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Domination in interval-valued fuzzy graphs

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Abstract. The purpose of this paper is to introduce the concept of domination in interval-valued fuzzy graphs. Order of an interval-valued fuzzy graph has been defined and its relation with domination number has been established. Also we give characterization for minimal dominating set and find relations between independent sets and dominating sets. Further, the notion of total dominating set has been introduced and some important results are proved.

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1. Introduction

The earliest idea of domination occurred in the game of chess where the problem was to place minimum number of chess pieces so as to dominate all the squares of the chess board. Mathematical research on the theory of domination for crisp graphs was initiated by Ore [16]. Cockayne and Hedetnieme [5] further developed the concept. Since then an extensive research has been done in this field. Rosenfeld [17] introduced the notion of fuzzy graphs in 1975. Some important works in fuzzy graph theory can be found in [2, 3, 4, 13, 14, 15]. Domination in fuzzy graphs was introduced and studied by Somasundaram and Somasundaram in [20]. Recently, the concept has been studied by Mohideen and Ismayil [12].

Zadeh [24] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets [23] which gives a more precise tool to model uncertainty in real life situations. Some recent work of Zadeh in connection with the importance of fuzzy logic may be found in [25, 26]. Interval-valued fuzzy sets have been widely used in many areas of science and engineering, e.g., in approximate reasoning [6, 7], medical diagnosis [18], multivalued logic [22], intelligent control [11], topological spaces [19] etc. Hongmei and Lianhua introduced the definition of interval-valued fuzzy graphs in [9]. Recently, Akram and Dudek [1] have studied several properties and operations on interval-valued fuzzy graphs. Isomorphism on interval valued fuzzy graph has been studied by Talebi and Rashmanlou [21].

In this paper, our aim is to introduce and study the theory of domination in the setting of interval-valued fuzzy graphs. Cardinality of an intuitionistic fuzzy set was defined by Lou and Yu in [10]. Following the same line, we have defined cardinality of an interval-valued fuzzy graph.

2. Preliminaries

Throughout this paper a graph will denote a graph without loops. For graph theoretic notations and terminologies, the readers are referred to Hararay [8]. First we collect some definitions to be used in this paper.

Definition 2.1. An interval-valued fuzzy set A on a set V is defined by

$$A = \{(x, [\mu_A^-(x), \mu_A^+(x)]) : x \in V\},$$

where μ_A^- and μ_A^+ are fuzzy subsets of V such that $\mu_A^-(x) \leq \mu_A^+(x)$ for all $x \in V$.

If $G^* = (V, E)$ is a crisp graph, then by an interval-valued fuzzy relation B on V we mean an interval-valued fuzzy set on E such that

$$\mu_B^-(xy) \leq \min\{\mu_A^-(x), \mu_A^-(y)\} \text{ and } \mu_B^+(xy) \leq \max\{\mu_A^+(x), \mu_A^+(y)\}$$

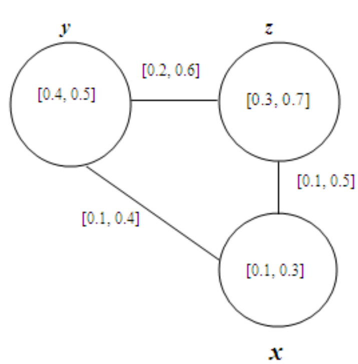
for all $xy \in E$ and we write $B = \{xy, [\mu_B^-(xy), \mu_B^+(xy)] : xy \in E\}$.

Definition 2.2. An interval-valued fuzzy graph (in short, IVFG) of a graph $G^* = (V, E)$ is a pair $G = (A, B)$, where $A = [\mu_A^-, \mu_A^+]$ is an interval-valued fuzzy set on V and $B = [\mu_B^-, \mu_B^+]$ is an interval-valued fuzzy relation on V.

Example 2.3. Consider the graph $G^* = (V, E)$, where $V = \{x, y, z\}$ and $E = \{xy, yz, zx\}$. Let A be an interval-valued fuzzy set on V and let B be an interval-valued fuzzy set on $E \subseteq V \times V$ defined by

$$A = \left\langle \left(\frac{x}{0.1}, \frac{y}{0.4}, \frac{z}{0.3}\right), \left(\frac{x}{0.3}, \frac{y}{0.5}, \frac{z}{0.7}\right) \right\rangle,$$
$$B = \left\langle \left(\frac{xy}{0.1}, \frac{yz}{0.2}, \frac{zx}{0.1}\right), \left(\frac{xy}{0.4}, \frac{yz}{0.6}, \frac{zx}{0.5}\right) \right\rangle.$$

Then $G = (A, B)$ is an IVFG of $G^* = (V, E)$.



Definition 2.4. The order p and size q of an IVFG $G = (A, B)$ of a graph $G^* = (V, E)$ are defined to be

$$p = \sum_{v \in V} \frac{1 + \mu_A^+(v) - \mu_A^-(v)}{2}$$

and

$$q = \sum_{xy \in E} \frac{1 + \mu_B^+(xy) - \mu_B^-(xy)}{2}.$$

Definition 2.5. Let $G = (A, B)$ be an IVFG on $G^* = (V, E)$ and $S \subseteq V$. Then the cardinality of S is defined to be

$$\sum_{v \in S} \frac{1 + \mu_A^+(v) - \mu_A^-(v)}{2}.$$

Definition 2.6. An IVFG $G = (A, B)$ of a graph $G^* = (V, E)$ is said to be complete if $\mu_B^-(xy) = \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $\mu_B^+(xy) = \max\{\mu_A^+(x), \mu_A^+(y)\}$ for all $xy \in E$ and is denoted by K_{μ_A}.

Definition 2.7. The complement of an IVFG $G = (A, B)$ of a graph $G^* = (V, E)$ is the IVFG $\overline{G} = (\overline{A}, \overline{B})$, where $\overline{A} = [\mu_A^-, \mu_A^+]$ and $\overline{B} = [\overline{\mu_B^-}, \overline{\mu_B^+}]$ is defined by

$$\overline{\mu_B^-}(xy) = \min\{\mu_A^-(x), \mu_A^-(y)\} - \mu_B^-(xy),$$

$$\overline{\mu_B^+}(xy) = \max\{\mu_A^+(x), \mu_A^+(y)\} - \mu_B^+(xy)$$

for all $xy \in E$.

Definition 2.8. An IVFG $G = (A, B)$ of a graph $G^* = (V, E)$ is said to be bipartile if the vertex set V can be partitioned into two nonempty sets V_1 and V_2 such that $\mu_B^-(xy) = 0$ and $\mu_B^+(xy) = 0$ if $x, y \in V_1$ or $x, y \in V_2$. Further if $\mu_B^-(xy) = \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $\mu_B^+(xy) = \max\{\mu_A^+(x), \mu_A^+(y)\}$ for all $x \in V_1$ and $y \in V_2$, then G is called a complete bipartile graph and is denoted by $K_{\mu_A^-, \mu_A^+}$, where μ_A^- and μ_A^+ are restrictions of μ_A^- and μ_A on V_1 and V_2 respectively.

Definition 2.9. An edge $e = xy$ of an IVFG G is called an effective edge if $\mu_B^-(xy) = \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $\mu_B^+(xy) = \max\{\mu_A^+(x), \mu_A^+(y)\}$. In this case, the vertex x is called a neighbor of y and conversely.

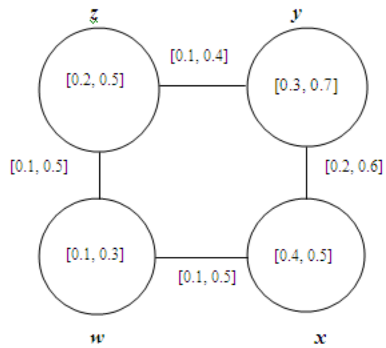
$N(x) = \{y \in V : y$ is a neighbor of $x\}$ is called the neighborhood of x.

Example 2.10. Consider the graph $G^* = (V, E)$, where $V = \{w, x, y, z\}$ and $E = \{wx, xy, yz, zw\}$. Let A be an interval-valued fuzzy set on V and let B be an interval-valued fuzzy set on $E \subseteq V \times V$ defined by

$$A = \left\langle \left(\frac{w}{0.1}, \frac{x}{0.4}, \frac{y}{0.3}, \frac{z}{0.2} \right), \left(\frac{w}{0.3}, \frac{x}{0.5}, \frac{y}{0.7}, \frac{z}{0.5} \right) \right\rangle,$$

$$B = \left\langle \left(\frac{wx}{0.1}, \frac{xy}{0.2}, \frac{yz}{0.1}, \frac{zw}{0.1} \right), \left(\frac{wx}{0.5}, \frac{xy}{0.6}, \frac{yz}{0.4}, \frac{zw}{0.5} \right) \right\rangle.$$

Then $G = (A, B)$ is an IVFG of $G^* = (V, E)$.



In this example, wx and zw are effective edges. Also, $N(w) = \{x, z\}$, $N(x) = \{w\}$, $N(z) = \{w\}$, $N(y) = \phi$ (the empty set).

3. Domination in interval valued fuzzy graphs

We now obtain our main results.

Definition 3.1. Let $G = (A, B)$ be an IVFG on V and $x, y \in V$. We say x dominates y if $\mu_B^-(xy) = \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $\mu_B^+(xy) = \max\{\mu_A^+(x), \mu_A^+(y)\}$.

A subset S of V is called a dominating set in G if for every $v \notin S$, there exists $u \in S$ such that u dominates v.

The minimum cardinality of a dominating set in G is called the domination number of G and is denoted by $\gamma(G)$ or by simply γ.

Remark 3.2. (i) For any $x, y \in V$, if x dominates y then y dominates x and as such domination is a symmetric relation.

(ii) If $\mu_B^-(xy) < \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $\mu_B^+(xy) < \max\{\mu_A^+(x), \mu_A^+(y)\}$ for all $x, y \in V$, then the only dominating set in G is V.

Example 3.3. (i) Since $\{v\}$ is a dominating set of K_{μ_A} for each $v \in V$, we have $\gamma(K_{\mu_A}) = \min_{v\in V} \frac{1+\mu_A^+(v)-\mu_A^-(v)}{2}$.

(ii) $\gamma(\overline{K_{\mu_A}}) = p$.

(iii) $\gamma(K_{\mu_{A^1}, \mu_{A^2}}) = \min_{x\in V_1} \frac{1+\mu_A^+(x)-\mu_A^-(x)}{2} + \min_{y\in V_2} \frac{1+\mu_A^+(y)-\mu_A^-(y)}{2}$.

Theorem 3.4. *For any IVFG G, $\gamma + \overline{\gamma} \leq 2p$, where $\overline{\gamma}$ is the domination number of $\overline{G}$ and the equality holds if and only if $0 < \mu_B^-(xy) < \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $0 < \mu_B^+(xy) < \max\{\mu_A^+(x), \mu_A^+(y)\}$ for all $x, y \in V$.*

Proof. The inequality immediately follows. Now $\gamma = p$ if and only if the only domination set in G is V. i.e., if and only if $\mu_B^-(xy) < \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $\mu_B^+(xy) < \max\{\mu_A^+(x), \mu_A^+(y)\}$.

Again $\overline{\gamma} = p$ if and only if $\min\{\mu_A^-(x), \mu_A^-(y)\} - \mu_B^-(xy) < \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $\max\{\mu_A^+(x), \mu_A^+(y)\} - \mu_B^+(xy) < \max\{\mu_A^+(x), \mu_A^+(y)\}$, which is equivalent to $\mu_B^-(xy) > 0$ and $\mu_B^+(xy) > 0$. Combining the above two, the result follows. □

Definition 3.5. A dominating set S of an IVFG G is said to be a minimal dominating set if no proper subset of S is a dominating set of G.

The following theorem is analogous to a result in [16] which gives a characterization of minimal dominating sets.

Theorem 3.6. *A dominating set D of an IVFG G is a minimal dominating set if and only if for each $d \in D$ one of the following two conditions holds.*

(i) $N(d) \cap D = \phi$.

(ii) There is a vertex $x \in V \setminus D$ such that $N(x) \cap D = \{d\}$.

Proof. Let D be a minimal dominating set of G. Then for every vertex $d \in D$, $D \setminus \{d\}$ is not a dominating set and so there exists $x \in V \setminus (D \setminus \{d\})$ which is not dominated by any vertex in $D \setminus \{d\}$. If $x = d$, then (i) holds. If $x \neq d$, then x is not dominated by $D \setminus \{d\}$, but is dominated by D, i.e., x is dominated only by d in D. Hence $N(x) \cap D = \{d\}$.

Conversely, let D be a dominating set and for each vertex $d \in D$, one of the two conditions holds. Suppose D be not a minimal dominating set. Then there exists a vertex $d \in D$ such that $D \setminus \{d\}$ is a dominating set. Thus d is dominated by at least one vertex in $D \setminus \{d\}$ and so, the condition (i) does not hold. Again if $D \setminus \{d\}$ is a dominating set, then every vertex in $V \setminus D$ is dominated by at least one vertex in $D \setminus \{d\}$ which implies that the condition (ii) does not hold. This leads to a contradiction. Hence D must be a minimal dominating set. □

Definition 3.7. A vertex u of an IVFG G is said to be an isolated vertex if $\mu_B^-(uv) < \min\{\mu_A^-(u), \mu_A^-(v)\}$ and $\mu_B^+(uv) < \max\{\mu_A^+(u), \mu_A^+(v)\}$ for all $v \in V \setminus \{u\}$ such that there is an edge between u and v, i.e., $N(u) = \phi$.

Example 3.8. In the example 2.10, y is an isolated vertex.

Remark 3.9. An isolated vertex does not dominate any other vertex in G.

Theorem 3.10. *Let G be an IVFG without isolated vertices. Let D be a minimal dominating set of G. Then $V \setminus D$ is a dominating set of G.*

Proof. Let D be a minimal dominating set and $d \in D$. Since G has no isolated vertices, there is a vertex $x \in N(d)$. Using similar arguments as in the proof of Theorem 3.6, we get that $x \in V \setminus D$. Thus every element of D is dominated by some element of $V \setminus D$ and consequently $V \setminus D$ is a dominating set. □

Definition 3.11. A set S of vertices of an IVFG G is said to be independent if $\mu_B^-(uv) < \min\{\mu_A^-(u), \mu_A^-(v)\}$ and $\mu_B^+(uv) < \max\{\mu_A^+(u), \mu_A^+(v)\}$ for all $u, v \in S$.

Next two theorems establish relations between independent sets and dominating sets.

Theorem 3.12. *An independent set is a maximal independent set of an IVFG G if and only if it is independent and dominating set.*

Proof. Let D be a maximal independent set of G. Thus for every $v \in V \setminus D$, the set $D \cup \{v\}$ is not independent. So, for every vertex $v \in V \setminus D$, there is a vertex $u \in D$ such that u is dominated by v. Thus D is a dominating set. Hence D is independent and dominating.

Conversely, let D be independent and dominating. If possible, suppose D is not maximal independent. Then there exists $v \in V \setminus D$ such that the set $D \cup \{v\}$

is independent. Then no vertex in D is dominated by v. Thus D can not be a dominating set, which is a contradiction. Hence D must be a maximal independent set. □

Theorem 3.13. *In an IVFG G, every maximal independent set is a minimal dominating set.*

Proof. Let S be a maximal independent set in G. By Theorem 3.12, S is a dominating set. Suppose S be not a minimal dominating set. Then there exists at least one vertex $v \in S$ for which $S \setminus \{v\}$ is a dominating set. But if $S \setminus \{v\}$ dominates $V \setminus (S \setminus \{v\})$, then at least one vertex in $S \setminus \{v\}$ must dominate v. This contradicts the fact that S is an independent set of G. Hence S must be a minimal dominating set. □

Definition 3.14. Let G be an IVFG without isolated vertices. A subset D of V is said to be a total dominating set if every vertex in V is dominated by a vertex in D.

The minimum cardinality of a total dominating set is called the total domination number of G and is denoted by γ_t.

Proof of the following is obvious.

Theorem 3.15. *For any IVFG G, $\gamma_t = p$ if and only if every vertex of G has a unique neighbor.*

Theorem 3.16. *If $\gamma_t = p$, then number of vertices in G is even.*

Proof. If possible let G has $2n+1$, i.e., odd number of vertices. Since G has no loops, for every vertex u, we get a unique vertex v distinct from u. Thus we can get n number of distinct pair of vertices such that in each pair one vertex is neighbor of the other vertex. Finally we are left with a single vertex which does not have a unique neighbor. This leads to a contradiction. Thus G must have even number of vertices. □

Theorem 3.17. *Let G be an IVFG without isolated vertices. Then $\gamma_t + \overline{\gamma_t} \leq 2p$ and the equality holds if and only if*

(i) the number of vertices in G is even, say $2n$.
(ii) there is a set S_1 of n mutually disjoint effective edges in G.
(iii) there is a set S_2 of n mutually disjoint effective edges in $\overline{G}$.
(iv) for any edge $xy \notin S_1 \cup S_2$, $0 < \mu_B^-(xy) < \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $0 < \mu_B^+(xy) < \max\{\mu_A^+(x), \mu_A^+(y)\}$.

Proof. Since $\gamma_t \leq p$ and $\overline{\gamma_t} \leq p$, the inequality follows.

(i) $\gamma_t + \overline{\gamma_t} = 2p$ if and only if $\gamma_t = \overline{\gamma_t} = p$. Using Corollary 3.16, we have that the number of vertices in G is even, say $2n$.

(ii) Since $\gamma_t = p$, there is a set S_1 of n disjoint effective edges in G.

(iii) Since $\overline{\gamma_t} = p$, there is a set S_2 of n disjoint effective edges in $\overline{G}$.

(iv) If $xy \notin S_1 \cup S_2$, then clearly $0 < \mu_B^-(xy) < \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $0 < \mu_B^+(xy) < \max\{\mu_A^+(x), \mu_A^+(y)\}$.

The converse is obvious. □

4. Conclusions

Research in the area of domination theory is interesting due to the diversity of applications and wide variety of domination parameters that can be defined. In this paper, the concept of dominating sets, independent sets, domination number etc. have been introduced for interval-valued fuzzy graphs and some interesting results have been proved. Other domination parameters can be defined and investigated in the similar setting as future work.

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References

[1] M. Akram and W. A. Dudek, Interval-valued fuzzy graphs, Comput. Math. Appl. 61 (2011) 289–299.
[2] P. Bhattacharya, Some remarks on fuzzy graphs, Pattern Recognit. Lett. 6 (1987) 297–302.
[3] K. R. Bhutani and A. Battou, On M-strong fuzzy graphs, Inform. Sci. 155 (2003) 103–109.
[4] K. R. Bhutani and A. Rosenfeld, Strong arcs in fuzzy graphs, Inform. Sci. 155 (2003) 103–109.
[5] E. J. Cockayne and S. T. Hedetnieme, Towards a theory of domination in graphs, Networks 7 (1977) 247–261.
[6] M. B. Gorzalczany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy Sets and Systems 21 (1987) 1–17.
[7] M. B. Gorzalczany, An interval-valued fuzzy inference method: Some basic properties, Fuzzy Sets and Systems 31 (1989) 243–251.
[8] F. Hararay, Graph Theory, 3rd Edition, Addison-Wesley, Reading, MA, 1972.
[9] J. Hongmei and W. Lianhua, Interval-valued fuzzy subsemigroups and subgroups associated by interval-valued fuzzy graphs, WRI Global Congress on Intelligent Systems (2009) 484–487.
[10] Y. Lou and C. Yu, A fuzzy optimization method for multi-criteria decision making problem based on the inclusion degrees of intuitionistic fuzzy sets, J. Inf. Computing Sci. 3(2) (2008) 146–152.
[11] J. M. Mendel, Uncertain rule-based fuzzy logic systems: Introduction and new directions, Prentice-Hall, New Jersey, 2001.
[12] S. I. Mohideen and A. M. Ismayil, Domination in fuzzy graph: A new approach, Int. J. Comput. Sci. Math. 2(3) (2010) 101–107.
[13] J. N. Mordeson, Fuzzy line graphs, Pattern Recognit. Lett. 14 (1993) 381–384.
[14] J. N. Mordeson and P. S. Nair, Fuzzy Graphs and Fuzzy Hypergraphs, Physica Verlag, Heidelberg, 1998.
[15] J. N. Mordeson and C. S. Peng, Operations on fuzzy graphs, Inform. Sci. 79 (1994) 159–170.
[16] O. Ore, Theory of Graphs, Amer. Math. Soc. Colloq. Publ. 38 (1962) Providence.
[17] A. Rosenfeld, Fuzzy Graphs, in: L. A. Zadeh, K. S. Fu, M. Shimura (Eds.), Fuzzy Sets and Their Applications, Academic Press (1975) 77–95.
[18] M. K. Roy and R. Biswas, I-V fuzzy relations and Sanchez's approach for medical diagnosis, Fuzzy Sets and Systems 47 (1992) 35–38.
[19] S. Saleh, On category of interval valued fuzzy topological spaces, Ann. Fuzzy Math. Inform. 4(2) (2012) 385–392.
[20] A. Somasundaram and S. Somasundaram, Domination in fuzzy graphs-I, Pattern Recognit. Lett. 19 (1998) 787–791.
[21] A. A. Talebi and H. Rashmanlou, Isomorphism on interval valued fuzzy graphs, Ann. Fuzzy Math. Inform. 6(1) (2013) 47–58.
[22] I. B. Turksen, Interval valued fuzzy sets based on normal forms, Fuzzy Sets and Systems 20 (1986) 191–210.
[23] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

[24] L. A. Zadeh, The concept of a linguistic and application to approximate reasoning I, Inform. Sci. 8 (1975) 149–249.
[25] L. A. Zadeh, Toward a generalized theory of uncertainty (GTU) - an outline, Inform. Sci. 172(1-2) (2005) 1–40.
[26] L. A. Zadeh, Is there a need for fuzzy logic?, Inform. Sci. 178 (2008) 2751–2779.

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