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Linear equations of generalised triangular fuzzy numbers

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Abstract. A necessary and sufficient condition for the existence of solution of linear equation by Buckley [Fuzzy Sets and Systems 38, (1990), 43-59] are generalized on generalized triangular fuzzy numbers. The arithmetic of generalised fuzzy numbers is used to show the existence of solution of the linear equation with such types fuzzy numbers.

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1. Introduction

The necessary and sufficient condition for the existence of solution of systems fuzzy equations was proposed in [17]. Work that are done on solving linear and quadratic fuzzy equations are available in [1, 2, 3, 5, 6, 7, 8, 20]. In [20] necessary and sufficient conditions for the existences of solution of $\tilde{A} + \tilde{X} = \tilde{C}$ and $\tilde{A}\tilde{X} = \tilde{C}$ when $\tilde{A}$ and $\tilde{C}$ are arbitrary fuzzy subsets of the real are discussed. In [6] the author argued that only simple equations can be solved using the concept of α-cuts of fuzzy numbers. A new solution method based on the concept of the united extension of a function from interval analysis [18] was proposed in [2]. Buckley [4, 5] used evolutionary algorithms and neural nets to solve linear and quadratic equations of fuzzy numbers. Certain decomposition of the coefficient matrix of the system of fully fuzzy linear equations to obtain a new method for solving these systems is proposed in [19]. A brief review of generalised fuzzy numbers with applications and there basic arithmetic are available in [9, 12, 13, 14, 15].

In this paper, the existence of solution of linear equation of non-normal or generalised triangular fuzzy number are discussed. The necessary and sufficient condition for the existence of solution are being discussed, which are put forwarded in the form of theorem. The coefficient of the linear equations are considered generalised triangular fuzzy number.

2. Preliminaries

Chen [10] proposed the concept of generalised fuzzy number. Let $\tilde{A}$ be generalised fuzzy number such that $\tilde{A} = (a_1, a_2, a_3, a_4; w_{\tilde{A}})$ where a_1, a_2, a_3 and a_4 are real values, $a_1 < a_2 < a_3 < a_4$, $w_{\tilde{A}}$ is the height of the generalised fuzzy number and $0 \leq w_{\tilde{A}} < 1$. The generalised fuzzy number $\tilde{A}$ is defined in a closed bounded interval and it is convex. The generalised fuzzy number reduces to normal fuzzy number if the height $w_{\tilde{A}} = 1$. Moreover the generalised fuzzy number reduces to generalised triangular fuzzy number if $a_3 = a_4$ and a triangular fuzzy number if $w_{\tilde{A}} = 1$. The support of the generalised fuzzy number is defined as the real interval $[a_1, a_4]$. The main difference between generalised fuzzy number and a fuzzy number is that the height of a fuzzy number is always 1 and the height of generalised fuzzy number lies within the interval $[0, 1]$. A generalised triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3; w_{\tilde{A}}) > 0$ if $a_1 > 0$, $\tilde{A} \geq 0$ if $a_1 \leq 0$, $\tilde{A} < 0$ if $a_3 < 0$ and $\tilde{A} \leq 0$ if $a_1 \leq 0$.

Chen [11, 16] proposed the arithmetic of generalised fuzzy numbers. In this paper, only the arithmetic of generalised triangular fuzzy number are defined as they are being used. Let $\tilde{A} = (a_1, a_2, a_3; w_{\tilde{A}})$ and $\tilde{B} = (b_1, b_2, b_3; w_{\tilde{B}})$ be two generalised triangular fuzzy numbers, such that $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$ are real values, the arithmetic of generalised real fuzzy numbers are defined as follows:

(1) Generalised triangular fuzzy number addition.

$$\begin{aligned} \tilde{A} \oplus \tilde{B} &= (a_1, a_2, a_3; w_{\tilde{A}}) \oplus (b_1, b_2, b_3; w_{\tilde{B}}) \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3; min(w_{\tilde{A}}, w_{\tilde{B}})) \end{aligned}$$

(2) Generalised triangular fuzzy number subtraction.

$$\begin{aligned} \tilde{A} \ominus \tilde{B} &= (a_1, a_2, a_3; w_{\tilde{A}}) \ominus (b_1, b_2, b_3; w_{\tilde{B}}) \\ &= (a_1 - b_3, a_2 - b_2, a_3 - b_1; min(w_{\tilde{A}}, w_{\tilde{B}})) \end{aligned}$$

(3) Generalised triangular fuzzy number multiplication.

$$\begin{aligned} \tilde{A} \otimes \tilde{B} &= (a_1, a_2, a_3; w_{\tilde{A}}) \otimes (b_1, b_2, b_3; w_{\tilde{B}}) \\ &= (a, a_2 \times b_2, c; min(w_{\tilde{A}}, w_{\tilde{B}})) \end{aligned}$$

where, $a = min(a_1 \times b_1, a_1 \times b_3, a_3 \times b_1, a_3 \times b_3)$ and
$c = max(a_1 \times b_1, a_1 \times b_3, a_3 \times b_1, a_3 \times b_3)$

(4) Generalised triangular fuzzy number division.

$$\begin{aligned} \tilde{A} \oslash \tilde{B} &= (a_1, a_2, a_3; w_{\tilde{A}}) \oslash (b_1, b_2, b_3; w_{\tilde{B}}) \\ &= (a, a_2 \div b_2, c; min(w_{\tilde{A}}, w_{\tilde{B}})) \end{aligned}$$

where, $a = min(a_1 \div b_1, a_1 \div b_3, a_3 \div b_1, a_3 \div b_3)$ and
$c = max(a_1 \div b_1, a_1 \div b_3, a_3 \div b_1, a_3 \div b_3)$

3. Main results

Theorem 3.1. *The equation $\tilde{A} \oplus \tilde{X} = \tilde{C}$ has a solution $\tilde{X} = (x_1, x_2, x_3; w_{\tilde{X}})$ if and only if $c_1 - a_1 < c_2 - a_2 < c_3 - a_3$ and $w_{\tilde{X}} = min(w_{\tilde{C}}, w_{\tilde{A}})$, where $\tilde{A} = (a_1, a_2, a_3; w_{\tilde{A}})$ and $\tilde{C} = (c_1, c_2, c_3; w_{\tilde{C}})$ are the generalised triangular fuzzy number.*

Proof. Let us assume that the solution exist and it be $\tilde{X} = (x_1, x_2, x_3; w_{\tilde{X}})$, where $x_1 < x_2 < x_3$. This solution satisfies the equation $\tilde{A} \oplus \tilde{X} = \tilde{C}$. Therefore,

$$(a_1, a_2, a_3; w_{\tilde{A}}) \oplus (x_1, x_2, x_3; w_{\tilde{X}}) = (c_1, c_2, c_3; w_{\tilde{C}})$$
$$a_1 + x_1 = c_1$$
$$a_2 + x_2 = c_2$$
$$a_3 + x_3 = c_3$$
$$min(w_{\tilde{A}}, w_{\tilde{X}}) = w_{\tilde{C}}.$$

Now $x_1 < x_2 < x_3$, implies $c_1 - a_1 < c_2 - a_2 < c_3 - a_3$. Also $min(w_{\tilde{A}}, w_{\tilde{X}}) = w_{\tilde{C}}$, implies $w_{\tilde{A}} = w_{\tilde{C}}$ or $w_{\tilde{X}} = w_{\tilde{C}}$. If $w_{\tilde{A}} = w_{\tilde{C}}$ then $w_{\tilde{A}} = w_{\tilde{C}} < w_{\tilde{X}}$ which is invalid. If $w_{\tilde{X}} = w_{\tilde{C}}$ then $w_{\tilde{X}} = w_{\tilde{C}} < w_{\tilde{A}}$, implies $min(w_{\tilde{C}}, w_{\tilde{A}}) = w_{\tilde{X}}$.

For the sufficient condition, let $x_1 = c_1 - a_1$, $x_2 = c_2 - a_2$, and $x_3 = c_3 - a_3$. Since $x_1 < x_2 < x_3$ and $min(w_{\tilde{A}}, w_{\tilde{C}}) = w_{\tilde{X}}$, $\tilde{X} = (x_1, x_2, x_3; w_{\tilde{X}})$ is a generalised triangular fuzzy number. It is obvious that it satisfies the equation $\tilde{A} \oplus \tilde{X} = \tilde{C}$, implies that solution exist. □

Example 3.2. Let $\tilde{A} = (2, 5, 7; 0.5)$ and $\tilde{C} = (1, 2, 3; 0.7)$, the solution exist when $1 - 2 < 2 - 5 < 3 - 7$ but it is not true hence the solution doesn't exit for the linear equation $\tilde{A} \oplus \tilde{X} = \tilde{C}$.

Example 3.3. Let $\tilde{A} = (0, 4, 6; 0.7)$ and $\tilde{C} = (0, 6, 10; 0.5)$, the solution exist for the linear equation $\tilde{A} \oplus \tilde{X} = \tilde{C}$ and it is $\tilde{X} = (0, 2, 4; 0.5)$.

Theorem 3.4. *For generalised fuzzy numbers $\tilde{A}$ and $\tilde{C}$ such that zero doesn't belong to the support of $\tilde{A}$ and $\tilde{C}$, the linear equation $\tilde{A} \otimes \tilde{X} = \tilde{C}$ has a solution $\tilde{X} = (x_1, x_2, x_3; w_{\tilde{X}})$, where $\tilde{A} = (a_1, a_2, a_3; w_{\tilde{A}})$ and $\tilde{C} = (c_1, c_2, c_3; w_{\tilde{C}})$ are real valued generalised fuzzy number, if and only if*

(1) $\frac{c_1}{a_1} < \frac{c_2}{a_2} < \frac{c_3}{a_3}$ *when* $\tilde{A} > 0$, $\tilde{C} \geq 0$;
(2) $\frac{c_3}{a_3} < \frac{c_2}{a_2} < \frac{c_1}{a_1}$ *when* $\tilde{A} < 0$, $\tilde{C} \leq 0$;
(3) $\frac{c_1}{a_3} < \frac{c_2}{a_2} < \frac{c_3}{a_1}$ *when* $\tilde{A} > 0$, $\tilde{C} \leq 0$;
(4) $\frac{c_3}{a_1} < \frac{c_2}{a_2} < \frac{c_1}{a_3}$ *when* $\tilde{A} < 0$, $\tilde{C} \geq 0$;

and $min(w_{\tilde{A}}, w_{\tilde{C}}) = w_{\tilde{X}}$.

Proof. (1) Let solution exist when $\tilde{A} > 0$, $\tilde{C} \geq 0$ and it is $\tilde{X} = (x_1, x_2, x_3; w_{\tilde{X}})$, where $x_1 < x_2 < x_3$ and are real positive such that $x_1 \geq 0$. This solution satisfies the equation $\tilde{A} \otimes \tilde{X} = \tilde{C}$. Therefore,

$$(a_1, a_2, a_3; w_{\tilde{A}}) \otimes (x_1, x_2, x_3; w_{\tilde{X}}) = (c_1, c_2, c_3; w_{\tilde{C}})$$
$$a_1 \times x_1 = c_1$$
$$a_2 \times x_2 = c_2$$
$$a_3 \times x_3 = c_3$$

$$min(w_{\tilde{A}}, w_{\tilde{X}}) = w_{\tilde{C}}.$$

Now $x_1 < x_2 < x_3$, implies $\frac{c_1}{a_1} < \frac{c_2}{a_2} < \frac{c_3}{a_3}$. Also $min(w_{\tilde{A}}, w_{\tilde{X}}) = w_{\tilde{C}}$, implies $w_{\tilde{A}} = w_{\tilde{C}}$ or $w_{\tilde{X}} = w_{\tilde{C}}$. If $w_{\tilde{A}} = w_{\tilde{C}}$ then $w_{\tilde{A}} = w_{\tilde{C}} < w_{\tilde{X}}$ which is invalid. If $w_{\tilde{X}} = w_{\tilde{C}}$ then $w_{\tilde{X}} = w_{\tilde{C}} < w_{\tilde{A}}$, implies $min(w_{\tilde{C}}, w_{\tilde{A}}) = w_{\tilde{X}}$.

For the sufficient condition, let $x_1 = \frac{c_1}{a_1}$, $x_2 = \frac{c_2}{a_2}$, and $x_3 = \frac{c_3}{a_3}$. Since $x_1 < x_2 < x_3$ and $min(w_{\tilde{A}}, w_{\tilde{C}}) = w_{\tilde{X}}$, $\tilde{X} = (x_1, x_2, x_3; w_{\tilde{X}})$ is a generalised triangular fuzzy number. It is obvious that it satisfies the equation $\tilde{A} \otimes \tilde{X} = \tilde{C}$, implies that solution exist. □

Proof. (2) Let solution exist when $\tilde{A} < 0$, $\tilde{C} \leq 0$ and it is $\tilde{X} = (x_1, x_2, x_3; w_{\tilde{X}})$, where $x_1 < x_2 < x_3$ and are real positive such that $x_1 \geq 0$. This solution satisfies the equation $\tilde{A} \otimes \tilde{X} = \tilde{C}$. Therefore,

$$(a_1, a_2, a_3; w_{\tilde{A}}) \otimes (x_1, x_2, x_3; w_{\tilde{X}}) = (c_1, c_2, c_3; w_{\tilde{C}})$$
$$x_1 = \frac{c_3}{a_3}; x_2 = \frac{c_2}{a_2}; x_3 = \frac{c_1}{c_1}$$
$$min(w_{\tilde{A}}, w_{\tilde{X}}) = w_{\tilde{C}}.$$

Now $x_1 < x_2 < x_3$, implies $\frac{c_3}{a_3} < \frac{c_2}{a_2} < \frac{c_1}{a_1}$. The proof of $min(w_{\tilde{A}}, w_{\tilde{C}}) = w_{\tilde{X}}$ is similar to the proof in part (1).

For the sufficient condition, let $x_1 = \frac{c_3}{a_3}$, $x_2 = \frac{c_2}{a_2}$, and $x_3 = \frac{c_1}{a_1}$. Since $x_1 < x_2 < x_3$ and $min(w_{\tilde{A}}, w_{\tilde{C}}) = w_{\tilde{X}}$, $\tilde{X} = (x_1, x_2, x_3; w_{\tilde{X}})$ is a generalised triangular fuzzy number. It is obvious that it satisfies the equation $\tilde{A} \otimes \tilde{X} = \tilde{C}$, implies that solution exist. □

Proof. (3) Let solution exist when $\tilde{A} > 0$, $\tilde{C} \leq 0$ and it is $\tilde{X} = (x_1, x_2, x_3; w_{\tilde{X}})$, where $x_1 < x_2 < x_3$ and are real negative such that $x_3 \leq 0$. This solution satisfies the equation $\tilde{A} \otimes \tilde{X} = \tilde{C}$. Therefore,

$$(a_1, a_2, a_3; w_{\tilde{A}}) \otimes (x_1, x_2, x_3; w_{\tilde{X}}) = (c_1, c_2, c_3; w_{\tilde{C}})$$
$$x_1 = \frac{c_1}{a_3}; x_2 = \frac{c_2}{a_2}; x_3 = \frac{c_3}{c_1}$$
$$min(w_{\tilde{A}}, w_{\tilde{X}}) = w_{\tilde{C}}.$$

Now $x_1 < x_2 < x_3$, implies $\frac{c_1}{a_3} < \frac{c_2}{a_2} < \frac{c_3}{a_1}$. The proof of $min(w_{\tilde{A}}, w_{\tilde{C}}) = w_{\tilde{X}}$ is similar to the proof in part (1).

For the sufficient condition, let $x_1 = \frac{c_1}{a_3}$, $x_2 = \frac{c_2}{a_2}$, and $x_3 = \frac{c_3}{a_1}$. Since $x_1 < x_2 < x_3$ and $min(w_{\tilde{A}}, w_{\tilde{C}}) = w_{\tilde{X}}$, $\tilde{X} = (x_1, x_2, x_3; w_{\tilde{X}})$ is a generalised triangular fuzzy number. It is obvious that it satisfies the equation $\tilde{A} \otimes \tilde{X} = \tilde{C}$, implies that solution exist. □

Proof. (4) Let solution exist when $\tilde{A} < 0$, $\tilde{C} \geq 0$, and it is $\tilde{X} = (x_1, x_2, x_3; w_{\tilde{X}})$, where $x_1 < x_2 < x_3$ and are real negative such that $x_3 \leq 0$. This solution satisfies the equation $\tilde{A} \otimes \tilde{X} = \tilde{C}$. Therefore,

$$(a_1, a_2, a_3; w_{\tilde{A}}) \otimes (x_1, x_2, x_3; w_{\tilde{X}}) = (c_1, c_2, c_3; w_{\tilde{C}})$$
$$x_1 = \frac{c_3}{a_1}; x_2 = \frac{c_2}{a_2}; x_3 = \frac{c_1}{c_3}$$
$$min(w_{\tilde{A}}, w_{\tilde{X}}) = w_{\tilde{C}}.$$

Now $x_1 < x_2 < x_3$, implies $\frac{c_3}{a_2} < \frac{c_2}{a_2} < \frac{c_1}{a_3}$. The proof of $min(w_{\tilde{A}}, w_{\tilde{C}}) = w_{\tilde{X}}$ is similar to the proof in part (1).

For the sufficient condition, let $x_1 = \frac{c_3}{a_1}$, $x_2 = \frac{c_2}{a_2}$, and $x_3 = \frac{c_1}{a_3}$. Since $x_1 < x_2 < x_3$ and $min(w_{\tilde{A}}, w_{\tilde{C}}) = w_{\tilde{X}}$, $\tilde{X} = (x_1, x_2, x_3; w_{\tilde{X}})$ is a generalised triangular fuzzy number. It is obvious that it satisfies the equation $\tilde{A} \otimes \tilde{X} = \tilde{C}$, implies that solution exist. □

Theorem 3.5. *For generalised fuzzy numbers $\tilde{A}$ and $\tilde{C}$ such that zero belong to the support of $\tilde{C}(c_2 = 0)$, the linear equation $\tilde{A} \otimes \tilde{X} = \tilde{C}$ has a solution $\tilde{X} = (x_1, x_2, x_3; w_{\tilde{X}})$, where $\tilde{A} = (a_1, a_2, a_3; w_{\tilde{A}})$ and $\tilde{C} = (c_1, c_2, c_3; w_{\tilde{C}})$ are real valued generalised fuzzy number, if and only if zero belongs to the support of $\tilde{X}(x_2 = 0)$ and $min(w_{\tilde{A}}, w_{\tilde{C}}) = w_{\tilde{X}}$.*

Proof. The prove of the theorem is obvious. □

Example 3.6. Let $\tilde{A} = (-6, -5, -4; 0.5)$ and $\tilde{C} = (-1, 0, 1; 1)$, the solution of the equation $\tilde{A} \otimes \tilde{X} = \tilde{C}$ exist and it is $\tilde{X} = (\frac{-1}{4}, 0, \frac{1}{4}; 0.5)$.

4. Conclusions

The existence of solution of linear equation of single fuzzy variable is investigated when the coefficients are generalised triangular fuzzy number. The necessary and sufficient condition for the existence of the solution of linear equations are deduced. The method of α-cuts fails if the fuzzy coefficients are of different height, so the method of arithmetic of generalised fuzzy numbers serves a good tool to shows the existence of solution of linear equations with fuzzy coefficients.

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