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The generalized Hyers-Ulam-Rassias stability of a quadratic functional equation in fuzzy Banach spaces

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Abstract. In this following paper, we wise to appraise generalization of the stability theorem for generalize Hyers-Ulam-Rassias Stability of the quadratic functional equation

$$f(2x + y) - f(x + 2y) = 3f(x) - 3f(y) \tag{0.1}$$

in fuzzy Banach spaces.

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1. Introduction

The idea of studying stability problem of functional equations started with a well-known problem posed by Ulam [19] in 1940 concerning the stability of group homomorphisms. Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$ does there exist a $\delta > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $H : G_1 \longrightarrow G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$? In the next year Hyers [8] gave a partial acceptable answer to this question. He showed that if $\delta > 0$ and $f : E \longrightarrow E_1$ with E and E_1 Banach spaces, such that

$$\| f(x + y) - f(x) - f(y) \| \leqslant \delta$$

for all $x, y \in E$ then there exists a unique $g : E \longrightarrow E_1$ such that $g(x + y) = g(x) + g(y)$ and $\| f(x) - g(x) \| \leqslant \delta$ for all $x, y \in E$. The generalized result of T. Aoki, who elaborated and pondered on the Hyers-Ulam stability formulae in [1], is the consequence of extensive acting upon and furthering to in 1978 by Th. M. Rassias. His evolutionary definition determining the generalized hypothesis in [14] denotes Hyers theorem as a special case. Of late Maligranda's rectifying performance that the mapping f satisfies some continuity assumption that finds its concept of existence of unique additive mapping, commenced by T. Aoki [1]. The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

and therefore the equation (1.1) is called the quadratic functional equation. F. Skof [18] proved the Hyers-Ulam stability theorem for (1.1) for the function $f : E \to E_1$ where E is a normed space and E_1 is a Banach space. In fact, P. W. Cholewa [5], S. Czerwik [6] proved the Hyers-Ulam stability theorem for (1.1) replacing E_1 by an Abelian group. This result was further generalized by Th. M. Rassias [15], C. Borelli and G. L. Forti [3]. Later on, in the paper [9], the authors further generalized this result for the new quadratic functional equation (0.1).
Ever since the concept of fuzzy sets was introduced by Zadeh [20] in 1965 to describe the situation in which data are imprecise or vague or uncertain. It has a wide range of application in the field of population dynamics, chaos control, computer programming, medicine , etc. With the launch of the concept of Fuzzy metric space in 1975 by Kramosil and Michalek [11] it takes several vistas of more developmental in such spaces. The fuzzy norm was introduced by Katsaras [10]. The norm has also been exposed by some mathematicians from several standpoint on vector space. The idea of fuzzy norm by Cheng and Moderson [4], Bag and Samanta [2] was so precise to relate to fuzzy metric Kramosil and Michalek type [11]. However, Geoge andVeeramani [7]opine, it is a fact that an ordinary normed linear space is a special one of fuzzy normed linear space. Since them many [12, 13] made attempt on it to prove a general stability theory of functional equations in fuzzy Banach spaces.

In this paper, this is an effort to generalize the stability theorem of generalized Hyers-Ulam-Rassias Stability of the quadratic functional equation (0.1) in Fuzzy Banach Spaces.

2. Preliminaries

We quote some definitions and examples which will be needed in the sequel.

Definition 2.1 ([17]). A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is continuous t - norm if $*$ satisfies the following conditions :
(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a \quad \forall\, a \in [0, 1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0, 1]$.

Through out this article, we further assume that $a * a = a \;\forall\, a \in [0, 1]$.

Definition 2.2 ([12]). The 3-tuple $(X, N, *)$ is called a fuzzy normed linear space if X is a real linear space, $*$ is a continuous t - norm and N is a fuzzy set in $X \times (0, \infty)$ satisfying the following conditions :

(i) $N(x, t) > 0$;
(ii) $N(x, t) = 1$ if and only if $x = 0$;
(iii) $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;
(iv) $N(x, s) * N(y, t) \leqslant N(x + y, s + t)$;
(v) $N(x, \cdot) : (0, \infty) \to (0, 1]$ is continuous ;

for all $x, y \in X$ and $t, s > 0$.
Note that $N(x, t)$ can be thought of as the degree of nearness between x and null vector 0 with respect to t.

Example 2.3. Let $X = [0, \infty)$, $a * b = ab$ for every $a, b \in [0, 1]$ and $\|\cdot\|$ be a norm defined on X. Define $N(x, t) = e^{-\frac{\|x\|}{t}}$ for all x in X. Then clearly $(X, N, *)$ is a fuzzy normed linear space.

Example 2.4. Let $(X, \|\cdot\|)$ be a normed linear space, and let $a * b = ab$ or $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Let $N(x, t) = \frac{t}{t + \|x\|}$ for all $x \in X$ and $t > 0$. Then $(X, N, *)$ is a fuzzy normed linear space and this fuzzy norm N induced by $\|\cdot\|$ is called the standard fuzzy norm.

Note 2.1. According to George and Veeramani [7], it can be proved that every fuzzy normed linear space is a metrizable topological space. In fact, also it can be proved that if $(X, \|\cdot\|)$ is a normed linear space, then the topology generated by $\|\cdot\|$ coincides with the topology generated by the fuzzy norm N of example (2.4). As a result, we can say that an ordinary normed linear space is a special case of fuzzy normed linear space.

Remark 2.5. In fuzzy normed linear space $(X, N, *)$, for all $x \in X$, $N(x, \cdot)$ is non- decreasing with respect to the variable t.

Definition 2.6 ([16]). Let $(X, N, *)$ be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim\limits_{n \to \infty} N(x_n - x, t) = 1$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim\limits_{n \to \infty} x_n = x$.

Definition 2.7 ([16]). Let $(X, N, *)$ be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is called Cauchy sequence if for each $\varepsilon > 0$ and $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

3. The generalized Hyers-Ulam-Rassias stability of the functional equation (0.1):

In this section, let X be a real vector space and (Y, N) be a fuzzy Banach space.

Theorem 3.1. *Let $\phi : X^2 \to [0, \infty)$ be a function such that*

$$\widetilde{\phi}(x, 0) = \sum_{n=0}^{\infty} \frac{\phi(2^n x, 0)}{4^n} < \infty \text{ and } \lim_{n \to \infty} \frac{\phi(2^n x, 2^n y)}{4^n} = 0 \tag{3.1}$$

for all $x\,,\,y\,\in\,X$ *. Let* $f\;:\;X\to Y$ *be a mapping with* $f(0)\,=\,0$ *such that*

$$\lim_{t\to\infty} N\left(\,f\,(2x+y)-f\,(x+2y)-3f\,(x)+3f\,(y)\;,\;t\,\phi\,(x\,,\,y)\,\right)\;=\;1 \tag{3.2}$$

uniformly on $X\,\times\,X$ *. Then* $Q\,(x)\,:=\,N-\lim\limits_{n\to\infty}\frac{f(2^{\,n}\,x)}{4^{\,n}}$ *exists for each* $x\,\in\,X$ *and defines a quadratic mapping* $Q\,:\,X\,\to\,Y$ *such that if for some* $\delta\,>\,0\,,\,\alpha\,>\,0$

$$N\left(\,f\,(2x+y)-f\,(x+2y)-3f\,(x)+3f\,(y)\;,\;\delta\,\phi\,(x\;,\;y)\,\right)\;\geqslant\;\alpha \tag{3.3}$$

for all $x\,,\,y\,\in\,X$*, then*

$$N\left(f\,(x)-Q\,(x)\;,\;\delta\,\widetilde{\phi}\,(x\;,\;0\,)\right)\;\geqslant\;\alpha \tag{3.4}$$

for all $x\,\in\,X$ *. Furthermore, the quadratic mapping* $Q\;:\;X\,\to\,Y$ *is a unique function such that*

$$\lim_{t\to\infty} N\left(f\,(x)-Q\,(x)\;,\;t\,\widetilde{\phi}\,(x\,,\,0\,)\right)\;=\;1 \tag{3.5}$$

uniformly on X*.*

Proof. For a given $\varepsilon>0$, by (3.2), there exits some $t_0\,>\,0$ such that

$$N\left(\,f\,(2x+y)-f\,(x+2y)-3f\,(x)+3f\,(y)\;,\;t\,\phi\,(x\,,\,y)\,\right)\;\geq\;1-\varepsilon \tag{3.6}$$

for all $t\,\geqslant\,t_0$. By induction on positive integer n, we show that

$$N\left(4^{\,n}f\,(\,x\,)\;-\;f\,(2^{\,n}\,x\,)\;,\;\frac{t}{4}\sum_{k\,=\,0}^{n-1}4^{\,n-k}\,\phi\,(2^{\,k}\;x\,,\,0)\right)\;\geqslant\;1\;-\;\varepsilon \tag{3.7}$$

for all $x\in X$ and for all $t\,\geqslant\,t_0$.
Letting $y\,=\,0$ in (3.6), we get

$N\left(\,f\,(2x)-f\,(x)-3f\,(x)+3f\,(0)\;,\;t\,\phi\,(x\,,\,0)\,\right)\;\geqslant\;1\,-\,\varepsilon\,,$

i.e., $N\left(\,f\,(2x)-4f\,(x)\;,\;t\,\phi\,(x\,,\,0)\,\right)\;\geqslant\;1-\varepsilon\,,$

i.e., $N\left(\,4f\,(x)-f\,(2x)\;,\;t\,\phi\,(x\,,\,0)\,\right)\;\geqslant\;1-\varepsilon$

for all $x\,\in\,X$ and for all $t\,\geqslant\,t_0$.
Thus we get (3.7) for $n\,=\,1$. Assume that (3.7) holds for $n\,\in\,N$. Then

$$\begin{aligned}
&N\left(4^{\,n+1}f\,(\,x\,)\;-\;f\,(2^{\,n+1}\,x\,)\;,\;\frac{t}{4}\sum_{k\,=\,0}^{n}4^{\,n-k+1}\,\phi\,(2^{\,k}\;x\,,\,0)\right)\\
&\geqslant\;N\left(4^{\,n+1}f\,(\,x\,)\;-\;4\,f\,(2^{\,n}\,x\,)\;,\;\frac{t}{4}\sum_{k\,=\,0}^{n-1}4^{\,n-k+1}\,\phi\,(2^{\,k}\;x\,,\,0)\right)\\
&\qquad\qquad\qquad\qquad\ast\;N\left(4\,f\,(2^{\,n}\,x\,)\;-\;f\,(2^{\,n+1}\,x\,)\;,\;t\,\phi\,(2^{\,n}\,x\,,\,0)\right)\\
&\geqslant\;(1\,-\,\varepsilon\,)\,\ast\,(1\,-\,\varepsilon\,)\;=\;1-\varepsilon
\end{aligned}$$

This completes the proof of (3.7). Letting $t\,=\,t_0$ and replacing n and x by p and $2^{\,n}\,x$ in (3.7) respectively, we get

$$N\left(4^{\,p}f\,(2^{\,n}\,x\,)\;-\;f\,(2^{\,n+p}\,x\,)\;,\;\frac{t_0}{4}\sum_{k\,=\,0}^{p\,-\,1}4^{\,p-k}\phi\,(2^{\,n+k}\,x\,,\,0\,)\right)\;\geqslant\;1\,-\,\varepsilon,$$

which implies that

$$(3.8)\quad N\left(\frac{f(2^n x)}{4^n}-\frac{f(2^{n+p}x)}{4^{n+p}}\,,\,\frac{t_0}{4.\,4^{n+p}}\sum_{k=0}^{p-1}4^{p-k}\phi\left(2^{n+k}x\,,\,0\right)\right)\geqslant 1-\varepsilon$$

for all $n \geqslant 0$ and for all $p > 0$. It follows from (3.1) and the equality

$$\sum_{k=0}^{p-1}4^{-n-k}\phi\left(2^{n+k}x\,,\,0\right) = \sum_{k=n}^{n+p-1}4^{-k}\phi\left(2^{k}x\,,\,0\right)$$

that for a given $\delta > 0$ there exits $n_0 \in \mathbb{N}$ such that

$$\frac{t_0}{4}\sum_{k=n}^{n+p-1}4^{-k}\phi\left(2^{k}x\,,\,0\right) < \delta$$

for all $n \geqslant n_0$ and for all $p > 0$. Now we deduce from (3.8) that

$$N\left(\frac{f(2^n x)}{4^n}-\frac{f(2^{n+p}x)}{4^{n+p}}\,,\,\delta\right)$$

$$\geqslant N\left(\frac{f(2^n x)}{4^n}-\frac{f(2^{n+p}x)}{4^{n+p}}\,,\,\frac{t_0}{4.\,4^{n+p}}\sum_{k=0}^{p-1}4^{p-k}\phi\left(2^{n+k}x\,,\,0\right)\right)$$

$$\geqslant 1-\varepsilon$$

for each $n \geqslant n_0$ and $p > 0$. Thus the sequence $\left\{\frac{f(2^n x)}{4^n}\right\}$ is Cauchy in Y. Since Y is a fuzzy Banach space, the sequence $\left\{\frac{f(2^n x)}{4^n}\right\}$ converges to some $Q(x) \in Y$. So we can define a function $Q : X \to Y$ by $Q(x) := N - \lim\limits_{n\to\infty}\frac{f(2^n x)}{4^n}$, namely for each $t > 0$ and $x \in X$,

$$\lim_{n\to\infty} N\left(\frac{f(2^n x)}{4^n} - Q(x)\,,\,t\right) = 1\,.$$

Now we show that Q satisfies (0.1). Let $x, y \in X$ and fix $t > 0$ and $0 < \varepsilon < 1$. Since $\lim\limits_{n\to\infty}\frac{\phi(2^n x\,,\,2^n y)}{4^n} = 0$ there exists $n_1 \geqslant n_0$ such that

$$(3.9)\qquad t_0\,\frac{\phi(2^n x\,,\,2^n y)}{4^n} < \frac{t}{5}$$

for all $n \geqslant n_1$. Hence for each $n \geqslant n_1$,

$$\begin{aligned}(3.10)\qquad & N\left(Q(2x+y)-Q(x+2y)-3\,Q(x)+3\,Q(y)\,,\,t\right)\\ &\geqslant N\left(Q(2x+y)-\frac{f(2^n(2x+y))}{4^n}\,,\,\frac{t}{5}\right)*\\ &\quad N\left(Q(x+2y)-\frac{f(2^n(x+2y))}{4^n}\,,\,\frac{t}{5}\right)*\\ &\quad N\left(3\,Q(x)-3\,\frac{f(2^n x)}{4^n}\,,\,\frac{t}{5}\right)*\\ &\quad N\left(3\,Q(y)-3\,\frac{f(2^n y)}{4^n}\,,\,\frac{t}{5}\right)*\end{aligned}$$

$$N\left(\frac{f(2^n(2x+y))}{4^n} - \frac{f(2^n(x+2y))}{4^n} - 3\,\frac{f(2^n x)}{4^n} + 3\,\frac{f(2^n y)}{4^n}\,,\ \frac{t}{5}\right)$$

Replacing x, y by $2^n x$, $2^n y$ respectively in (3.6) and for $t = t_0$, we get

$$N\Big(f(2^n(2x+y)) - f(2^n(x+2y)) - 3f(2^n x) + 3f(2^n y)\,,$$
$$t_0\,\phi(2^n x\,,\,2^n y)\Big) \geqslant 1 - \varepsilon,$$

which implies that

$$(3.11)\quad N\left(4^{-n} f(2^n(2x+y)) - 4^{-n} f(2^n(x+2y)) - 3\,.\,4^{-n} f(2^n x) + 3\,.4^{-n} f(2^n y)\,,\ t_0 4^{-n}\,\phi(2^n x\,,\,2^n y)\right) \geqslant 1 - \varepsilon\,.$$

The first four terms on the RHS of (3.10) tend to 1 as $n \to \infty$ and the last term $\geqslant 1 - \varepsilon$ by (3.9) and (3.11). Thus,

$N(Q(2x+y) - Q(x+2y) - 3Q(x) + 3Q(y)\,,\,t) \geqslant 1-\varepsilon$ for all $t > 0$,

$\Rightarrow\ N(Q(2x+y) - Q(x+2y) - 3Q(x) + 3Q(y)\,,\,t) = 1$ for all $t > 0$,

$\Rightarrow\ Q(2x+y) - Q(x+2y) - 3Q(x) + 3Q(y) = 0$ for all $x, y \in X$.

Hence Q satisfies (0.1), i.e., $Q : X \to Y$ is quadratic . Now let for some positive $\delta > 0$, $\alpha > 0$. (3.3) holds. Let

$$\phi_n(x\,,0) = \sum_{k=0}^{n-1} 4^{-k}\phi\left(2^k x\,,0\right) \text{ for all } x \in X.$$

Let $x \in X$. By the same reasoning as in the beginning of the proof, we can deduce from (3.3) that

$$(3.12)\qquad N\left(4^n f(x) - f(2^n x)\,,\ \delta \sum_{k=0}^{n-1} 4^{n-k}\,\phi\left(2^k x\,,0\right)\right) \geqslant \alpha$$

for all positive integers n. Let $t > 0$. We have

$$(3.13)\qquad N(f(x) - Q(x)\,,\delta\phi_n(x\,,0) + t)$$
$$\geqslant N\left(f(x) - \frac{f(2^n x)}{4^n}\,,\delta\phi_n(x\,,0)\right) * N\left(\frac{f(2^n x)}{4^n} - Q(x)\,,\,t\right).$$

Combining (3.12) and the fact that

$\displaystyle\lim_{n\to\infty} N\left(\frac{f(2^n x)}{4^n} - Q(x)\,,\,t\right) = 1,$ we have

$$N(f(x) - Q(x)\,,\delta\phi_n(x\,,0) + t) \geqslant \alpha * 1 = \alpha$$

for large enough $n \in N$.

From the continuity of $N(f((x) - Q(x)\,,\cdot)$ and considering $t \to 0$, we get

$$N\left(f(x) - Q(x)\,,\ \delta\,\widetilde{\phi}(x\,,\,0)\right) \geqslant \alpha \text{ for all } x \in X.$$

Uniqueness: Let T be another quadratic mapping satisfying (0.1) and (3.5). Fix $c > 0$, given $\varepsilon > 0$, by (3.5) for Q and T we can find some $t_0 > 0$ such that

$$N\left(f(x) - Q(x)\,,\ t\,\widetilde{\phi}(x\,,0)\right) \geqslant 1 - \varepsilon,$$

$$N\left(f(x) - T(x)\ ,\ t\,\widetilde{\phi}(x, 0)\right) \geqslant 1 - \varepsilon$$

for all $x \in X$ and $t \geqslant t_0$.
Again, we see from (0.1) that $f(2x) = 4f(x)$ for $y = 0$, by induction, it implies that

(3.14) $\quad f(2^n x) = 4^n f(x)$.

Since Q and T satisfy (0.1), it follows from (3.14) that

$Q(2^n x) = 4^n Q(x)$ and $T(2^n x) = 4^n T(x)$.

Fix some $x \in X$ and find some integer n_0 such that

(3.15) $$t_0 \sum_{k=n}^{\infty} 4^{-k} \phi(2^k x, 0) < \frac{c}{2} \quad \text{for all } n \geqslant n_0.$$

Since

$$\sum_{k=n}^{\infty} 4^{-k} \phi(2^k x, 0) = \frac{1}{4^n} \sum_{k=n}^{\infty} 4^{-(k-n)} \phi(2^{k-n}(2^n x), 0)$$

$$= \frac{1}{4^n} \sum_{m=0}^{\infty} 4^{-m} \phi(2^m (2^n x), 0) = \frac{1}{4^n} \widetilde{\phi}(2^n x, 0),$$

we have

$$N(Q(x) - T(x)\ ,\ c)$$

$$= N\left(Q(x) - \frac{f(2^n x)}{4^n} + \frac{f(2^n x)}{4^n} - T(x)\ ,\ \frac{c}{2} + \frac{c}{2}\right)$$

$$\geqslant N\left(Q(x) - \frac{f(2^n x)}{4^n}\ ,\ \frac{c}{2}\right) * N\left(\frac{f(2^n x)}{4^n} - T(x)\ ,\ \frac{c}{2}\right)$$

$$= N\left(f(2^n x) - 4^n Q(x)\ ,\ 4^n \frac{c}{2}\right) * N\left(f(2^n x) - 4^n T(x)\ ,\ 4^n \frac{c}{2}\right)$$

$$= N\left(f(2^n x) - Q(2^n x)\ ,\ 4^n \frac{c}{2}\right) * N\left(f(2^n x) - T(2^n x)\ ,\ 4^n \frac{c}{2}\right)$$

$$\geqslant N\left(f(2^n x) - Q(2^n x)\ ,\ 4^n t_0 \sum_{k=n}^{\infty} 4^{-k} \phi(2^k x, 0)\right) *$$

$$N\left(f(2^n x) - T(2^n x)\ ,\ 4^n t_0 \sum_{k=n}^{\infty} 4^{-k} \phi(2^k x, 0)\right)$$

$$= N\left(f(2^n x) - Q(2^n x)\ ,\ t_0\,\widetilde{\phi}(2^n x, 0)\right) *$$

$$N\left(f(2^n x) - T(2^n x)\ ,\ t_0\,\widetilde{\phi}(2^n x, 0)\right)$$

$$\geqslant (1-\varepsilon) * (1-\varepsilon) = 1-\varepsilon$$

It follows that $N(Q(x) - T(x)\ ,\ c) = 1$ for all $c > 0$. Thus $Q(x) = T(x)$ for all $x \in X$. This completes the proof of the theorem. $\square$

Corollary 3.2. *Let* $\theta \geqslant 0$ *and* p *be a real number with* $0 < p < 2$. *Let* $f : X \to Y$ *be a function such that with* $f(0) = 0$ *and*

(3.16)
$$\lim_{t \to \infty} N\left(f(2x+y) - f(x+2y) - 3f(x) + 3f(y) \,,\, t\,\theta\left(\|x\|^{p} + \|y\|^{p}\right)\right) = 1$$

uniformly on $X \times X$. *Then* $Q(x) := N - \lim_{n \to \infty} \frac{f(2^{n} x)}{4^{n}}$ *exists for each* $x \in X$ *and defines a quadratic function* $Q : X \to Y$ *such that if for some* $\delta > 0$, $\alpha > 0$

(3.17)
$$N\left(f(2x+y) - f(x+2y) - 3f(x) + 3f(y) \,,\, \delta\,\theta\left(\|x\|^{p} + \|y\|^{p}\right)\right) \geqslant \alpha$$

for all $x, y \in X$, *then*

$$N\left(f(x) - Q(x) \,,\, \delta \frac{4\,\theta}{(4 - 2^{p})} \|x\|^{p} \right) \geqslant \alpha \tag{3.18}$$

for all $x \in X$. *Furthermore, the function* $Q : X \to Y$ *is a unique function such that*

$$\lim_{t \to \infty} N\left(f(x) - Q(x) \,,\, t \frac{4\,\theta}{(4 - 2^{p})} \|x\|^{p} \right) = 1$$

uniformly on X.

Proof. Define $\phi(x, y) = \theta\left(\|x\|^{p} + \|y\|^{p}\right)$. Now

$$\widetilde{\phi}(x, 0) = \sum_{n=0}^{\infty} \frac{\phi(2^{n} x, 0)}{4^{n}} = \sum_{n=0}^{\infty} \frac{\theta\left(\|2^{n} x\|^{p}\right)}{4^{n}} = \theta\,\|x\|^{p} \sum_{n=0}^{\infty} \left(\frac{2^{p}}{4}\right)^{n}$$

$$= \theta\,\|x\|^{p} \frac{1}{1 - \frac{2^{p}}{4}} = \theta\,\|x\|^{p} \frac{4}{4 - 2^{p}}.$$

It proves this corollary with the help of Theorem 3.1. □

Theorem 3.3. *Let* $\phi : X^{2} \to [0, \infty)$ *be a function such that* $\widetilde{\phi}(x, 0) = \sum_{n=0}^{\infty} 4^{n} \phi\left(\frac{x}{2^{n}}, 0\right) < \infty$; *for all* $x \in X$. *Let* $f : X \to Y$ *be a function with* $f(0) = 0$ *such that*

$$\lim_{t \to \infty} N\left(f(2x+y) - f(x+2y) - 3f(x) + 3f(y) \,,\, t\,\phi(x, y)\right) = 1$$

uniformly on $X \times X$. *Then* $Q(x) := N - \lim_{n \to \infty} 4^{n} f\left(\frac{x}{2^{n}}\right)$ *exists for each* $x \in X$ *and defines a quadratic function* $Q : X \to Y$ *such that if for some* $\delta > 0$, $\alpha > 0$

$$N\left(f(2x+y) - f(x+2y) - 3f(x) + 3f(y) \,,\, \delta\,\phi(x, y)\right) \geqslant \alpha$$

for all $x, y \in X$, *then*

$$N\left(f(x) - Q(x) \,,\, \delta\, \widetilde{\phi}(x, 0) \right) \geqslant \alpha \quad \text{for all } x \in X.$$

Furthermore, the function $Q : X \to Y$ *is a unique function such that*

$$\lim_{t \to \infty} N\left(f(x) - Q(x) \,,\, t\, \widetilde{\phi}(x, 0) \right) = 1$$

uniformly on X.

Corollary 3.4. *Let $\theta \geqslant 0$ and p be a real number with $0 < p < 2$. Let $f : X \to Y$ be a function such that with $f(0) = 0$ and*

$$\lim_{t \to \infty} N\left(f(2x+y) - f(x+2y) - 3f(x) + 3f(y)\,,\ t\,\theta\left(\|x\|^{p} + \|y\|^{p}\right)\right) = 1$$

uniformly on $X \times X$. Then $Q(x) := N - \lim_{n \to \infty} 4^{n} f\left(\frac{x}{2^{n}}\right)$ exists for each $x \in X$ and defines a quadratic function $Q : X \to Y$ such that if for some $\delta > 0$, $\alpha > 0$

$$N\left(f(2x+y) - f(x+2y) - 3f(x) + 3f(y)\,,\ \delta\,\theta\left(\|x\|^{p} + \|y\|^{p}\right)\right) \geqslant \alpha$$

for all $x, y \in X$, then

$$N\left(f(x) - Q(x)\,,\ \frac{2^{p}}{2^{p} - 4}\,\delta\,\theta\,\|x\|^{p}\right) \geqslant \alpha$$

for all $x \in X$. Furthermore, the function $Q : X \to Y$ is a unique function such that

$$\lim_{t \to \infty} N\left(f(x) - Q(x)\,,\ \frac{2^{p}}{2^{p} - 4}\,t\,\theta\,\|x\|^{p}\right) = 1 \quad \textit{uniformly on } X.$$

Proof. Define $\phi(x, y) = \theta\left(\|x\|^{p} + \|y\|^{p}\right)$ and

$$\widetilde{\phi}(x, 0) = \sum_{n=0}^{\infty} 4^{n} \phi\left(\frac{x}{2^{n}}, 0\right) = \sum_{n=0}^{\infty} 4^{n}\,\theta\left(\left\|\frac{x}{2^{n}}\right\|^{p}\right) = \theta\,\|x\|^{p} \sum_{n=0}^{\infty} \left(\frac{4}{2^{p}}\right)^{n}$$

$$= \theta\,\|x\|^{p}\,\frac{1}{1 - \frac{4}{2^{p}}} = \theta\,\|x\|^{p}\,\frac{2^{p}}{2^{p} - 4}.$$

It proves this corollary with the help of Theorem 3.1. □

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