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On convergence of inclined-valued fuzzy bidirectional associative memory neural networks with thresholds

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Abstract. In this paper, the incline-valued fuzzy bidirectional associative memory neural networks with thresholds (L-FBAMTNNs) are introduced. Some sufficient conditions for the L-FBAMTNNs to be strongly convergent and strongly stable are given. It is shown that the convergence index and the period of limit-cycles of an L-FBAMTNNs can be estimated by the indices and the periods of the product of connection weight matrices of the L-FBAMTNNs. Also the stable states and equilibria of an L-FBAMTNNs can be given by the standard eigenvectors of the product matrices of connection weight matrices.

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1. Introduction

Fuzzy neural networks (FNNs) have been first proposed by Lee and Lee in [9, 10]. As a hybrid intelligent system of soft computing technique, FNN is an efficient tool to deal with complex systems containing linguistic information and data information. FNNs have been applied in diverse areas such as pattern recognition, system modeling, image procession, control theory and so on.

Kosko studied fuzzy associative memory neural networks (FAMNNs) in [7] by introducing the max and min operators in associative memory networks and invented in [8] bidirectional associative memory neural networks (BAMNNs) which generalized the single layer associative memory networks. Since then, FBAMNNS based on max-min (resp., max-product, max-T, max-T_L, max-T_ξ) composition have extensively been studied [2, 3, 11, 13, 15, 16, 17, 18].

The fuzzy algebra [0,1] with max-min (resp., max-product, max-T, max-T_L, max-T_ξ) operations forms a special type of semirings, called an incline. Inclines and incline matrices have been applied in automata theory, medical diagnosis, informational systems and other fields. Therefore, combination of inclines and neural networks is a natural thing. Han firstly introduced the incline-valued fuzzy bidirectional associative memories (L-FBAMNNs) and investigated the convergence and stability of the L-FBAMNNs in [4].

It is well known that FAMNNs with thresholds generally have larger fault-tolerance than that without thresholds, so many achievements on FAMNNs with thresholds has been made [11, 12, 13]. In this paper, we set up the incline-valued bidirectional associative memories with thresholds (L-FBAMTNNs), and then discuss the strong convergence, strong stability, convergence index and period of limit-cycles, stable states and equilibria of the L-FBAMTNNs by the product matrices of connection weight matrices of a L-FBAMTNNs.

2. Preliminaries

Definition 2.1 ([1])**.** Let $+$ and $\cdot$ be two binary operations on a nonempty set L. An algebraic system $(L, +, \cdot)$ is called an *incline* if it satisfies the following conditions:

(1) $(L, +)$ is a semilattice,
(2) $(L, \cdot)$ is a semigroup,
(3) $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$ for all $x, y, z \in L$,
(4) $x + xy = xy + x = x$ for all $x, y \in L$.

In an incline L, define a relation $\leq$ by $x \leq y \Leftrightarrow x + y = y$. It is easy to see that $\leq$ is a partial order on L and $x + y$ is the least upper bound of $\{x, y\} \in L$. We say that $\leq$ is induced by the operation $+$. It follows that $xy \leq x$ and $yx \leq x$ for any $x, y \in L$.

We call the additive identity 0 and multiplicative identity 1 in L(if there exist such elements), respectively, the *zero* and the *identity* of L. It's easy to see that

$$0 + x = x + 0 = x, 0 \leq x, 0x = x0 = 0,\ x1 = 1x = x, x \leq 1, x + 1 = 1 + x = 1$$

for all $x \in L$. By an *incline with zero and identity* we mean an incline Lthat has both zero and identity satisfying $0 \neq 1$. In this paper, L always stands for an incline with zero and identity.

L is said to be *commutative* if $xy = yx$ for all $x, y \in L$.

An element $a \in L$ in is said to be *idempotent* if $a^2 = a$.We denote by $I(L)$ the set of all idempotent elements in L. Obviously, $0, 1 \in L$.

Theorem 2.2 ([5])**.** *If L is commutative, then $I(L)$ is a distributive lattice with the join and meet are, respectively, addition and multiplication of L.*

For any positive n,we denote by $\overline{n}$ and $[n]$, respectively, the set $\{1, 2, \ldots, n\}$ and the least common multiple of integers $1, 2, \ldots, n$.

Denote by $L^{m\times n}$,L^n and L_n, respectively, the set of all $m \times n$matrices, the set of all column vectors of order n and the set of all row vectors of order n over L.

For the addition, multiplication and scalar multiplication of incline matrices, the author may refer to [1].If there exit some positive integers k and d satisfying $A^k = A^{k+d}$,then the least such integers k and d are called the *index* and the *period* of A , and denoted by $i(A)$ and $p(A)$, respectively. In this case, we say that A *has index.* In particular, when $p(A) = 1$, we say that A *converges in finite steps.*

Theorem 2.3 ([5]). *If $A \in L^{n\times n}$ has index, then $p(A)|[n]$.*

Theorem 2.4 ([14]). *If D is a distributive lattice and $A \in L^{n\times n}$, then $i(A) \leq (n-1)^2 + 1$.*

Theorem 2.5 ([5]). *If $A \in I(L)^{n\times n}$ is symmetric, then $i(A) \leq 2n - 2$and $p(A) \leq 2$.*

Theorem 2.6 ([4]). *Let $A \in L^{n\times m}$ and $B \in L^{m\times n}$.If the matrix $AB \in L^{n\times n}$ has index, then the matrix $BA \in L^{m\times m}$ also has index, $p(AB) = p(BA)$ and $|i(AB) - i(BA)| \leq 1$.*

Let $A \in L^{n\times n}$.Denote by $\varepsilon(A)$, A_{*j} and $\mathcal{C}(A)$, respectively, the set of all standard eigenvectors of A, the jth column vector of A and the subsemimodule of L^n finitely generated by column vectors of A over L. It's easy to see that $\mathcal{C}(A)$ has the greatest element $\sum_{j\in \underline{n}} A_{*j}$, denoted by $\tau(A)$. Similarly, we denote by A_{i*} the row vector of A and by $\mathcal{R}(A)$ the subsemimodule of L_n finitely generated by row vectors of A over L. Then $\mathcal{R}(A)$ has the greatest element $\sum_{i\in \underline{n}} A_{i*}$, denoted by $\eta(A)$.For any positive integers k and l,we denote $A^{k,l} = A^k + A^{k+1} + \ldots + A^{k+l-1}$.

Theorem 2.7 ([6]). *If $A \in L^{n\times n}$ has index, then the following hold:*

(1) $\varepsilon(A) = \mathcal{C}(A^{i(A),p(A)})$,

(2) *$\tau(A^{i(A)})$ is the greatest standard eigenvector of A.*

3. Incline-valued FBAM With Thresholds

In this section, we introduce a threshold at each unit of the two-layer associative NNs, and get the incline-valued fuzzy bidirectional associative memory neural networks (L-FBAMNNs(W, R, P, Q)) with thresholds as follows:

$$\begin{cases} X^{(t+1)} = (Y^{(t)} + Q)R \\ Y^{(t+1)} = (X^{(t)} + P)W \end{cases}$$

where $t = 1, 2, \ldots$ is the iteration number and

$$X^{(t)} = (x_1^{(t)}, x_2^{(t)}, \ldots, x_n^{(t)}), Y^{(t)} = (y_1^{(t)}, y_2^{(t)}, \ldots, y_m^{(t)}),$$
$$P = (p_1, p_2, \ldots, p_n), Q = (q_1, q_2, \ldots, q_m),$$

m, n are natural numbers, and $W = (w_{ij})_{n\times m}$, $R = (r_{ji})_{m\times n}$ are the connection weight matrices. P, Q are called the threshold vectors.

Definition 3.1. Let $(X^{(0)}, Y^{(0)}) \in L_n \times L_m$ (we always assume that $Y^{(0)} = (X^{(0)} + P)W$) be any stimulating state of the L-FBAMNNs(W, R, P, Q). If there exist some positive integers k and l satisfying $(X^{(k)}, Y^{(k)}) = (X^{(k+l)}, Y^{(k+l)})$, then the L-FBAMNNs$(W, R, P, Q)$is said to be converge to a limit-cycle in finite steps for the stimulating state $(X^{(0)}, Y^{(0)})$, and the least such integers k and l are called the convergence index and the period of the limit-cycle of the L-FBAMNNs(W, R, P, Q) for $(X^{(0)}, Y^{(0)})$ and denoted by $i(X^{(0)}, Y^{(0)})$ and $p(X^{(0)}, Y^{(0)})$,respectively.

In particular, when $p(X^{(0)}, Y^{(0)}) = 1$, the L-FBAMNNs(W, R, P, Q) is said to be *converge to a state in finite steps* for the stimulating state $(X^{(0)}, Y^{(0)})$.

Definition 3.2. A L-FBAMNNs(W,R,P,Q) is said to be strongly convergent if it converges to a limit-cycle in finite steps for any stimulating state in $L_n \times L_m$.

Definition 3.3. A L-FBAMNNs(W,R,P,Q) is said to be strongly stable if it converges to a state in finite steps for any stimulating state in $L_n \times L_m$.

It is clear that very strongly stable L-FBAMNNs(W,R,P,Q) is strongly convergent.

Put

$$i_{L\text{-FBAM}} := \max\{i(X^{(0)},Y^{(0)}) | (X^{(0)},Y^{(0)}) \in L_n \times L_m\},$$
$$p_{L\text{-FBAM}} := \text{l.c.m.}\{p(X^{(0)},Y^{(0)}) | (X^{(0)},Y^{(0)}) \in L_n \times L_m\},$$

where 'l.c.m.' mean the least common multiple. The positive integers $i_{L-\text{FBAM}}$ and $p_{L-\text{FBAM}}$ are called the *convergence index* and the *period* of limit-cycles of the L-FBAMNNs(W,R,P,Q), respectively.

Definition 3.4. A pattern $(X,Y) \in L_n \times L_m$ is called a stable state of the L-FBAMNNs(W,R,P,Q) if $X = ((X+P)W+Q)R$ and $Y = ((Y+Q)R+P)W$.

Definition 3.5. A pattern $(X,Y) \in L_n \times L_m$ is called an equilibrium of the L-FBAMNNs(W,R,P,Q) if $X = (Y+Q)R$ and $Y = (X+P)W$.

Note that an equilibrium is a stable state, but a stable state is not necessarily an equilibrium, which can be seen from the following example.

Example 3.6. Let $L = ([0,1],\vee,\wedge)$. Then L is an incline. Consider the following matrices

$$W = R = \begin{pmatrix} 1 & 0.2 \\ 0.2 & 1 \end{pmatrix}, P = Q = \begin{pmatrix} 0 & 0 \end{pmatrix}, X = \begin{pmatrix} 0.1 & 0.1 \end{pmatrix}, Y = \begin{pmatrix} 0.2 & 02 \end{pmatrix},$$

It's easy to see that $XWR = X$ and $YRW = Y$. But $XW = \begin{pmatrix} 0.1 & 0.1 \end{pmatrix} \neq Y$ and $YR = \begin{pmatrix} 0.2 & 0.2 \end{pmatrix} \neq X$. Hence (X,Y) is a stable state but not an equilibrium of L-FBAMNNs(W,R,P,Q).

Theorem 3.7. *For an L-FBAMNNs(W,R,P,Q), if the matrix $WR \in L^{n\times n}$ or $RW \in L^{m\times m}$ has index, then the L-FBAMNNs(W,R,P,Q) is strongly convergent, and $i_{L-\text{FBAM}} \le \max\{i(MR), i(RM)+p(WR)\}, p_{L-\text{FBAM}} | p(WR)$.*

Proof. Put $k := \max\{i(WR), i(RW)\}, l := p(WR)$. Then $(RW)^k = (RW)^{k+l}$ and $(WR)^k = (WR)^{k+l}$.Let $(X^{(0)},Y^{(0)})$ be any stimulating state of the L-FBAMNNs(W,R,P,Q). We have that

$$\begin{aligned}
X^{(k)} &= X^{(0)}(WR)^k + P\textstyle\sum_{t=1}^{k}(WR)^t + QR\sum_{t=0}^{k-1}(WR)^t,\\
X^{(k+l)} &= X^{(0)}(WR)^{k+l} + P\textstyle\sum_{t=1}^{k+l}(WR)^t + QR\sum_{t=0}^{k+l-1}(WR)^t\\
&= X^{(0)}(WR)^k + P\textstyle\sum_{t=1}^{k+l-1}(WR)^t + QR\sum_{t=0}^{k+l-1}(WR)^t,\\
X^{(k+2l)} &= X^{(0)}(WR)^{k+2l} + P\textstyle\sum_{t=1}^{k+2l}(WR)^t + QR\sum_{t=0}^{k+2l-1}(WR)^t\\
&= X^{(0)}(WR)^k + P\textstyle\sum_{t=1}^{k+l-1}(WR)^t + QR\sum_{t=0}^{k+2l-1}(WR)^t\\
&= X^{(k+l)}
\end{aligned}$$

Similarly, we have that $Y^{(k+l)} = Y^{(k+2l)}$. Hence $(X^{(k+2l)},Y^{(k+2l)}) = (X^{(k+l)},Y^{(k+l)})$. Since $(X^{(0)},Y^{(0)})$ is any stimulating state,we have that L-FBAMNNs(W,R,P,Q) is strongly convergent, and

$$i_{L-\mathrm{FBAM}} \leq \max\{i(MR), i(RM)\} + p(WR), p_{L-\mathrm{FBAM}}|p(WR).$$

This completes the proof. □

The converse of Theorem 3.7 does not hold, which can be seen from the following example.

Example 3.8. Let $L = ([0,1], \vee, \times)$, where $\times$ is the ordinary multiplication. Then L is an incline. Consider the following matrices

$$W = R = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}, P = Q = \begin{pmatrix} 0.8 & 0.8 \end{pmatrix}.$$

It's easy to see that

$$(WR)^k = \begin{pmatrix} 0.1^{2k} & 0.1^{2k} \\ 0.1^{2k} & 0.1^{2k} \end{pmatrix}.$$

Hence WR does not converge in finite steps, and so has no index and period. But for any stimulating state $(X^{(0)}, Y^{(0)})$ (where $Y^{(0)} = (X^{(0)} + P)W$), we have that

$$\begin{aligned} X^{(1)} &= X^{(0)}WR + PWR + QR \\ &= \begin{pmatrix} 0.08 & 0.08 \end{pmatrix} \\ X^{(k)} &= X^{(0)}(WR)^k + P\textstyle\sum_{t=1}^{k}(WR)^t + QR\sum_{t=0}^{k-1}(WR)^t \\ &= X^{(0)}(WR)^k + PWR + QR\textstyle\sum_{t=0}^{k-1}(WR)^t \\ &= \begin{pmatrix} 0.08 & 0.08 \end{pmatrix} \text{(for any } k \geq 2) \end{aligned}$$

Hence $X^{(1)} = X^{(k)}, k \geq 2$. Similarly,$Y^{(1)} = Y^{(k)}, k \geq 2$. Thus the L-FBAMNNs($W$, R, P, Q) converges in finite steps and has index 1 and period 1.This shows the converse of Theorem 3.7 does not hold.

Corollary 3.9. *If $W \in I(L)^{n\times m}$ and $R \in I(L)^{m\times n}$, then the following hold:*

(1) *the L-FBAMNNs(W, R, P, Q) is strongly convergent,*
(2) $i_{L-\mathrm{FBAM}} \leq min\{(n-1)^2 + 2 + p(WR), (m-1)^2 + 2 + p(WR)\}$.

Proof. (1) By theorem 2.4, $WR \in I(L)^{n\times n}$ and so WR has index. Hence, by Theorem 3.1, the L-FBAMNNs(W, R, P, Q) is strongly convergent.

(2) It follows from Theorem 2.4, Theorem 2.6 and Theorem 3.7. □

Corollary 3.10. *If $W \in I(L)^{n\times m}$ and $R = W^T$, then the following hold:*

(1) *the L-FBAMNNs(W, R, P, Q) is strongly convergent,*
(2) $i_{L-\mathrm{FBAM}} \leq min\{2n - 1 + p(WR), 2m - 1 + p(WR)\}$ *and* $p_{L-\mathrm{FBAM}} \leq 2$.

Proof. Obviously, the matrix $WR \in I(L)^{n\times m}$ is symmetric. By Theorem 2.5, $i(WR) \leq 2n - 2$ and $p(WR) \leq 2$. Similarly, we have that $i(RW) \leq 2m - 2$ and $p(RW) \leq 2$. Hence (1) and (2) hold from Theorem 3.7. □

Corollary 3.11. *For any L-FBAMNNs(W, R, P, Q),if the matrix WR converges in finite steps, then the L-FBAMNNs(W, R, P, Q) is strongly stable.*

Theorem 3.12. *Let $(X, Y), (X_0, Y_0) \in L_n \times L_m$. If $X = XWR, Y = YRW$ and (X_0, Y_0) is a stable state of the L-FBAMNNs(W, R, P, Q), then $(X + X_0, Y + Y_0)$ is also a stable state of the L-FBAMNNs(W, R, P, Q).*

Proof. $(((X+X_0)+P)W+Q)R = XWR + X_0WR + PWR + QR = X + X_0$ and $(((Y+Y_0)+Q)R+P)W = YRW + Y_0RW + QRW + PW = Y + Y_0$. So, $(X+X_0, Y+Y_0)$ is also a stable state. □

Theorem 3.13. *A pattern* $(X,Y) \in L_n \times L_m$ *is an equilibrium of a L-FBAMNNs(*W,R,P,Q*) if and only if* $X = XWR+PWR+QR$ *and* $Y=(X+P)W$.

Proof. (X,Y) is an equilibrium of a L-FBAMNNs(W,R,P,Q) if and only if $X = XWR + PWR + QR$ and $Y = YRW + QRW + PW$ if and only if $X = XWR + PWR + QR$ and $Y = (X+P)W$. □

Theorem 3.14. *For a L-FBAMNNs(*W,R,P,Q*), if* WR *has index and* (X_0,Y_0) *is a stable state of the L-FBAMNNs(*W,R,P,Q*), then*

(1) *every element in* $\left\{(X+X_0, Y+Y_0) | X \in \mathcal{R}\left((WR)^{i(WR),p(WR)}\right), Y \in \mathcal{R}\left((RW)^{i(RW),p(RW)}\right)\right\}$ *(denoted by* $\mathfrak{M}$*, for short) is a stable state of the network,*

(2) $\left(\left(\eta(WR)^{i(WR)}\right)+X_0, \left(\eta(RW)^{i(RW)}\right)+Y_0\right)$ *is the greast stable state in* $\mathfrak{M}$*,*

(3) *every element in* $\left\{(X+X_0, (X+X_0+P)W) | X \in \mathcal{R}\left((WR)^{i(WR),p(WR)}\right)\right\}$ *(denoted by* $\mathfrak{N}$*, for short) is an equilibrium of the network,*

(4) $\left(\left(\eta(WR)^{i(WR)}\right)+X_0, \left(\left(\eta(RW)^{i(RW)}\right)+X_0+P\right)W\right)$ *is the greatest equilibrium in* $\mathfrak{N}$.

Proof. (1) By Definition 3.4, we have $X_0 = ((X_0+P)W+Q)R$ and $Y_0 = ((Y_0+Q)R+P)W$. For any $(X+X_0, Y+Y_0) \in \mathfrak{M}$, we have that $XWR = X, Y = YRW$, and hence

$$(((X+X_0)+P)W+Q)R = XWR + X_0WR + PW + QR = X + X_0,$$

$$(((Y+Y_0)+Q)R+P)W = YRW + Y_0RW + QRW + PW = Y + Y_0.$$

So $(X+X_0, Y+Y_0)$ is a stable state.

(2) By Theorem 2.7, $\left(\left(\eta(WR)^{i(WR)}\right), \left(\eta(RW)^{i(RW)}\right)\right)$ is the greatest element in $\mathcal{R}\left((WR)^{i(WR),p(WR)}\right) \times \mathcal{R}\left((RW)^{i(RW),p(RW)}\right)$, hence

$$\left(\left(\eta(WR)^{i(WR)}\right)+X_0, \left(\eta(RW)^{i(RW)}\right)+Y_0\right)$$

is the greatest stable state in $\mathfrak{M}$.

(3) It follows from Theorem 3.13 and Theorem 2.7.

(4) It follows from (3) and Theorem 2.7. □

4. Conclusions

In the paper, we obtain the facts that the L-FBAMNNs(W,R,P,Q) is strongly convergent if the product matrices WR and RW have indices, and strongly stable if WR and RW have indices and have period 1. It is shown that the convergence index and the period of limit-cycles of the L-FBAMNNs(W,R,P,Q) can be estimated by the indices and the periods of the product matrices WR and RW. In addition, we obtained some stable states and equilibria of the L-FBAMNNs(W,R,P,Q) when the product matrices WR and RW have indices. For the L-FBAMNNs(W,R,P,Q)

whose product of connection weight matrices have no indices, can we find the structure of all its stable states and equilibria? This problem is left for future studies.

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