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Finite dimensional fuzzy normed linear spaces

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Abstract. In this paper we consider general t-norm in the definition of fuzzy normed linear space which is introduced by the authors in an earlier paper. It is proved that if t-norm is chosen other than "min" then decomposition theorem of a fuzzy norm into a family of crisp norms may not hold. We study some basic results on finite dimensional fuzzy normed linear spaces in general t-norm setting.

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1. Introduction

It is Katsaras [10] who, while studying fuzzy topological vector spaces, first introduced in 1984 the idea of fuzzy norm on a linear space. In 1992, Felbin [6], introduced an idea of a fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space so that the corresponding fuzzy metric associated to this fuzzy norm is of Kaleva & Seikkala type [9]. In 1994, Cheng & Mordeson [5] introduced another idea of a fuzzy norm on a linear space in such a manner that the corresponding fuzzy metric is of Kramosil & Michalek type [12].

Following Cheng & Mordeson, we have introduced in [1] a definition of a fuzzy norm whose associated fuzzy metric is similar to Kramosil & Michalek type [12]. The novelty of this definition is the validity of a decomposition theorem for this type of fuzzy norm into a family of crisp norms as developed by us [1]. This concept has been used in developing fuzzy functional analysis and its applications and a large number of papers by different authors have been published (for reference please see [2, 3, 4, 7, 8]). It is to be noted here that though the decomposition theorem facilitates the study on functional analysis in a long way but it requires a strong restriction on the particular choice of "*min*" for t-norm of the triangle inequality (N4) and this "*min*" t-norm plays a vital role in the formation of the decomposition theorem. So a natural query arises-how far the results of fuzzy normed linear spaces can be established with the fuzzy norm in its general form i.e. waiving the restricted "*min*" t-norm in the triangle inequality.

In this paper we have attempted to deal with this problem and have been able to establish some important results involving completeness and compactness of finite dimensional fuzzy normed linear spaces including the celebrated Riesz Lemma.

The organization of the paper is as in the following:
Section 2 comprises some preliminary results. In Section 3, we introduce a definition of fuzzy norm by using general t-norm and illustrate by an example the situation where the decomposition theorem for fuzzy norm into a family of crisp norms is not valid. Some basic results of finite dimensional fuzzy normed linear spaces are established in Section 4. In the last Section 5, Riesz lemma is established in general t-norm setting.

2. Preliminaries

In this section some definitions and preliminary results are given which are used in this paper.

Definition 2.1 ([1]). Let U be a linear space over a field $\mathbb{F}$. A fuzzy subset N of $U \times \mathbb{R}$ is called a fuzzy norm on U if $\forall x, u \in U$ and $c \in \mathbb{F}$,
(N1) $\forall t \in \mathbb{R}$ with $t \leq 0,\ N(x\ ,\ t) = 0$;
(N2) $(\forall t \in \mathbb{R}, t > 0, N(x\ ,\ t) = 1)$ iff $x = \underline{0}$;
(N3) $\forall t \in \mathbb{R},\ t > 0,\ N(cx\ ,\ t) = N(x\ ,\ \frac{t}{|c|})$ if $c \neq 0$;
(N4) $\forall s, t \in \mathbb{R};\ x, u \in U$;
$N(x + u\ ,\ s + t) \geq\ min\{N(x\ ,\ s),\ N(u\ ,\ t)\}$;
(N5) $N(x\ ,\ .)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x\ ,\ t) = 1$.
The pair $(U\ ,\ N)$ will be referred to as a fuzzy normed linear space.

Theorem 2.2 ([1]). (Decomposition theorem) *Let $(U\ ,\ N)$ be a fuzzy normed linear space. Assume that,*
(N6) $\forall t > 0,\ N(x\ ,\ t) > 0\ \Rightarrow x = \underline{0}$.
Define $||x||_\alpha\ = \bigwedge\{t > 0:\ N(x\ ,\ t) \geq \alpha\},\ \alpha \in (0\ ,\ 1)$. *Then* $\{||\ ||_\alpha : \alpha \in (0\ ,\ 1)\}$ *is an ascending family of norms on U and they are called α-norms on U corresponding to the fuzzy norm on U.*

Definition 2.3 ([1]). Let $(U\ ,\ N)$ be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in U. Then $\{x_n\}$ is said to be convergent if $\exists x \in U$ such that

$$\lim_{n\to\infty} N(x_n - x\ ,\ t) = 1\ \forall t > 0.$$

In that case x is called the limit of the sequence $\{x_n\}$ and is denoted by $lim x_n$.

Definition 2.4 ([1]). A sequence $\{x_n\}$ in U is said to be a Cauchy-sequence if $\lim_{n\to\infty} N(x_{n+p} - x_n\ ,\ t) = 1\ \forall t > 0$ and $p = 1, 2, 3, \ldots\ldots$

Definition 2.5 ([1]). Let $(U\ ,\ N)$ be a fuzzy normed linear space. A subset F of U is said to be closed if for any sequence $\{x_n\}$ in F converges to x. i.e.,

$$\lim_{n\to\infty} N(x_n - x\ ,\ t) = 1\ \forall t > 0 \text{ implies that } x \in F.$$

Definition 2.6 ([1]). Let $(U\ ,\ N)$ be a fuzzy normed linear space. A subset B of U is said to be the closure of F if for any $x \in B$, $\exists$ a sequence $\{x_n\}$ in F such that $\lim_{n\to\infty} N(x_n - x\ ,\ t) = 1\ \forall t > 0$. We denote the set B by $\bar{F}$.

Definition 2.7 ([1]). Let $(U\ ,\ N)$ be a fuzzy normed linear space. A subset A of U is said to be compact if any sequence $\{x_n\}$ in A has a subsequence converging to an element of A.

Definition 2.8 ([2]). Let $(U\ ,\ N)$ be a fuzzy normed linear space. We define a set $B(x,\alpha,t)$ as $B(x,\alpha,t) = \{y\ : N(x-y\ ,\ t) > 1-\alpha\}$.

Theorem 2.9 ([2]). *Let $(U\ ,\ N)$ be a fuzzy normed linear space. If we define $\tau = \{G \subset U :\ x \in G$ iff $\exists t > 0$ and $0 < \alpha < 1$ such that $B(x,\alpha,t) \subset G\}$, then τ is a topology on $(U\ ,\ N)$.*

Definition 2.10 ([2]). Members of τ are called open sets in $(U\ ,\ N)$.

Note 2.11 ([2]). *It is to be noted that the underlying topology of the "closeness" and "compactness" given above is τ.*

Definition 2.12 ([11]). A binary operation $* : [0\ ,\ 1] \times [0\ ,\ 1] \to [0\ ,\ 1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative;
(2) $a * 1 = a\ \ \forall a \in [0\ ,\ 1]$;
(3) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for each $a,b,c,d \in [0\ ,\ 1]$.

If $*$ is continuous then it is called continuous t-norm. The following are examples of some t-norms that are frequently used as fuzzy intersections defined for all $a,\ b \in\ [0,\ 1]$.

(i) Standard intersection: $a * b = min(a,\ b)$.
(ii) Algebraic product: $a * b = ab$.
(iii) Bounded difference: $a * b = max(0,\ a + b - 1)$.

$$a * b = \begin{cases} a & \text{for } b = 1 \\ b & \text{for } a = 1 \\ 0 & \text{for otherwise.} \end{cases}$$

The relations among the t-norms are

$$a * b(\text{Drastic}) \le\ max(0,\ a + b - 1) \le\ ab \le\ min(a,\ b).$$

3. General definition of fuzzy normed linear spaces

In this section we take definition of a fuzzy norm on a linear space in its general form in the sense that here the "*min*" of the condition (N4) of Definition 2.1 is replaced by a t-norm.

Definition 3.1. Let U be a linear space over a field $\mathbb{F}$. A fuzzy subset N of $U \times \mathbb{R}$ is called a fuzzy norm on U if $\forall x, u \in U$ and $c \in \mathbb{F}$,

(N1) $\forall t \in \mathbb{R}$ with $t \le 0$, $N(x\ ,\ t) = 0$;
(N2) $(\forall t \in \mathbb{R}, t > 0, N(x\ ,\ t) = 1)$ iff $x = \underline{0}$;
(N3) $\forall t \in \mathbb{R},\ t > 0,\ N(cx\ ,\ t) = N(x\ ,\ \frac{t}{|c|})$ if $c \neq 0$;
(N4) $\forall s, t \in \mathbb{R};\ x, u \in U$;

$$N(x+u\ ,\ s+t) \geq\ N(x\ ,\ s) * N(u\ ,\ t);$$

(N5) $N(x\ ,\ .)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x\ ,\ t) = 1$.

The pair $(U\ ,\ N)$ will be referred to as a fuzzy normed linear space.

The following example shows that $(X\ ,\ N)$ is not a fuzzy normed linear space w.r.t. the continuous t-norm "*min*" but it is a fuzzy normed linear space w.r.t.the continuous "t"-norm "*product*" but decomposition theorem from fuzzy norm into a family of crisp norms does not hold.

Example 3.2. Let $(X = \mathbb{R}^2)$ be a linear space over the field $\mathbb{R}$(the set of real numbers),

$x = (x_1,\ x_2) \in\ X$ and $N : X \times \mathbb{R} \to [0,\ 1]$ defined by

$$N(x,\ t) = \begin{cases} \frac{t^2}{(t+|x_1|)(t+|x_2|)} & \text{for } t\ > 0 \\ 0 & \text{for } t \leq\ 0. \end{cases}$$

$$a * b = \begin{cases} a & \text{for } b = 1 \\ b & \text{for } a = 1 \\ 0 & \text{for otherwise.} \end{cases}$$

Then $(X,\ N,\ *)$ is not a fuzzy normed linear space w.r.t. the continuous t-norm "*min*" but it is a fuzzy normed linear space w.r.t. the continuous "t"-norm "*product*".

Proof. First we shall show that $(X,\ N)$ is not a fuzzy normed linear space w.r.t. the continuous t-norm "*min*". Let $x = (1,\ 0),\ y = (0,\ 1),\ t = s = 1$ then $N(x,\ t) = \frac{1}{2} = N(y,\ s)$ and $N(x+y,\ t+s) = \frac{4}{9}$. Thus $N(x+y,\ t+s) < min\{N(x,\ t),\ N(y,\ s)\}$. So N does not satisfy the condition $(N4)$. Hence $(X,\ N)$ is not a fuzzy normed linear space w.r.t. the continuous t-norm "*min*".

Next we shall show that $(X,\ N,\ *)$ is a fuzzy normed linear space for continuous t-norm "*product*".

(N1). $\forall t \in \mathbb{R}$ with $t \leq 0,\ \ N(x,\ t) = 0$ (from definition).

(N2). $\forall t \in \mathbb{R},\ \ t >\ 0,\ \ N(x,\ t) = 1$

$$\Rightarrow\ \frac{t^2}{(t+|x_1|)(t+|x_2|)} = 1$$
$$\Rightarrow\ t^2 = t^2 + t(|x_1| + |x_2|) + |x_1x_2|$$
$$\Rightarrow\ t(|x_1| + |x_2|) + |x_1x_2| = 0\ \ \forall\ t > 0$$
$$\Rightarrow\ |x_1| = 0,\ \ |x_2| = 0 \text{ i.e. } x = \theta.$$

Conversely if $x = \theta$ then $N(x,\ t) = 1\ \ \forall t \in \mathbb{R},\ \ t >\ 0$ (from definition).

(N3). $\forall\ t \in R,\ \ t >\ 0,\ \ N(cx,\ t) = \frac{t^2}{(t+|cx_1|)(t+|cx_2|)}$

$$= \frac{(\frac{t}{c})^2}{(\frac{t}{|c|}+|x_1|)(\frac{t}{|c|}+|x_2|)}$$
$$= N(x,\ \tfrac{t}{|c|}) \text{ if } c \neq\ 0.$$

(N4). Let $t > 0,\ s > 0$ then

$$N(x+y,\ t+s) - N(x,\ t) * N(y,\ s)$$
$$= \frac{(t+s)^2}{(t+s+|x_1+y_1|)(t+s+|x_2+y_2|)} - \frac{t^2s^2}{(t+|x_1|)(t+|x_2|)(s+|y_1|)(s+|y_2|)}$$
$$\geq\ \frac{t^2+2st+s^2}{(t+s+|x_1|+|y_1|)(t+s+|x_2|+|y_2|)} - \frac{t^2s^2}{(t^2+t|x_1|+t|x_2|+|x_1x_2|)(s^2+s|y_1|+s|y_2|+|y_1y_2|)}$$
$$= \big\{(t^2 + 2st + s^2)(t^2s^2 + t^2s|y_1| + t^2s|y_2| + t^2|y_1y_2| + ts^2|x_1| + ts|x_1y_1|$$

$$+ts|x_1y_2| + t|x_1y_1y_2| + ts^2|x_2| + ts|x_2y_1| + ts|x_2y_2| + t|x_2y_1y_2| + s^2|x_1x_2|$$
$$+s|x_1x_2y_1| + s|x_1x_2y_2| + |x_1x_2y_1y_2|) - t^2s^2(t^2 + st + t|x_2| + t|y_2| + st + s^2$$
$$+s|x_2| + s|y_2| + t|x_1| + s|x_1| + |x_1x_2| + |x_1y_2| + t|y_1| + s|y_1| + |y_1x_2| + |y_1y_2|)\Big\}$$
$$\times \frac{1}{(t+s+|x_1|+|y_1|)(t+s+|x_2|+|y_2|)(t^2+t|x_1|+t|x_2|+|x_1x_2|)(s^2+s|y_1|+s|y_2|+|y_1y_2|)}$$

$$= \Big\{(t^4s^2 + t^4s|y_1| + t^4s|y_2| + t^4|y_1y_2| + t^3s^2|x_1| + t^3s|x_1y_1| + t^3s|x_1y_2| + t^3|x_1y_1y_2|$$
$$+t^3s^2|x_2| + t^3s|x_2y_1| + t^3s|x_2y_2| + t^3|x_2y_1y_2| + t^2s^2|x_1x_2| + t^2s|x_1x_2y_1| + t^2s|x_1x_2y_2|$$
$$+t^2|x_1x_2y_1y_2| + 2s^3t^3 + 2s^2t^3|y_1| + 2s^2t^3|y_2| + 2st^3|y_1y_2| + +2s^3t^2|x_1| + 2t^2s^2|x_1y_1|$$
$$+2t^2s^2|x_1y_2| + 2st^2|x_1y_1y_2| + 2t^2s^3|x_2| + 2t^2s^2|x_2y_1| + 2s^2t^2|x_2y_2| + 2st^2|x_2y_1y_2|$$
$$+2s^3t|x_1x_2| + 2s^2t|y_1x_1x_2| + 2s^2t|y_2x_1x_2| + 2st|y_1y_2x_1x_2| + s^4t^2 + s^3t^2|y_1|$$
$$+s^3t^2|y_2| + s^2t^2|y_1y_2| + s^4t|x_1| + s^3t|y_1x_1| + s^3t|x_1y_2| + s^2t|y_1x_1y_2| + s^4t|x_2|$$
$$+s^3t|y_1x_2| + s^3t|y_2x_2| + s^2t|y_1y_2x_2| + s^4|x_1x_2| + s^3|y_1x_1x_2| + s^3|y_2x_1x_2|$$
$$+s^2|y_1y_2x_1x_2|) - \big(s^2t^4 + s^3t^3 + s^2t^3|x_2| + s^2t^3|y_2| + s^3t^3 + s^4t^2 + s^3t^2|x_2|$$
$$+s^3t^2|y_2| + s^2t^3|x_1| + s^3t^2|x_1| + s^2t^2|x_1x_2| + s^2t^2|x_1y_2| + s^2t^3|y_1| + s^3t^2|y_1|$$
$$+s^2t^2|y_1x_2| + s^2t^2|y_1y_2|\big)\Big\}$$
$$\times \Big\{\frac{1}{(t+s+|x_1|+|y_1|)(t+s+|x_2|+|y_2|)(t^2+t|x_1|+t|x_2|+|x_1x_2|)(s^2+s|y_1|+s|y_2|+|y_1y_2|)}\Big\}$$

$$= \Big\{(\underbrace{t^4s^2}_{(i)} + t^4s|y_1| + t^4s|y_2| + t^4|y_1y_2| + \underbrace{t^3s^2|x_1|}_{(ii)} + t^3s|x_1y_1| + t^3s|x_1y_2| + t^3|x_1y_1y_2|$$
$$+\underbrace{t^3s^2|x_2|}_{(iii)} + t^3s|x_2y_1| + t^3s|x_2y_2| + t^3|x_2y_1y_2| + \underbrace{t^2s^2|x_1x_2|}_{(iv)} + t^2s|x_1x_2y_1| + t^2s|x_1x_2y_2|$$
$$+t^2|x_1x_2y_1y_2| + \underbrace{2s^3t^3}_{(v)} + \underbrace{2s^2t^3|y_1|}_{(vi)} + \underbrace{2s^2t^3|y_2|}_{(vii)} + 2st^3|y_1y_2| + \underbrace{2s^3t^2|x_1|}_{(viii)} + 2t^2s^2|x_1y_1|$$
$$+\underbrace{2t^2s^2|x_1y_2|}_{(ix)} + 2st^2|x_1y_1y_2| + \underbrace{2t^2s^3|x_2|}_{(x)} + \underbrace{2t^2s^2|x_2y_1|}_{(xi)} + 2s^2t^2|x_2y_2| + 2st^2|x_2y_1y_2|$$
$$+2s^3t|x_1x_2| + 2s^2t|y_1x_1x_2| + 2s^2t|y_2x_1x_2| + 2st|y_1y_2x_1x_2| + \underbrace{s^4t^2}_{(xii)} + \underbrace{s^3t^2|y_1|}_{(xiii)} + \underbrace{s^3t^2|y_2|}_{(xiv)}$$
$$+\underbrace{s^2t^2|y_1y_2|}_{(xv)} + s^4t|x_1| + s^3t|y_1x_1| + s^3t|x_1y_2| + s^2t|y_1x_1y_2| + s^4t|x_2| + s^3t|y_1x_2|$$
$$+s^3t|y_2x_2| + s^2t|y_1y_2x_2| + s^4|x_1x_2| + s^3|y_1x_1x_2| + s^3|y_2x_1x_2| + s^2|y_1y_2x_1x_2|)$$
$$-\Big(\underbrace{s^2t^4}_{(i)} + \underbrace{2s^3t^3}_{(ii)} + \underbrace{s^2t^3|x_2|}_{(iii)} + \underbrace{s^2t^3|y_2|}_{(iv)} + \underbrace{s^4t^2}_{(v)} + \underbrace{s^3t^2|x_2|}_{(vi)} + \underbrace{s^3t^2|y_2|}_{(vii)} + \underbrace{s^2t^3|x_1|}_{(viii)}$$
$$+\underbrace{s^3t^2|x_1|}_{(ix)} + \underbrace{s^2t^2|x_1x_2|}_{(x)} + \underbrace{s^2t^2|x_1y_2|}_{(ix)} + \underbrace{s^2t^3|y_1|}_{(xii)} + \underbrace{s^3t^2|y_1|}_{(xiii)} + \underbrace{s^2t^2|y_1x_2|}_{(xiv)} + \underbrace{s^2t^2|y_1y_2|}_{(xv)}\Big)\Big\}$$

$\times\{\frac{1}{(t+s+|x_1|+|y_1|)(t+s+|x_2|+|y_2|)(t^2+t|x_1|+t|x_2|+|x_1x_2|)(s^2+s|y_1|+s|y_2|+|y_1y_2|)}\} > \ 0.$

Other cases are trivially satisfied.

(N5). $\lim_{t\to\infty} N(x,\ t) = \lim_{t\to\infty} \dfrac{t^2}{(t+|x_1|)(t+|x_2|)} = 1$. Hence $(X\ ,\ N)$ is a fuzzy normed linear space w.r.t.the continuous "t"-norm "*product*".

Now, $N(x\ ,\ t) \geq \alpha \Rightarrow \frac{t^2}{(t+|x_1|)(t+|x_2|)} \geq \alpha$
$\Rightarrow t^2 \geq \alpha t^2 + t\alpha(|x_1| + |x_2|) + \alpha|x_1||x_2|$
$\Rightarrow (1-\alpha)t^2 - t\alpha(|x_1| + |x_2|) - \alpha|x_1||x_2| \geq 0$
$\Rightarrow (t-a)(t-b) \geq 0,$

where

$$a(= \tfrac{\alpha(|x_1|+|x_2|)+\sqrt{\alpha^2(|x_1|+|x_2|)^2+4\alpha(1-\alpha)|x_1||x_2|}}{2(1-\alpha)})$$

and

$$b(= \tfrac{\alpha(|x_1|+|x_2|)-\sqrt{\alpha^2(|x_1|+|x_2|)^2+4\alpha(1-\alpha)|x_1||x_2|}}{2(1-\alpha)})$$

are the roots of the equation $(1-\alpha)t^2 - t\alpha(|x_1| + |x_2|) - \alpha|x_1||x_2| = 0$. So

$$||x||_\alpha = \ \bigwedge\{t > 0 :\ N(x\ ,\ t) \geq \alpha\} = \tfrac{\alpha(|x_1|+|x_2|)+\sqrt{\alpha^2(|x_1|+|x_2|)^2+4\alpha(1-\alpha)|x_1||x_2|}}{2(1-\alpha)}$$

for all $\alpha \in (0\ ,\ 1)$. Take $x = (1\ ,\ 0)$ and $y = (0\ ,\ 1)$. Then

$$||x||_\alpha = \tfrac{\alpha}{1-\alpha}\ ,\quad ||y||_\alpha = \tfrac{\alpha}{1-\alpha}\ \text{ and }\ ||x+y||_\alpha = \tfrac{\alpha+\sqrt{\alpha}}{1-\alpha}$$

for $\alpha \in (0\ ,\ 1)$. Now $||x||_\alpha + ||y||_\alpha = \frac{2\alpha}{1-\alpha} = \frac{\alpha+\alpha}{1-\alpha} < \frac{\alpha+\sqrt{\alpha}}{1-\alpha} = ||x+y||_\alpha\ ,\ \forall\alpha \in (0\ ,\ 1)$. Hence $||x||_\alpha$ fails to satisfy the triangle inequality of a norm. Thus decomposition theorem from fuzzy norm into a family of crisp norms does not hold. □

Remerk 3.3. *Above example is a fuzzy normed linear space X with respect to the continuous t-norm "product" but for $x \in X$, the function from X to non-negative real number given by $\bigwedge\{t > 0;\ N(x\ ,\ t) \geq \alpha\}$ is not a norm function on X. If , however N is a fuzzy norm on X with respect to a t-norm $*$ satisfying the property $\alpha * \alpha = \alpha$ for some $\alpha \in (0\ ,\ 1)$, for that α, $\bigwedge\{t > 0;\ N(x\ ,\ t) \geq \alpha\},\ x \in X$ will be a norm on X.*

4. Finite dimensional fuzzy normed linear spaces

In this section we study completeness and compactness properties over finite dimensional fuzzy normed linear spaces. Firstly consider the following Lemma which plays the key role in studying properties of finite dimensional fuzzy normed linear spaces.

Lemma 4.1. *Let $(U\ ,\ N)$ be a fuzzy normed linear space with the underlying t-norm $*$ continuous at (1 , 1) and $\{x_1,\ x_2,\ ....,x_n\}$ be a linearly independent set* *of vectors in U. Then* $\exists c > 0$ *and* $\delta \in (0\ ,\ 1)$ *such that for any set of scalars* $\{\alpha_1,\ \alpha_2, ..., \alpha_n\}$*;*

$$N(\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n,\ c\sum_{j=1}^{n} |\alpha_j|)\ <\ 1 - \delta. \tag{4.1}$$

Proof. Let $s = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$. If $s = 0$ then $\alpha_j = 0\ \forall j = 1, 2, ..., n$ and the relation (4.1) holds for any $c > 0$ and $\delta \in (0\ ,\ 1)$. Next we suppose that $s > 0$. Then (4.1) is equivalent to

$$N(\beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n,\ c)\ <\ 1 - \delta \tag{4.2}$$

for some $c > 0$ and $\delta \in (0\ ,\ 1)$, and for all scalars β's with $\sum_{j=1}^{n} |\beta_j| = 1$. If possible suppose that (4.2) does not hold. Thus for each $c > 0$ and $\delta \in (0\ ,\ 1)$, $\exists$ a set of scalars $\{\beta_1,\ \beta_2, ..., \beta_n\}$ with $\sum_{j=1}^{n} |\beta_j| = 1$ for which

$$N(\beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n,\ c)\ \geq 1 - \delta.$$

Then for $c = \delta = \frac{1}{m},\ m = 1, 2, ...,$ $\exists$ a set of scalars $\{\beta_1^{(m)},\ \beta_2^{(m)}, ..., \beta_n^{(m)}\}$ with $\sum_{j=1}^{n} |\beta_j^{(m)}| = 1$ such that $N(y_m\ ,\ \frac{1}{m})\ \geq 1 - \frac{1}{m}$ where

$$y_m =\ \beta_1^{(m)} x_1 + \beta_2^{(m)} x_2 + ... + \beta_n^{(m)} x_n.$$

Since $\sum_{j=1}^{n} |\beta_j^{(m)}| = 1$, we have $0 \leq |\beta_j^{(m)}|\ \ \leq 1$ for $j = 1, 2, ..., n$. So for each fixed j the sequence $\{\beta_j^{(m)}\}$ is bounded and hence $\{\beta_1^{(m)}\}$ has a convergent subsequence. Let β_1 denote the limit of that subsequence and let $\{y_{1,m}\}$ denote the corresponding subsequence of $\{y_m\}$. By the same argument $\{y_{1,m}\}$ has a subsequence $\{y_{2,m}\}$ for which the corresponding subsequence of scalars $\{\beta_2^{(m)}\}$ converges to β_2 (say). Continuing in this way, after n steps we obtain a subsequence $\{y_{n,m}\}$ where

$$y_{n,m} =\ \sum_{j=1}^{n} \gamma_j^{(m)} x_j \text{ with } \sum_{j=1}^{n} |\gamma_j^{(m)}| = 1 \text{ and } \gamma_j^{(m)} \rightarrow \beta_j \text{ as } m \rightarrow \infty.$$

Let $y = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n$. Thus we have

$$\lim_{m\to\infty} N(y_{n,m} - y\ ,\ t) = 1\ \forall t > 0. \tag{4.3}$$

Now for $k > 0$, choose m such that $\frac{1}{m}\ < k$. We have

$$\begin{aligned} N(y_{n,m}\ ,\ k) &= N(y_{n,m} + \underline{0}\ ,\ \frac{1}{m} + k - \frac{1}{m}) \\ &\geq N(y_{n,m}\ ,\ \frac{1}{m}) * N(\underline{0}\ ,\ k - \frac{1}{m}) \\ &\geq (1 - \frac{1}{m}) * N(\underline{0}\ ,\ k - \frac{1}{m}). \end{aligned}$$

i.e. $N(y_{n,m}\ ,\ k) \geq (1 - \frac{1}{m}) * N(\underline{0}\ ,\ k - \frac{1}{m})$.
i.e. $\lim_{m\to\infty} N(y_{n,m}\ ,\ k) \geq 1$. i.e.

$$\lim_{m\to\infty} N(y_{n,m}\ ,\ k)\ = 1. \tag{4.4}$$

Now

$N(y\ ,\ 2k) = N(y - y_{n,m} + y_{n,m}\ ,\ k + k)\ \geq N(y - y_{n,m}\ ,\ k) * N(y_{n,m}\ ,\ k)$
$\Rightarrow N(y\ ,\ 2k) \geq \lim_{m\to\infty} N(y - y_{n,m}\ ,\ k) * \lim_{m\to\infty} N(y_{n,m}\ ,\ k)$
(by the continuity of t-norm $*$ at $(1,\ 1)$).
$\Rightarrow N(y\ ,\ 2k) \geq 1 * 1$ by (4.3) & (4.4)
$\Rightarrow N(y\ ,\ 2k)\ = 1 * 1 = 1$.

Since $k > 0$ is arbitrary, by (N2) it follows that $y = \underline{0}$. Again since $\sum_{j=1}^{n} |\beta_j^{(m)}| = 1$ and $\{x_1,\ x_2,\ ....,x_n\}$ are linearly independent set of vectors, so

$$y = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n \neq \underline{0}.$$

Thus we arrive at a contradiction and the lemma is proved. □

Theorem 4.2. *Every finite dimensional fuzzy normed linear space $(U,\ N)$ with the continuity of the underlying t-norm $*$ at $(1,\ 1)$ is complete.*

Proof. Let (U, N) are fuzzy normed linear space and dimU=k (say). Let $\{e_1, e_2, \cdots, e_k\}$ be a basis for U and $\{x_n\}$ be a Cauchy sequence in U. Let

$$x_n =\ \beta_1^{(n)} e_1 + \beta_2^{(n)} e_2 + ... + \beta_k^{(n)} e_k$$

where $\beta_1^{(n)},\ \beta_2^{(n)}, ...,\beta_k^{(n)}$ are suitable scalars. So

$$\lim_{m,n\to\infty} N(x_m - x_n\ ,\ t) = 1\ \forall t > 0. \tag{4.5}$$

Now from Lemma 4.1, it follows that $\exists c > 0$ and $\delta \in (0, 1)$ such that

$$N\left(\sum_{i=1}^{k} (\beta_i^{(m)} - \beta_i^{(n)}) e_i\ ,\ c \sum_{i=1}^{k} |\beta_i^{(m)} - \beta_i^{(n)}|\right)\ < 1 - \delta. \tag{4.6}$$

Again for $1 > \delta > 0$, from (4.5), it follows that $\exists$ a positive integer $n_0(\delta, t)$ such that

$$N\left(\sum_{i=1}^{k} (\beta_i^{(m)} - \beta_i^{(n)}) e_i\ , t\right) > 1 - \delta\ \forall m, n \geq n_0(\delta, t). \tag{4.7}$$

Now from (4.6) and (4.7), we have

$$N\left(\sum_{i=1}^{k} (\beta_i^{(m)} - \beta_i^{(n)}) e_i\ ,\ t\right) > 1 - \delta$$
$$> N\left(\sum_{i=1}^{k} (\beta_i^{(m)} - \beta_i^{(n)}) e_i\ ,\ c \sum_{i=1}^{k} |\beta_i^{(m)} - \beta_i^{(n)}|\right)\ \ \forall m, n \geq n_0(\delta, t)$$

$\Rightarrow c\sum_{i=1}^{k}|\beta_i^{(m)}-\beta_i^{(n)}| \ < \ t \ \forall m,n \geq n_0(\delta,t)$ (since $N(x,.)$ is nondecreasing in t).
$\Rightarrow \sum_{i=1}^{k}|\beta_i^{(m)}-\beta_i^{(n)}| \ < \ \frac{t}{c} \ \ \forall m,n \geq n_0(\delta,t)$
$\Rightarrow |\beta_i^{(m)}-\beta_i^{(n)}| \ < \ \frac{t}{c} \ \ \forall m,n \geq n_0(\delta,t)$ and $i=1,2,....,k$.
Since $t>0$ is arbitrary, from above we have,
$\lim_{m,n\to\infty}|\beta_i^{(m)}-\beta_i^{(n)}|=0$ for $i=1,2,.....,k$.
$\Rightarrow \{\beta_i^{(n)}\}$ is a Cauchy sequence of scalars for each $i=1,2,.....,k$.
So each sequence $\{\beta_i^{(n)}\}$ converges. Let $\lim_{n\to\infty}\beta_i^{(n)}=\beta_i$ for $i=1,2,.....,k$ and $x=\sum_{i=1}^{k}\beta_i e_i$. Clearly $x\in U$. Now $\forall t>0$,

$$N(x_n-x\ ,\ t)=N\left(\sum_{i=1}^{k}\beta_i^{(n)}e_i-\sum_{i=1}^{k}\beta_i e_i\ ,\ t\right)=N\left(\sum_{i=1}^{k}(\beta_i^{(n)}-\beta_i)e_i\ ,\ t\right).$$

i.e.

$$N(x_n-x\ ,\ t)\geq N\left(e_1\ ,\ \frac{t}{k|\beta_1^{(n)}-\beta_1|}\right)*N\left(e_2\ ,\ \frac{t}{k|\beta_2^{(n)}-\beta_2|}\right)*....$$
$$(4.8)\qquad ....*N\left(e_k\ ,\ \frac{t}{k|\beta_k^{(n)}-\beta_k|}\right).$$

When $n\to\infty$ then $\frac{t}{k|\beta_i^{(n)}-\beta_i|}\to\infty$ (since $\beta_i^{(n)}\to\beta_i$) for $i=1,2,....,k$ and $\forall t>0$.
From (4.8) we get, using the continuity of t-norm $*$ at (1 , 1),
$\lim_{n\to\infty}N(x_n-x\ ,\ t)\geq 1*1*........*1\ \forall t>0$
$\Rightarrow \lim_{n\to\infty}N(x_n-x\ ,\ t)\ =1\ \forall t>0$
$\Rightarrow x_n\to x$
$\Rightarrow$ U is complete. □

Definition 4.3. Let $(U\ ,\ N)$ be a fuzzy normed linear space and $A\subset U$. A is said to be fuzzy bounded if for each r, $0<r<1$, $\exists t>0$ such that $N(x\ ,\ t)\ >1-r\ \ \forall x\in A$.

Theorem 4.4. *Let $(U\ ,\ N)$ be a finite dimensional fuzzy normed linear space in which the underlying t-norm $*$ is continuous at (1 , 1). Then a subset A is compact iff A is closed and bounded.*

Proof. First we suppose that A is compact. We have to show that A is closed and bounded. Let $x\in\bar{A}$. Then $\exists$ a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty}x_n=x$. Since A is compact, $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to a point in A. Again $\{x_n\}\to x$ so $\{x_{n_k}\}\to x$ and hence $x\in A$. So A is closed. If possible suppose that A is not bounded. Then $\exists r=r_0,\ 0<r_0<1$, such that for each positive integer n, $\exists x_n\in A$ such that $N(x_n\ ,\ n)\ \leq 1-r_0$. Since A is compact, $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some element $x\in A$. Thus $\lim_{k\to\infty}N(x_{n_k}-x\ ,\ t)=1\ \ \forall t>0$. Also

$N(x_{n_k}\ ,\ n_k) \leq 1 - r_0$. Now $1 - r_0 \geq N(x_{n_k}\ ,\ n_k) = N(x_{n_k} - x + x\ ,\ n_k - t + t)$ where $t > 0$

$\Rightarrow 1 - r_0 \geq N(x_{n_k} - x\ ,\ t) * N(x\ ,\ n_k - t)$

$\Rightarrow 1 - r_0 \geq \lim_{k\to\infty} N(x_{n_k} - x\ ,\ t) * \lim_{k\to\infty} N(x\ ,\ n_k - t)$

$\Rightarrow 1 - r_0 \geq 1 * 1 = 1$ (using the continuity of t-norm at (1 , 1))

$\Rightarrow r_0 \leq 0$ which is a contradiction. Hence A is bounded.

Conversely suppose that A is closed and bounded and we have to show that A is compact. Let dim U=n and $\{e_1,\ ,e_2, ..., e_n\}$ be a basis for U. Choose a sequence $\{x_k\}$ in A and suppose $x_k = \beta_1^{(k)}e_1 + \beta_2^{(k)}e_2 + ... + \beta_n^{(k)}e_n$ where $\beta_1^{(k)},\ \beta_2^{(k)}, ..., \beta_n^{(k)}$ are scalars. Now from Lemma 4.1, $\exists c > 0$ and $\delta \in (0, 1)$ such that

$$N\left(\sum_{i=1}^{n}\beta_i^{(k)}e_i\ ,\ c\sum_{i=1}^{n}|\beta_i^{(k)}|\right)\ <\ 1-\delta. \tag{4.9}$$

Again since A is bounded, for $\delta \in (0\ ,\ 1)$ such that $N(x\ ,\ t)\ >\ 1-\delta\ \ \forall x \in A$. So

$$N\left(\sum_{i=1}^{n}\beta_i^{(k)}e_i\ ,\ t\right)\ >\ 1-\delta. \tag{4.10}$$

From (4.9) and (4.10) we get,

$N(\sum_{i=1}^{n}\beta_i^{(k)}e_i\ ,\ c\sum_{i=1}^{n}|\beta_i^{(k)}|)\ <\ 1-\delta\ < N(\sum_{i=1}^{n}\beta_i^{(k)}e_i\ ,\ t)$

$\Rightarrow N(\sum_{i=1}^{n}\beta_i^{(k)}e_i\ ,\ c\sum_{i=1}^{n}|\beta_i^{(k)}|)\ <\ N(\sum_{i=1}^{n}\beta_i^{(k)}e_i\ ,\ t)$

$\Rightarrow c\sum_{i=1}^{n}|\beta_i^{(k)}|\ <\ t$ (since $N(x\ ,\ .)$ is non-decreasing)

$\Rightarrow |\beta_i^{(k)}|\ \leq \frac{t}{c}$ for $k = 1, 2, ...$ and $i = 1, 2, ..., n$.

So each sequence $\{\beta_i^{(k)}\}$ $(i = 1, 2, ..., n)$ is bounded. By repeated applications of Bolzano-Weierstrass theorem, it follows that each of the sequences $\{\beta_i^{(k)}\}$ has a convergent subsequence say $\{\beta_i^{k_l}\},\ \forall i = 1, 2, ..., n$. Let

$$x_{k_l}\ =\ \beta_1^{(k_l)}e_1 + \beta_2^{(k_l)}e_2 + ... + \beta_n^{(k_l)}e_n$$

and $\{\beta_1^{(k_l)}\},\ \{\beta_2^{(k_l)}\}, ..., \{\beta_n^{(k_l)}\}$ are all convergent. Let

$$\beta_i = \lim_{l\to\infty}\beta_i^{(k_l)},\ i = 1, 2, ..., n \text{ and } x\ =\ \beta_1e_1 + \beta_2e_2 + ... + \beta_ne_n.$$

Now for $t > 0$ we have

$$\begin{aligned} N(x_{k_l} - x\ ,\ t) &= N\left(\sum_{i=1}^{n}(\beta_i^{k_l} - \beta_i)e_i\ ,\ t\right) \\ &\geq N\left(e_1\ ,\ \frac{t}{n|\beta_1^{k_l} - \beta_1|}\right) * \cdots * N\left(e_n\ ,\ \frac{t}{n|\beta_n^{k_l} - \beta_n|}\right) \end{aligned}$$

$\Rightarrow \lim_{l\to\infty} N(x_{k_l} - x\ ,\ t)\ \geq 1 * 1 * .... * 1$ $(\beta_i^{k_l} \to \beta_i$ as $l \to \infty)$

(using the continuity of t-norm $*$ at $(1\ ,\ 1)$)
$\Rightarrow \lim\limits_{l\to\infty} N(x_{k_l} - x\ ,\ t)\ = 1.$
Since $t > 0$ is arbitrary, it follows that $\lim\limits_{l\to\infty} x_{k_l}\ = 1$. i.e. $\{x_{k_l}\}$ is a convergent subsequence of $\{x_k\}$ and converges to x. Since A is closed and $\{x_k\}$ is a sequence in A, it follows that $x \in A$. Thus every sequence in A has a convergent subsequence and converges to an element of A. Hence A is compact. □

5. Riesz lemma

In this section Riesz lemma is established in t-norm $(*)$ setting.

Lemma 5.1. *(Riesz) Let Y and Z be subspaces of a fuzzy normed linear space $(X\ ,\ N)$ and Y is a closed and proper subset of Z. Then for every real number $\theta \in (0\ ,\ 1)$, $\exists z \in Z$ such that $N(z\ ,\ 1) > 0$ and $N(z - y\ ,\ \theta) = 0\ \ \forall y \in Y$.*

Proof. Since Y is a proper subset of Z, $\exists v \in Z - Y$. Denote

$$d = \bigwedge_{y\in Y} \bigwedge\{t > 0\ : N(v - y\ ,\ t) > 0\}. \tag{5.1}$$

We claim that $d > 0$. If $d = 0$, i.e. $\bigwedge\limits_{y\in Y} \bigwedge\{t > 0\ : N(v - y\ ,\ t) > 0\} = 0$
$\Rightarrow$ for a given $\epsilon > 0$, $\exists y(\epsilon) \in Y$ such that $\bigwedge\{t > 0\ : N(v - y\ ,\ t) > 0\} < \epsilon$
$\Rightarrow N(v - y\ ,\ \epsilon) > 0.$
Choose $\alpha \in (0\ ,\ 1)$ such that $N(v - y\ ,\ \epsilon) > 1 - \alpha$. i.e. $y \in B(v, 1 - \alpha, \epsilon)$. Sine $\epsilon > 0$ is arbitrary, it follows that v is in the closure of Y. Since Y is closed, it implies that $v \in Y$ which is a contradiction. Thus $d\ > 0$. We now take $\theta \in (0\ ,\ 1)$. So $\frac{d}{\theta} > d$. Thus for some $y_0 \in Y$, we have

$$d \le \bigwedge\{t > 0\ : N(v - y_0\ ,\ t) > 0\}\ < k' < \frac{d}{\theta}. \tag{5.2}$$

Let $z = \frac{v - y_0}{k'}$. Now $N(z\ ,\ 1) = N(\frac{v-y_0}{k'}\ ,\ 1)$. i.e.

$$N(z\ ,\ 1) = N(v - y_0\ ,\ k'). \tag{5.3}$$

Then $N(v - y_0\ ,\ k') > 0$. From (5.3) we have $N(z\ ,\ 1) > 0$. Now for $y \in Y$,

$$\begin{aligned}\bigwedge\{t > 0\ : N(z - y\ ,\ t) > 0\} &= \bigwedge\{t > 0\ : N(v - y_0 - k'y\ ,\ k't) > 0\}\\ &= \frac{1}{k'}\bigwedge\{s > 0\ : N(v - y_0 - k'y\ ,\ s) > 0\}.\end{aligned}$$

i.e. $\bigwedge\{t > 0\ : N(z - y\ ,\ t) > 0\}\ \ge \frac{d}{k'}$ (since $y_0 + k'y \in Y$)
$\Rightarrow \bigwedge\{t > 0\ : N(z - y\ ,\ t) > 0\}\ > \theta$ by (5.2).
i.e. $N(z - y\ ,\ \theta) \le 0$
$\Rightarrow N(z - y\ ,\ \theta)\ = 0\ \forall y \in Y.$ □

Theorem 5.2. *Let $(X\ ,\ N)$ be a fuzzy normed linear space and $N(x\ ,\ .)$ $(x \ne \underline{0})$. If the set $M = \{x\ : N(x\ ,\ 1) > 0\}$ is compact then X is finite dimensional.*

Proof. If possible suppose that dim X=∞. Take $x_1 \in X$ such that $N(x_1\ ,\ 1) > 0$. Suppose X_1 is the subspace of X generated by x_1. Since dim $X_1 = 1$, it is a closed and proper subset of X. Thus by the Lemma 5.1, $\exists x_2 \in X$ such that $N(x_2\ ,\ 1) > 0$ and $N(x_2 - x_1\ ,\ \frac{1}{2})\ = 0$. The elements $x_1\ , x_2$ generate a two dimensional proper closed subspace of X. By Lemma 5.1, $\exists x_3 \in X$ with $N(x_3\ ,\ 1) > 0$ such that $N(x_3 - x_1\ ,\ \frac{1}{2})\ = 0$,
$N(x_3 - x_2\ ,\ \frac{1}{2})\ = 0$. Proceeding in the same way, we obtain a sequence $\{x_n\}$ of elements $x_n \in M$ such that

$$N(x_n\ ,\ 1) > 0 \text{ and } N(x_n - x_m\ ,\ \frac{1}{2})\ = 0\ (m \neq n). \tag{5.4}$$

From (5.4), it follows that neither the sequence $\{x_n\}$ nor its any subsequence converges. This contradicts the compactness of M. Hence dim X is finite. □

6. Conclusion

In this paper, we consider general t-norm replacing the particular "*min*" t-norm in our earlier definition of fuzzy normed linear space. Though the particular t-norm "*min*" facilitates the study of fuzzy functional analysis in many ways, specially because of the validity of unique decomposition of fuzzy norms into α-norms, but it creates a great annoyance that the theory works only for this particular choice of "min" t-norm in the triangle inequality of the fuzzy norm. This is a bit of undesirable phenomenon, specially in the context of fuzzy theory, where validity of a result on a wide choice of t-norms is of considerable importance. So it is a natural query-how far the results of fuzzy normed linear spaces can be extended to the case of fuzzy norm in its general form. In this paper we have attempted this problem in the context of finite dimensional fuzzy normed linear spaces and have been able to extended some important concepts with the relaxation of the "*min*" condition of the t-norm to a great extent. We think that there is a large scope of developing more results of fuzzy functional analysis in general t-norm setting.

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