If possible let $\exists\ \alpha_0 \in (0,\ 1)$ such that $\langle x,\ y\rangle^1_{\alpha_0} \ne \langle x,\ y\rangle^2_{\alpha_0}$.
Without loss of generality we can choose $k_1,\ k_2,\ k_3$ such that
$\langle x,\ y\rangle^1_{\alpha_0} > k_1 > k_2 > k_3 > \langle x,\ y\rangle^2_{\alpha_0}$.
Therefore $\mathfrak{F}_1(x,\ y,\ k_1) < \alpha_0,\ \mathfrak{F}_2(x,\ y,\ k_3) \ge \alpha_0$.
Thus $\alpha_0 > \mathfrak{F}_1(x,\ y,\ k_1) \ge \mathfrak{F}_1(x,\ y,\ k_2+),\ \mathfrak{F}_2(x,\ y,\ k_2-) \ge \mathfrak{F}_2(x,\ y,\ k_3) \ge \alpha_0$.
Combining we have $\mathfrak{F}_1(x,\ y,\ k_2+) < \alpha_0 \le \mathfrak{F}_2(x,\ y,\ k_2-) \le \mathfrak{F}_2(x,\ y,\ k_2+)$
$\Rightarrow \mathfrak{F}_1(x,\ y,\ k_2+) < \mathfrak{F}_2(x,\ y,\ k_2+)$ - contradiction.
Therefore $\langle x,\ y\rangle^1_\alpha = \langle x,\ y\rangle^2_\alpha,\ \ \forall\ \alpha \in (0,\ 1)\ \ \forall\ x \in X$. □

Definition 4.6. Let X be a linear space over R and $\mathfrak{F}_1,\ \mathfrak{F}_2$ be two fuzzy real inner products on X. Then $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are said to be equipotent if
$\mathfrak{F}_1(x,\ y,\ t-) = \mathfrak{F}_2(x,\ y,\ t-)$ and $\mathfrak{F}_1(x,\ y,\ t+) = \mathfrak{F}_2(x,\ y,\ t+)\ \forall\ t \in R,\ \ \forall\ x,\ y \in X$.

Theorem 4.7. *Let $(X,\ \mathfrak{F})$ be a fuzzy real inner product space satisfying (FI-8) and $\langle\ ,\ \rangle_\alpha,\quad \alpha \in (0,\ 1)$ denotes the α-inner product of $\mathfrak{F}$. Define,*
for $c \geq 0$

$$\mathfrak{F}'(cx,\ y,\ t) = \begin{cases} 0 & t = 0 \text{ and either } x = 0 \text{ or } y = 0. \\ \vee\{\alpha \in (0,\ 1) : \langle cx,\ y\rangle_\alpha \leq t\} & \text{otherwise} \end{cases}$$

for $c < 0$

$$\mathfrak{F}'(cx,\ y,\ t) = \begin{cases} 1 & t = 0 \text{ and either } x = 0 \text{ or } y = 0. \\ 1 - \vee\{\alpha \in (0,\ 1) : \langle cx,\ y\rangle_\alpha \geq t\} & \text{otherwise.} \end{cases}$$

Then $\mathfrak{F}'$ is a fuzzy real inner product on X and $\mathfrak{F},\ \mathfrak{F}'$ are equipotent.

Proof. It has been proved that $\mathfrak{F}'$ is a fuzzy inner product on X.
We have $\langle x,\ y\rangle_\alpha = \wedge\{t \in R : \mathfrak{F}(x,\ y,\ t) \geq \alpha\},\quad \alpha \in (0,\ 1)$
If possible suppose that for some $t = t_0 \in R,\ \mathfrak{F}(x,\ y,\ t_0-) \neq \mathfrak{F}'(x,\ y,\ t_0-)$.
Without loss of generality we may suppose $\mathfrak{F}(x,\ y,\ t_0-) < \mathfrak{F}'(x,\ y,\ t_0-)$.
Choose β such that $\mathfrak{F}(x,\ y,\ t_0-) < \beta < \mathfrak{F}'(x,\ y,\ t_0-)$.
For $t_0 - \epsilon < t < t_0\ (\epsilon > 0),\ \mathfrak{F}(x,\ y,\ t) < \beta < \mathfrak{F}'(x,\ y,\ t)$.
Now for $t_0 - \epsilon < t < t_0,\ \mathfrak{F}(x,\ y,\ t) < \beta\ \Rightarrow\ \langle x,\ y\rangle_\beta \geq t_0$ and
$\mathfrak{F}'(x,\ y,\ t) > \beta\ \Rightarrow\ \langle x,\ y\rangle_\beta \leq t\quad \forall\, t \in (t_0 - \epsilon,\ t_0)$.
Thus we arrive at a contradiction.
Therefore $\mathfrak{F}(x,\ y,\ t_0-) = \mathfrak{F}'(x,\ y,\ t_0-)$.
Similarly we can prove $\mathfrak{F}(x,\ y,\ t_0+) = \mathfrak{F}'(x,\ y,\ t_0+)$.
Hence $\mathfrak{F}$ and $\mathfrak{F}'$ are equipotent. □

Theorem 4.8. *Let $(X,\ \mathfrak{F})$ be a fuzzy real inner product space satisfying (FI-8). Assume that*
(FI-9) *For $x \neq \underline{0} \neq y,\quad \mathfrak{F}(x,\ y,\ .)$ is a continuous function of R.*
Let $\langle x,\ y\rangle_\alpha = \wedge\{t \in R : \mathfrak{F}(x,\ y,\ t) \geq \alpha\},\quad \alpha \in (0,\ 1)$ and
define a function $\mathfrak{F}' : X \times X \times R \to [0,\ 1]$ as
for $c \geq 0$

$$\mathfrak{F}'(cx,\ y,\ t) = \begin{cases} 0 & t = 0 \text{ and either } x = 0 \text{ or } y = 0. \\ \vee\{\alpha \in (0,\ 1); \langle cx,\ y\rangle_\alpha \leq t\} & \text{otherwise} \end{cases}$$

for $c < 0$

$$\mathfrak{F}'(cx,\ y,\ t) = \begin{cases} 1 & t = 0 \text{ and either } x = 0 \text{ or } y = 0. \\ 1 - \vee\{\alpha \in (0,\ 1); \langle cx,\ y\rangle_\alpha \geq t\} & \text{otherwise.} \end{cases}$$

Then
(i) $\{\langle\ ,\ \rangle_\alpha :\ \alpha \in (0,\ 1)\}$ is an ascending family of inner products on X.
(ii) $\mathfrak{F}'$ is a fuzzy real inner product on X.
(iii) $\mathfrak{F}' = \mathfrak{F}$.

To prove this theorem first we prove the following lemma.

Lemma 4.9. *Let $(X,\ \mathfrak{F})$ be a fuzzy real inner product space satisfying (FI-8), $x_0,\ y_0(\neq \underline{0}) \in X$ and $\langle\ ,\ \rangle_\alpha$ be the α-inner product on X, $\alpha \in (0,\ 1)$. Then*
(1) if $\mathfrak{F}(x_0,\ y_0,\ .)$ is upper semicontinous and if for $t_0 \in R,\ \mathfrak{F}(x_0,\ y_0,\ t_0) = \alpha_0 \in$

(0, 1) *then* $\mathfrak{F}(x_0,\ y_0,\ \langle x_0, y_0\rangle_{\alpha_0}) = \alpha_0$.
(2) if $\mathfrak{F}(x_0,\ y_0,\ .)$ *is continuous, then for any* $\alpha \in (0,\ 1)$, $\mathfrak{F}(x_0,\ y_0,\ \langle x_0, y_0\rangle_{\alpha}) = \alpha$.

Proof. (1) From definition
$\langle x_0, y_0\rangle_{\alpha_0} = \wedge\{t \in R : \mathfrak{F}(x_0,\ y_0,\ t) \geq \alpha_0\}$.
Since $\mathfrak{F}(x_0,\ y_0,\ t_0) = \alpha_0$, we get $\langle x_0,\ y_0\rangle_{\alpha_0} \leq t_0$.
Since $\mathfrak{F}(x_0,\ y_0,\ .)$ is nondecreasing, we have $\alpha_0 = \mathfrak{F}(x_0,\ y_0,\ t_0) \geq \mathfrak{F}(x_0,\ y_0,\ \langle x_0, y_0\rangle_{\alpha_0})$
$\Rightarrow \mathfrak{F}(x_0,\ y_0,\ \langle x_0, y_0\rangle_{\alpha_0} \leq \alpha_0$.
If possible suppose that $\mathfrak{F}(x_0,\ y_0,\ \langle x_0, y_0\rangle_{\alpha_0}) < \alpha_0$.
Then by the upper semicontinuity of $\mathfrak{F}(x_0,\ y_0,\ .)$, $\exists\ t' > \langle x_0, y_0\rangle_{\alpha_0}$ such that $\mathfrak{F}(x_0,\ y_0,\ t') < \alpha_0$.
Then $\langle x_0, y_0\rangle_{\alpha_0} = \wedge\{t \in R : \mathfrak{F}(x_0,\ y_0,\ t) \geq \alpha_0\} \geq t' > \langle x_0, y_0\rangle_{\alpha_0}$ -a contradiction.
Therefore $\mathfrak{F}(x_0,\ y_0,\ \langle x_0, y_0\rangle_{\alpha_0} = \alpha_0$.

(2) Since $\mathfrak{F}(x_0,\ y_0,\ .)$ is continuous, by (FI-6) for each $\alpha \in (0,\ 1)$, $\exists\ t \in R$ such that $\mathfrak{F}(x_0,\ y_0,\ t) = \alpha$.
Then by (1), the proof follows. □

Now we prove the Theorem 4.8.

Proof. Points (i) and (ii) follows from Theorem 4.1 and Theorem 4.3.
For (iii) consider the following cases.
(Case-I) Let $x_0 = \underline{0} = y_0$ and $t_0 = 0$.
Then $\mathfrak{F}(x_0,\ y_0,\ t_0) = 0 = \mathfrak{F}'(x_0,\ y_0,\ t_0)$ [by definition].
(Case-II) Let $x_0 = \underline{0} = y_0$ and $t_0 \neq 0$.
$\mathfrak{F}(x_0,\ y_0,\ t_0) = 1$ if $t_0 > 0$ and $= 0$ if $t_0 < 0$.
Now $\langle x_0, y_0\rangle_{\alpha} = 0\ \ \forall\ \alpha \in (0,\ 1)$
$\mathfrak{F}'(x_0,\ y_0,\ t_0) = \vee\{\alpha \in (0,\ 1); 0 \leq t_0\}$
If $t_0 > 0$, then $\mathfrak{F}'(x_0,\ y_0,\ t_0) = \vee\{\alpha \in (0,\ 1);\ \alpha \in (0,\ 1)\} = 1$
If $t_0 < 0$, then $\mathfrak{F}'(x_0,\ y_0,\ t_0) = \vee\{\phi\} = 0$
(Case-III) Either x_0 or y_0 is zero and $t_0 \in R$.
Then $\mathfrak{F}(x_0,\ y_0,\ t_0) = H(t_0)$.
By definition $\mathfrak{F}'(x_0,\ y_0,\ t_0) = 0$ if $t_0 = 0$ and
$\langle x_0, y_0\rangle_{\alpha} = 0\ \ \forall\ \alpha \in (0,\ 1)$.
Therefore if $t_0 > 0$ then $\mathfrak{F}'(x_0,\ y_0,\ t_0) = 1$ and if $t_0 < 0$ then $\mathfrak{F}'(x_0,\ y_0,\ t_0) = \vee\phi = 0$.
Thus $\mathfrak{F}'(x_0,\ y_0,\ t_0) = H(t_0)$.
(Case-IV) If $x_0 \neq \underline{0} \neq y_0$, $t_0 \in R$ and $\mathfrak{F}(x_0,\ y_0,\ t_0) = 0$
For $\alpha \in (0,\ 1)$, $\langle x_0, y_0\rangle_{\alpha} = \wedge\{t \in R : \mathfrak{F}(x_0,\ y_0,\ t) \geq \alpha\}$
By Lemma 4.9(2), we have $\forall\ \alpha \in (0,\ 1)$, $\mathfrak{F}(x_0,\ y_0,\ \langle x_0, y_0\rangle_{\alpha}) = \alpha$
Since $\mathfrak{F}(x_0,\ y_0,\ t_0) = 0 < \alpha$, it follows that $t_0 < \langle x_0, y_0\rangle_{\alpha}$, $\ \forall\ \alpha \in (0,\ 1)$
Therefore $\mathfrak{F}'(x_0,\ y_0,\ t_0) = \vee\{\alpha \in (0,\ 1); \langle x_0, y_0\rangle_{\alpha} \leq t_0\} = \vee\{\phi\} = 0$.
Hence $\mathfrak{F}(x_0,\ y_0,\ t_0) = \mathfrak{F}'(x_0,\ y_0,\ t_0) = 0$.
(Case-V) When $x_0 \neq \underline{0} \neq y_0$, $t_0 \in R$ and $0 < \mathfrak{F}(x_0,\ y_0,\ t_0) < 1$.
Let $\mathfrak{F}(x_0,\ y_0,\ t_0) = \alpha_0$. Then $0 < \alpha_0 < 1$.
Now $\mathfrak{F}'(x,\ y,\ t) = \vee\{\alpha \in (0,\ 1);\ \langle x,\ y\rangle_{\alpha} \leq t\}$ when $x \neq \underline{0} \neq y$
$\langle x,\ y\rangle_{\alpha} = \wedge\{t \in R : \mathfrak{F}(x,\ y,\ t) \geq \alpha\}$, $\ x,\ y \in X,\ 0 < \alpha < 1$.

Since $\mathfrak{F}(x_0,\ y_0,\ t_0) = \alpha_0$, we have $\langle x_0, y_0\rangle_{\alpha_0} \leq t_0$ and hence $\mathfrak{F}'(x_0,\ y_0,\ t_0) \geq \alpha_0$
$\Rightarrow \mathfrak{F}'(x_0,\ y_0,\ t_0) \geq \mathfrak{F}(x_0,\ y_0,\ t_0)$. (4.8.1)
Now from Lemma 4.9(1), we have $\langle x_0, y_0\rangle_{\alpha_0} = t_0$
For $1 > \alpha > \alpha_0$, let $\langle x_0, y_0\rangle_{\alpha} = t'$. Then $t' \geq t_0$.
By Lemma 4.9(2), $\mathfrak{F}(x_0,\ y_0,\ t') = \alpha$.
So, $\mathfrak{F}(x_0,\ y_0,\ t') = \alpha > \alpha_0 = \mathfrak{F}(x_0,\ y_0,\ t_0)$.
Since $\mathfrak{F}(x_0,\ y_0,\ .)$ is nondecreasing, $\mathfrak{F}(x_0,\ y_0,\ t') > \alpha_0 = \mathfrak{F}(x_0,\ y_0,\ t_0)\ \Rightarrow t' > t_0$.
So, for $1 > \alpha > \alpha_0$, $\langle x_0, y_0\rangle_{\alpha} = t' > t_0$.
Hence $\mathfrak{F}'(x_0,\ y_0,\ t_0) \leq \alpha_0 = \mathfrak{F}(x_0,\ y_0,\ t_0)$. (4.8.2)
By (4.8.1) and (4.8.2) we have $\mathfrak{F}'(x_0,\ y_0,\ t_0) = \mathfrak{F}(x_0,\ y_0,\ t_0)$.
(Case-VI) When $x_0 \neq \underline{0} \neq y_0$, $t_0 \in R$ and $\mathfrak{F}(x_0,\ y_0,\ t_0) = 1$.
Here $\langle x_0, y_0\rangle_{\alpha} \leq t_0\ \ \forall\ \alpha \in (0,\ 1)\ \ \Rightarrow \mathfrak{F}'(x_0,\ y_0,\ t_0) = 1$.
Thus $\mathfrak{F}'(x_0,\ y_0,\ t_0) = \mathfrak{F}(x_0,\ y_0,\ t_0) = 1$.
Hence $\mathfrak{F}' = \mathfrak{F}$ in any case. □

5. Conclusion.

Following the definition of fuzzy inner product introduced by Goudarzi & Vaezpour, we introduce a definition of fuzzy inner product using particular t-norm as "min". From this definition we are able to establish a decomposition theorem from a fuzzy inner product into an ascending family of crisp inner product. We see that only a few number of papers have been published in fuzzy inner product spaces by using the previous definitions introduced by different authors. There is a wide scope to develop fuzzy inner product space viz. fuzzy operator theory, fuzzy spectral theory etc. We think that decomposition theorem which are established in this paper will be helpful to study fuzzy inner product spaces to a large extent.

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