Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 6, No. 2, (September 2013), pp. 251–261
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Kleene's fuzzy similarity and measure of similarity

Ismat Beg, Samina Ashraf

Received 23 July 2012; Accepted 25 September 2012

Abstract. Using Kleene Dienes implication operator we define fuzzy similarity between fuzzy subsets of a crisp universe as a fuzzy subset of the given universe. Properties of pointwise character of fuzzy similarity are studied. It is shown that the fuzzy similarity is a local equivalence relation on X. The measure of fuzzy similarity is then defined as composite of Sugeno's fuzzy measure and the fuzzy set of similarity. Many examples of measure of similarity are also constructed.

2010 AMS Classification: 03E72, 46S40, 68T37

Keywords: Fuzzy set of similarity, Measure of fuzzy similarity.

Corresponding Author: Ismat Beg (begismat@yahoo.com)

1. Introduction

The concept of similarity and dissimilarity measures is an important aspect of all sciences. There are several similarity measures that are proposed and used for various purposes in the fields of arts and sciences ranging from anthropology to zoology (see [1], [8], [16], [24] and [26]). Fuzzy similarity measures enjoy a lot of advantages on their crisp counterparts. This is why fuzzy similarity measures are widely studied (see [2], [4], [5], [6], [12], [17], [18], [19], [20] and [22]). Generally, the value of a scalar-valued similarity measure of two fuzzy sets is determined by comparing the corresponding membership values for each element in the universe. The result of this comparison produces a value between 0 and 1, that represents the degree to which the two sets are identical. Dubois and Prade (see [11, chapter 7.3]) use a symmetric difference operation and a scalar evaluator for generation of scalar-valued fuzzy similarity measure, which together satisfy certain axioms. Cross and Sudkamp [9] improved the approach of Dubois and Prade by defining fuzzy-valued assessment of the similarity of fuzzy sets over a universe X. Kehagias and Konstantinidou [17] have extended the range of similarity mapping to a Boolean lattice. Moreover, the comparisons of different fuzzy similarity measures as well as their aggregations have also been studied by Wang et al [23] and Fonck et al [13].

In Section 2, we present preliminaries and set our notations. In section 3, we reformulate the definition of fuzzy similarity mapping as a mapping on $F(X) \times F(X) \longrightarrow F(X)$, where $F(X)$ denotes the set of all fuzzy subsets of a given universe X. The fuzzy set of similarity stands as an element in $F(X)$. The concept of fuzzy set of similarity is a t-norm as well as fuzzy inclusion dependent. The fuzzy inclusion is defined in terms of a fuzzy implicator and hence is highly sensitive to the properties of the fuzzy implicator being used to define it. A lot of variations are observed in the properties of these operators . Instead of working with general fuzzy implicators, it seems appropriate to particularize about the implicator and the t-norm in use. This is why our study is restricted to the fuzzy set of similarity defined in terms of Kleene's implicator and the respective t-norm and t-conorm. To highlight this fact we name the newly defined concept Kleene's fuzzy similarity. Many important properties of fuzzy set of similarity are established and it is proved that Kleene's fuzzy set of similarity is a local fuzzy equivalence relation which is highly transitive. In section 4, we focus on [0,1]-valued judgement of the similarity of two fuzzy subsets, for this purpose we consider a measure of fuzzy similarity on $F(X)$, defined as the composite application by Sugeno's [21] fuzzy measure and the fuzzy set of similarity previously locally defined. Some new properties and examples of measures of fuzzy similarity are also constructed. Two types of fuzzy equivalence relations are defined and used in this paper: 1) fuzzy local equivalence relation and 2) fuzzy equivalence relation. The first one is related with degree of equivalence at each point of space and the other about allocation of a single degree of equivalence to the two fuzzy sets. It is observed that the fuzzy similarity is a fuzzy local equivalence relation without any additional condition imposed on it, while only the use of specific measures can make measure of fuzzy of similarity, a fuzzy equivalence relation.

2. Preliminaries

Definition 2.1 ([25]). Let $F(X)$ be the set of all fuzzy subsets of a universe X. For all $A, B \in F(X)$, A is said to be a *subset* of B if $A(x) \leq B(x)$ for all $x \in X$, where $A(x)$ and $B(x)$ represent the membership grades of x in A and B respectively. In this case we write $A \subseteq B$ and call it the inclusion in Zadeh's sense. Two fuzzy sets A and B are said to be *equal* if and only if $A(x) = B(x)$ for all $x \in X$. The *min* and *max* operators will be used for the construction of fuzzy sets $M(A, B)$ and $M^*(A, B)$ which represent the intersection and union of fuzzy sets A and B i.e., for all $x \in X$

$$M(A, B)(x) = \min(A(x), B(x)) \text{ and } M^*(A, B)(x) = \max(A(x), B(x)).$$

Moreover $ker(A)$, the *kernel of a fuzzy set* A and $supp(B)$, the *support of a fuzzy set* B are defined as: $ker(A) = \{x \in X \mid A(x) = 1\}$ and $supp(B) = \{x \in X \mid B(x) > 0\}$.

Definition 2.2 ([14]). A *negator* N is an order-reversing $[0, 1] \to [0, 1]$ mapping such that $N(0) = 1$ and $N(1) = 0$. The negators are used to model pointwise complements in the literature of fuzzy sets. Throughout this paper A^c, the complement of a fuzzy set A, will be calculated by the standard negator which is defined as: $N(x) = 1 - x$.

Definition 2.3 ([10]). A *fuzzy implicator* I is a binary operation on $[0,1]$ with order reversing first partial mappings and order preserving second partial mappings satisfying the boundary conditions:]

$$I(0,1) = I(0,0) = I(1,1) = 1 \text{ and } I(1,0) = 0.$$

The *Kleene's implicator* (I_M) is defined as:

$$I_M(x,y) = \max(1-x, y) \text{ for all } x, y \in [0,1].$$

Definition 2.4 ([3]). A binary operation E on $F(X)$ is an $\epsilon-$*local fuzzy equivalence* on $F(X)$, if the following conditions hold for all $x \in X$ and for all $A, B, C \in F(X)$:
1. *Fuzzy Reflexivity at* x: $E(A,A)(x) \in [0,1]$;
2. *Symmetry at* x: $E(A,B)(x) = E(B,A)(x)$ for all $x \in X$;
3. $\epsilon-$*fuzzy transitivity at* x:
$I_M(\min(E(A,B)(x), E(B,C)(x)), E(A,C)(x)) = \epsilon$.

If $\epsilon > 0$, then E is called *fuzzy transitive at* x, if $\epsilon \in [0.5, 1]$, then E will be called *strong fuzzy transitive at* x and it is called *weak fuzzy transitive at* x otherwise.

Definition 2.5 ([3]). A fuzzy relation R on $F(X)$ is called an $\epsilon-$*fuzzy equivalence relation* on $F(X)$ if for all $A, B, C \in F(X)$ following are satisfied:
1. *Fuzzy reflexivity*: $R(A,A) \in [0,1]$;
2. *Fuzzy symmetry*: $R(A,B) = R(B,A)$ for all $A, B \in F(X)$;
3. $\epsilon-$*Fuzzy transitivity*: $I_M(\min(R(A,B), R(B,C)), R(A,C)) = \epsilon$.

If $\epsilon > 0$, then R is called *fuzzy transitive*, if $\epsilon \in [0.5, 1]$, then R will be called *strong fuzzy transitive,* otherwise it is called *weak fuzzy transitive* .

Definition 2.6 ([7]). The *fuzzy inclusion* is a mapping $Inc : F(X) \times F(X) \to F(X)$ which assigns to every $A, B \in F(X)$ a fuzzy set $Inc(A,B) \in F(X)$ defined as:

$$Inc(A,B)(x) = I(A(x), B(x)) \text{ for all } x \in X. \tag{2.1}$$

Remark 2.7. If I_M is used in the definition of fuzzy inclusion (2.1), then the following statements are true for any $A, B, C, D \in F(X)$:

1a. For any $x \in X$, $Inc(A,B)(x) = Inc(B,A)(x) \Leftrightarrow A(x) = B(x)$.
1b. For any $x \in X$, $Inc(A,B)(x) = 1 \Leftrightarrow x \in (supp(A))^c \cup Ker(B)$.
1c. For any $x \in X$, $Inc(A,B)(x) = 0 \Leftrightarrow x \in Ker(A) \cap (supp(B))^c$.
1d. For all $x \in X$, $Inc(A,A)(x) \geq 0.5$.
2. $Inc(A,B) = \underline{1} \Leftrightarrow X = (suppA)^c \cup Ker(B)$.
3. $Inc(A,B) = \underline{0} \Leftrightarrow A = X$ and $B = \varnothing$.
4. $Inc(A, A^c) = \underline{0} \Leftrightarrow A = X$.
5. The fuzzy inclusion defined in Definition 2.6 is strong fuzzy transitive for all $x \in X$ and for all $A, B, C \in F(X)$.
6. $Inc(A,B) = Inc(B^c, A^c)$.
7. $M^*(Inc(A,B), Inc(B,A)) = \underline{1}$.
$\Leftrightarrow X = (supp(A))^c \cup Ker(B) \cup Ker(A) \cup (supp(B)^c)$.
8. $B \subseteq C \Rightarrow Inc(A,B) \subseteq Inc(A,C)$.
9. $B \subseteq C \Rightarrow Inc(C,A) \subseteq Inc(B,A)$.

10. $M(Inc(A,B), Inc(C,D))$

$$\subseteq M[Inc(M(A,C), M(B,D)), Inc(M^*(A,C), M^*(B,D))] \tag{2.2}$$

$$\subseteq M^*[Inc(M(A,C), M(B,D)), Inc(M^*(A,C), M^*(B,D))] \tag{2.3}$$

$$\subseteq M^*(Inc(A,B), Inc(C,D)). \tag{2.4}$$

11. $Inc(A, M(B,C)) = M(Inc(A,B), Inc(A,C))$.
12. $Inc(A, M^*(B,C)) = M^*(Inc(A,B), Inc(A,C))$.
13. $Inc(A,B) \subseteq M[Inc(M(A,C), M(B,C)), Inc(M^*(A,C), M^*(B,C))]$.

3. Kleene's fuzzy similarity

Definition 3.1. Let T be a t-norm, and Inc be fuzzy inclusion as defined in (2.1), then a *fuzzy similarity mapping* S is an : $F(X) \times F(X) \to F(X)$ mapping which allocates to all $A, B \in F(X)$ a fuzzy set $S_{T,Inc}(A,B)$ on X defined as:

$$S_{T,Inc}(A,B)(x) = T(Inc(A,B)(x), Inc(B,A)(x)), \text{ for all } x \in X. \tag{3.1}$$

In this case, $S_{T,Inc}(A,B)$ is called a *fuzzy set of similarity* between A and B. The Definition 3.1 is a fuzzy inclusion as well as a t-norm dependent and consequently it will assign different fuzzy sets of similarity to the same pair of fuzzy sets for different fuzzy sets of inclusions and t-norms. In this section the definition of *Kleene's fuzzy set of similarity* is obtained by taking $T = M$, the *min* operator and I_M be used in Inc. We shall drop the subscripts for this specific choice in the Definition 3.1 i.e., $S(A,B)$ is a fuzzy set defined for all $x \in X$ as follows:

$$\begin{aligned} S(A,B)(x) &= \min(Inc(A,B)(x), Inc(B,A)(x)) \\ &= \min(I_M(A(x), B(x)), I_M(B(x), A(x))). \end{aligned} \tag{3.2}$$

The first proposition develops the conditions for the maximum and minimum degrees of similarity at any point $x \in X$. Fuzzy sets X and $\varnothing$ will be denoted by $\underline{1}$ and $\underline{0}$ respectively to specifically denote the maximal and minimal elements of the range.

Proposition 3.2. *For all $A, B \in F(X)$ and for all $x \in X$ we have:*
1. $S(A,B)(x) = 1 \Leftrightarrow x \in [(supp(A))^c \cap (supp(B))^c] \cup [Ker(A) \cap Ker(B)]$.
2. $S(A,B)(x) = 0 \Leftrightarrow x \in [(supp(A))^c \cup (supp(B))^c] \cap [Ker(A) \cup Ker(B)]$.
3. $S(A,B)(x) = S(B,A)(x)$.
4. $S(A,A)(x) \geq 0.5$.

Proof. For all $A, B \in F(X)$ and for any $x \in X$,
1. $S(A,B)(x) = 1 \Leftrightarrow \min(Inc(A,B)(x), Inc(B,A)(x)) = 1$
$\Leftrightarrow Inc(A,B)(x) = 1$ and $Inc(B,A)(x) = 1$
$\Leftrightarrow x \in [(supp(A))^c \cup Ker(B)] \cap [Ker(A) \cup (supp(B))^c]$ by Remark 2.7 (1b)
$\Leftrightarrow x \in [(supp(A))^c \cap (supp(B))^c] \cup [Ker(A) \cap Ker(B)]$, using distributivity of $\cap$ and $\cup$.
2. $S(A,B)(x) = 0 \Leftrightarrow \min(Inc(A,B)(x), Inc(B,A)(x)) = 0$
$\Leftrightarrow$ either $Inc(A,B)(x) = 0$ or $Inc(B,A)(x) = 0$
$\Leftrightarrow x \in (Ker(A) \cap (supp(B))^c) \cup (Ker(B) \cap (supp(A))^c)$ by Remark 2.7 (1c)
$\Leftrightarrow x \in [(supp(A))^c \cup (supp(B))^c] \cap [Ker(A) \cup Ker(B)]$, using distributivity of $\cap$ and $\cup$.
3. Follows from commutativity of *min*.

4. $S(A,A)(x) = \min(Inc(A,A)(x), Inc(A,A)(x)) = Inc(A,A)(x) \geq 0.5$, by Remark 2.7 (1d). □

Theorem 3.3. *The Kleene's fuzzy set of similarity S is a strong transitive local fuzzy equivalence relation on $F(X)$ at all $x \in X$.*

Proof. Local fuzzy reflexivity is a consequence of the Definition 3.1 and local fuzzy symmetry is obtained in Proposition 3.2(3). For strong fuzzy transitivity, suppose otherwise, that is there exists an $x \in X$ and $A, B, C \in F(X)$ such that

$I_M(M(S(A,B), S(B,C)), S(A,C))(x) < 0.5$
$\Leftrightarrow max(1 - M(S(A,B), S(B,C))(x), S(A,C)(x)) < 0.5$
$\Leftrightarrow 1 - M(S(A,B), S(B,C))(x) < 0.5$ and $S(A,C)(x) < 0.5$
$\Leftrightarrow \min(S(A,B)(x), S(B,C)(x)) > 0.5$ and $S(A,C)(x) < 0.5$
$\Leftrightarrow S(A,B)(x) > 0.5$ and $S(B,C)(x) > 0.5$ and $S(A,C)(x) < 0.5$
$\Leftrightarrow \min(Inc(A,B)(x), Inc(B,A)(x)) > 0.5$
and $\min(Inc(B,C)(x), Inc(C,B)(x)) > 0.5$
and $\min(Inc(A,C)(x), Inc(C,A)(x)) < 0.5$
$\Leftrightarrow [Inc(A,B)(x) > 0.5$ and $Inc(B,A)(x) > 0.5]$
and $[Inc(B,C)(x) > 0.5$ and $Inc(C,B)(x) > 0.5]$
and [either $Inc(A,C)(x) < 0.5$ or $Inc(C,A)(x) < 0.5]$
$\Leftrightarrow$ Inc is weak fuzzy transitive at x for $A, B, C \in F(X)$, a contradiction to Remark 2.7(5). □

Proposition 3.4. *For all $A, B \in F(X)$, we have*
1. *$S(A,B) = \underline{1} \Leftrightarrow X = [((supp(A))^c \cap (supp(B))^c] \cup [Ker(A) \cap Ker(B)]$ i.e., A and B are equal crisp sets.*
2. *$S(A,B) = \underline{0} \Leftrightarrow X = [(supp(A))^c \cup (supp(B))^c] \cap [Ker(A) \cup Ker(B)]$.*
3. *$S(A,B) = S(B,A)$.*

Proof. For all $A, B \in F(X)$

1. $S(A,B) = \underline{1} \Leftrightarrow S(A,B)(x) = 1$ for all $x \in X$
$\Leftrightarrow x \in [((supp(A))^c \cap (supp(B))^c] \cup [Ker(A) \cap Ker(B)]$ for all $x \in X$ by Proposition 3.2(1).
$\Leftrightarrow X = [(supp(A))^c \cap (supp(B)^c] \cup [Ker(A) \cap Ker(B)]$
i.e., A and B are equal crisp sets.
2. $S(A,B) = \underline{0} \Leftrightarrow S(A,B) = (x) = 0$ for all $x \in X$
$\Leftrightarrow x \in [(supp(A))^c \cup (supp(B))^c] \cap [Ker(A) \cup Ker(B)]$ for all $x \in X$ by Proposition 3.2(2).
$\Leftrightarrow X = [(supp(A))^c \cup (supp(B))^c] \cap [Ker(A) \cup Ker(B)]$
$\Leftrightarrow A$ and B are crisp sets such that $A = B^c$.
3. For all $x \in X$, $S(A,B)(x) = S(B,A)(x)$, by Proposition 3.2(3).
Hence $S(A,B) = S(B,A)$. □

Corollary 3.5. *For all $A \in F(X)$, Proposition 3.4 leads to the following conclusions:*
1. *There does not exist any set $A \in F(X)$ such that $S(A, A^c) = \underline{1}$.*
2. *$S(A, A^c) = \underline{0} \Leftrightarrow A$ is a crisp set.*

Proposition 3.6. *For any* $A, B \in F(X)$, *we have:*

$$S(A, B) = S(B^c, A^c).$$

Proof. For any $A, B \in F(X)$ and for all $x \in X$
$S(A, B)(x) = \min(Inc(A, B)(x), Inc(B, A)(x))$
$= \min(Inc(B^c, A^c)(x), Inc(A^c, B^c)(x))$ by Remark 2.7 (6)
$= S(B^c, A^c)(x)$. □

Proposition 3.7. *For any* $A, B, C \in F(X)$, $A \subseteq B \subseteq C$ *implies that:*

$$S(A, C) \subseteq \begin{cases} S(A, B); \\ S(B, C). \end{cases}$$

Proof. The assumption $A \subseteq B \subseteq C$ implies that for all $x \in X$, $A(x) \leq B(x) \leq C(x)$.
It implies that $S(A, B)(x) = \min(Inc(A, B)(x), Inc(B, A)(x))$
$= Inc(B, A)(x) \supseteq Inc(C, A)(x)$ for all $x \in X$, by Remark 2.7 (9)
$= \min(Inc(A, C)(x), Inc(C, A)(x)) = S(A, C)(x)$.
Therefore, $S(A, B) \supseteq S(A, C)$.
$S(A, C) \subseteq S(B, C)$ can be proved similarly using Remark 2.7 (8). □

Theorem 3.8. *For all* $A, B, C, D \in F(X)$,

$$M(S(A, B), S(C, D)) \subseteq \begin{cases} S(M^*(A, C), M^*(B, D); \\ S(M(A, C), M(B, D)). \end{cases}$$

Proof. Using Remark 2.7(10) inequality (2.2), we get

$$M(Inc(A, B), Inc(C, D)) \subseteq Inc(M(A, C), M(B, D)) \tag{3.3}$$

Interchanging the roles of A and B as well as of C and D, we get

$$M(Inc(B, A), Inc(D, C)) \subseteq Inc(M(B, D), M(A, C)) \tag{3.4}$$

M is increasing in both variables so, (3.3) and (3.4) together imply that

$$\begin{aligned} & M[M\{Inc(A, B), Inc(C, D)\}, M\{Inc(B, A), Inc(D, C)\}] \\ \subseteq \quad & M[Inc(M(A, C), M(B, D)), Inc(M(B, D), M(A, C))]. \end{aligned}$$

Re arranging terms and applying associativity of M, we get

$$M(S(A, B), S(C, D)) \subseteq S(M(A, C), M(B, D)).$$

Similarly, we can prove that $M(S(A, B), S(C, D)) \subseteq S(M^*(A, C), M^*(B, D))$. □

Corollary 3.9. *For all* $A, B, C \in F(X)$,

$$M(S(A, B), S(A, C)) \subseteq S(A, M(B, C))$$

Proof. This can be easily obtained by putting $C = A$ and $D = C$ in Theorem 3.8. □

Corollary 3.10. *If* C *is a crisp set, then for all* $A, B \in F(X)$,

$$S(A, B) \subseteq \begin{cases} S(M^*(A, C), M^*(B, C)). \\ S(M(A, C), M(B, C)). \end{cases}$$

Proof. Since C is crisp, by Proposition 3.4 (1) $S(C,C)=1$.

By putting $D=C$ in Theorem 3.8 we get

$$M(S(A,B),S(C,C)) = M(S(A,B),1) \subseteq \begin{cases} S(M^*(A,C),M^*(B,C)); \\ S(M(A,C),M(B,C). \end{cases}$$

Using the fact that $M(S(A,B),1)=S(A,B)$, we get

$$S(A,B) \subseteq \begin{cases} S(M^*(A,C),M^*(B,C)); \\ S(M(A,C),M(B,C)). \end{cases}$$

□

Proposition 3.11. *For all $A,B\in F(X)$,*

1. If $A\subseteq B$, then $S(A,M^*(A,B))=S(B,M(A,B))$.

2. If $B\subseteq A$, then $S(A,M(A,B))=S(B,M^*(A,B))$.

Proof. For all $A,B\in F(X)$,

1. $A\subseteq B$ implies that for all $x\in X$, $A(x)\le B(x)$,
so $M^*(A,B)=B$ and $M(A,B)=A$.
It implies that $S(A,M^*(A,B))=S(A,B)=S(B,A)=S(B,M(A,B))$.
Hence $S(A,M^*(A,B))=S(B,M(A,B))$.

2. $B\subseteq A$ implies that for all $x\in X$, $B(x)\le A(x)$,
so, $M^*(A,B)(x)=A$ and $M(A,B)=B$.
It implies that $S(A,M(A,B))=S(A,B)=S(B,A)=S(B,M^*(A,B))$.
Hence $S(A,M(A,B))=S(B,M^*(A,B))$. □

Proposition 3.12. *For all $A,B\in F(X)$,we have:*

(i) $M(S(M(A,B),A),S(A,M^*(A,B)))\subseteq S(A,B)$.

(ii) $M(S(M(A,B),B),S(B,M^*(A,B)))\subseteq S(A,B)$.

(iii) $S(M(A,B),M^*(A,B))=S(A,B)$.

(iv) $M(S(A,M^*(A,B)),S(B,M^*(A,B)))\subseteq S(A,B)$.

(v) $M(\ S(A,M(A,B)),S(B,M(A,B)))\subseteq S(A,B)$.

Proof. For all $A,B\in F(X)$ and for any $x\in X$,in the following we construct the proof only for the case when $A(x)\le B(x)$. The other situatation can be proved in a similar way. If $A(x)\le B(x)$, then $M^*(A,B)(x)=B(x)$ and $M(A,B)=A(x)$ so,

(i) $\min[S(M(A,B)(x),A(x)),S(A(x),M^*(A,B)(x))]$
$=\min[S(A(x),A(x)),S(A,B)(x)]\le S(A,B)(x)$.

(ii) $M(S(M(A,B)(x),B(x)),S(B(x),M^*(A,B)(x)))$
$=M(S(A,B)(x),S(B,B)(x))\le S(A,B)(x)$.

(iii) $S(M(A,B)(x),M^*(A,B)(x))=S(A,B)(x)$.

(iv) $M(S(A(x),M^*(A,B)(x)),S(B(x),M^*(A,B)(x)))$
$=M(Inc(M^*(A,B)(x),A(x)),Inc(M^*(A,B)(x),B(x))$
$=M(Inc(B(x),A(x)),Inc(B(x),B(x)))$.
$\le S(B,A)(x)=S(A,B)(x)$ by symmetry of S.

(v) $M[S(A(x),M(A,B)(x)),S(B(x),M(A,B)(x))]$
$=M(S(A,A)(x),S(B,A)(x))\le S(A,B)(x)$. □

4. Measure of fuzzy similarity

Definition 4.1. [15, Definition 2.7] Let (X,ρ) be a measurable space. A function $m:\rho\to[0,\infty[$ is a *fuzzy measure* if it satisfies the following properties:

m1: $m(\varnothing) = 0$, and $m(X) = 1$; *m2*: $A \subseteq B$ implies that $m(A) \leq m(B)$.

The concept of measure considers that $\rho \subseteq \{0, 1\}^X$, but this consideration can be extended to a set of fuzzy subsets $\Im$ of X, $\Im \subseteq F(X)$, satisfying the properties of measurable space $(F(X), \Im)$.

Definition 4.2. A *measure of fuzzy similarity* is a fuzzy relation on $F(X)$ defined as:

$$m_S(A, B) = m(S(A, B)) \text{ for all } A, B \in F(X), \tag{4.1}$$

where, $S(A, B)$ is the fuzzy set of similarity of A and B and m is Sugeno's fuzzy measure defined in 4.1.

Example 4.3. Here are some examples of the measure m: for all $A \in F(X)$,
1a. $m_1(A) = Plinth(A) = \inf_{x \in X} A(x)$.
2a. $m_2(A) = \sup_{x \in X} A(x)$.
3a. $m_3(A) = \frac{1}{2}[Plinth(A) + Height(A)] = \frac{1}{2}[\inf_{x \in X} A(x) + \sup_{x \in X} A(x)]$.
4a. In case of finite universes,

$$m_4(A) = \frac{|A|}{|X|}.$$

5a. In case of bounded universes equipped with a measure m,

$$m_5(A) = \frac{\int A(x) dm}{m(X)}.$$

where $|A|$, the *scalar cardinality* of a fuzzy subset A of a finite universe X, is defined as:

$$|A| = \sum_{x \in X} A(x).$$

Applying these measures on the fuzzy set of similarity defined in (3.2), we get
1b. $m_{1S}(A, B) = \inf_{x \in X} \min[I_M(A(x), B(x)), I_M(B(x), A(x))]$.
2b. $m_{2S}(A, B) = \sup_{x \in X} \min[I_M(A(x), B(x)), I_M(B(x), A(x))]$.
3b.

$$\begin{aligned} m_{3S}(A, B) &= \frac{1}{2}[\inf_{x \in X} \min[I_M(A(x), B(x)), I_M(B(x), A(x))] \\ &\quad + \sup_{x \in X} \min[I_M(A(x), B(x)), I_M(B(x), A(x))]]. \end{aligned}$$

4b. In case of finite universes

$$m_{4S}(A, B) = \frac{|\min[\max(1 - A(x), B(x)), \min(1 - B(x), A(x))]|}{|X \times X|}.$$

5b. In case of bounded universes equipped with a measure m the measure $m_{4S}(A, B)$ is defined as:

$$m_{5S}(A, B) = \frac{\int \min[I_M(A(x), B(x)), I_M(B(x), A(x))] dm}{m(X \times X)}.$$

Example 4.4. Let $X = \{1, 2, 3, 4\}$, $A = \{(1, 0.3), (2, 0.8), (3, 0.8), (4, 0.3)\}$ and $B = \{(1, 0.2), (2, 0.4), (3, 0.6), (4, 0.8)$. Then

$$\begin{aligned} Inc(A, B) &= \{(1, 0.7), (2, 0.4), (3, 0.6), (4, 0.8)\}, \\ Inc(B, A) &= \{(1, 0.8), (2, 0.8), (3, 0.8), (4, 0.3)\}, \end{aligned}$$

Hence

$$S(A, B) = \{(1, 0.7), (2, 0.4), (3, 0.6), (4, 0.3)\},$$

where, the first coordinate of every ordered pair represents the element and the second coordinate represents its membership in the respective fuzzy set. Applying different measures on $S(A, B)$ we get

1. $m_{1S}(A, B) = 0.3$, **2.** $m_{2S}(A, B) = 0.7$,
3. $m_{3S}(A, B) = 0.5$, **4.** $m_{4S}(A, B) = \frac{2}{4} = 0.5$.

The following remarkable properties of measure or degree of inclusion, when studied with reference to the Kleene's implicator can be obtained by application of monotone fuzzy measure on the inequalities obtained in Propositions 3.4-3.11.

Proposition 4.5. *For all $A, B \in F(X)$,*
1. If A and B are equal crisp sets, then $m_{1S}(A, B) = 1$.
2. If A and B are such that $[((supp(A))^c \cap (supp(B))^c] \cup [Ker(A) \cap Ker(B)] \neq \varnothing$, then $m_{2S}(A, B) = 1$.
3. $m_S(A, B) = m_S(B, A)$.
4. There does not exist any set $A \in F(X)$ such that $m_S(A, A^c) = 1$.
5. If A and B are crisp sets such that $A = B^c$, then $m_S(A, B) = 0$;
6. $m_S(A, B) = m_S(A^c, B^c)$;
7. If $A \subseteq B \subseteq C$, then for any $A, B, C \in F(X)$,

$$m_S(A, C) \leq m_S(A, B) \text{ and } m_S(A, C) \leq m_S(B, C).$$

8. $m(M(S(A, B), S(C, D))) \leq \begin{cases} m_S(M^*(A, C), M^*(B, D); \\ m_S(M(A, C), M(B, D)). \end{cases}$
9. $m_S(A, B) = m_S(M(A, B), M^*(A, B))$.

Theorem 4.6. *If $m \in \{m_1,\ m_2\}$, then m_S is strong transitive fuzzy equivalence relation on $F(X)$.*

Proof. For any $A, B, C \in F(X)$,
Reflexivity: $m_S(A, A) = m(M(S(A, A), S(A, A))) \in [0, 1]$ by Proposition 3.2 (4).
Fuzzy symmetry: $m_S(A, B) = m(S(A, B)) = m(S(B, A))$ by Proposition 3.4 (3) $= m_S(B, A)$.
Strong fuzzy transitivity: From Theorem 3.3 it follows that
$I_M(M(S(A, B), S(B, C)), S(A, C))(x) \geq 0.5$ for all $x \in X$.
Taking inf and sup over x we get the result. □

Remark 4.7. If $A, B \in F(X)$ are such that $M^*(A, B) = X$, then $m_{4S}(A, B) = \frac{|A \cap B|}{|A \cup B|}$.

Proof. $M^*(A, B) = X \Rightarrow M^*(A, B)(x) = 1$ for all $x \in X$. It follows from 3.12(iii) that

$S(A, B) = S(M(A, B), M^*(A, B)) = Inc(M^*(A, B), M(A, B))$.

So for any $x \in X$, $Inc(M^*(A, B), M(A, B))(x)$

$= \max(1 - M^*(A,B)(x), M(A,B)(x))$
$= \max(1-1, M(A,B)(x)) = \max(0, M(A,B)(x)) = M(A,B)(x).$
Hence $Inc(M^*(A,B), M(A,B)) = M(A,B)$
so, $m_{4S}(A,B) = m_{4S}(M(A,B), M^*(A,B)) = m_4(S(M(A,B), M^*(A,B)))$
$= m_4(M(A,B)) = \frac{|M(A,B)|}{|X|} = \frac{|M(A,B)|}{|M^*(A,B)|} = \frac{|A\cap B|}{|A\cup B|}.$ □

5. Conclusion

The perception of similarity as a fuzzy set and exploring its properties as a point-wise mapping are highly beneficial as already discussed by Cross and Sudkamp [9]. The approach presented in this paper differs from previous approaches in the sense that it is two dimensional. Once the properties of fuzzy set of similarity are established we propose the application of a normal fuzzy measure to the results obtained and get many other results about measures of fuzzy similarity. The comparison of results with different measures and different implicators is another assignment at hand. So we expect that the fuzzy measures of similarity obtained purely from application of fuzzy measure to the fuzzy set of similarity may also be converted to many other forms by using the properties studied in section 3 and 4.

References

[1] F. G. Ashby and N. A. Perrin, Towards a unified theory of similarity and recognition, Psychological Review 95(1) (1988) 124–150.
[2] I. Beg and S. Ashraf, Fuzzy similarity and measure of similarity with Lukasiewicz implicator, New Math. Nat. Comput. 4(2) (2008) 191–206.
[3] I. Beg and S. Ashraf, Fuzzy equivalence relations, Kuwait J. Sci. Engrg. (35)(1A) (2008) 33–51.
[4] I. Beg and S. Ashraf, Fuzzy inclusion and fuzzy similarity with Gödel implication operator, New Math. Nat. Comput. 5(3) (2009) 617–633.
[5] I. Beg and S. Ashraf, Similarity measures for fuzzy sets, Appl. Comput. Math. 8(2) (2009) 192–202.
[6] I. Beg and S. Ashraf, Fuzzy transitive relations, Fuzzy Systems Math. 24(4) (2010) 162–169.
[7] I. Beg and S. Ashraf, Fuzzy set of inclusion under Kleene's implicator, Appl. Comput. Math. 10(1) (2011) 65–77.
[8] L. Biacino, Fuzzy subsethood and belief functions of fuzzy events, Fuzzy Sets and Systems 158 (2007) 38–49.
[9] V. Cross and T. Sudkamp, Similarity and Compatibility in fuzzy set theory: Assessments and Applications, Physica-Verlag Heidelberg, 2002.
[10] B. De Baets and E. E. Kerre, Fuzzy relations and applications, Advances in Electronics and Electron Physics 89 (1994) 255–324.
[11] D. Dubois and H. Prade, A unifying view of comparison indices in a fuzzy set-theoretic framework. in R. Yager, editor, Fuzzy Set and Possibility Theory Recent Developments, Pergamon Press, New York, NY, (1982) 3–13.
[12] J. Fan, W. Xie and J. Pie, Subsethood measure: new definitions, Fuzzy Sets and Systems 106 (1999) 201–209.
[13] P. Fonck, J. Fodor and M. Roubens, An application of aggregation procedures to the definition of measures of similarity between fuzzy sets, Fuzzy Sets and Systems 97 (1988) 67–74.
[14] J. Fodor and R. R. Yager, Fuzzy Set-theoretic Operators and Quantifiers (Chapter 1.2 in: D. Dubois and H. Prade, Eds., Handbook of Fuzzy Sets and Possibility Theory Vol. 1. 125–193.
[15] L. Garmendia, The evolution of the concept of fuzzy measure, Preprint (Web Page www.fdi.ucm.es/professor/lgarmend).
[16] R. Jain, S. N. Jayaram Murthy, P. L-J Chen and S. Chatterjee, Similarity measures for image databases, IEEE Int. Conference on Fuzzy Systems 3 (1995) 1247–1254.

[17] A. Kehagias and M. Konstantinidou, L-Fuzzy valued inclusion measure, L-Fuzzy similarity and L-fuzzy distance, Fuzzy Sets and Systems 136(3) (2003) 313–332.
[18] M. Köppen, Ch. Nowack and G. Rösel, Fuzzy-subsethood based color image processing, preprint http://citeseer.ist.psu.edu/627259.html.
[19] B. Kosko, Neural networks and fuzzy systems, Prentice Hall, 1992.
[20] S. Sanitini and R. Jain, Similarity is a geometer, Multimedia Tools and Applications 5(3) (1997) 277–306.
[21] M. Sugeno, Fuzzy measures and fuzzy integrals a survey. In *Fuzzy Automata and Decision Processes* (editors: M.M. Gupta, G.N. Saridis and B.R. Gaines) North Holland, Amsterdam, (1977) 89–102.
[22] W. J. Wang, New similarity measures on fuzzy sets and elements, Fuzzy Sets and Systems 85 (1997) 305–309.
[23] X. Wang, B. De Baets and E. E. Kerre, A comparative study of similarity measures, Fuzzy Sets and Systems 73 (1995) 259–268.
[24] J. Williams and N. Steele, Difference, distance and similarity as a basis for fuzzy decision support based on prototypical decision classes, Fuzzy Sets and Systems 131 (2002) 35–46.
[25] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[26] R. Zwick, E. Carlstein and D. V. Budesco, Measures of similarity amongst fuzzy concepts: A comparative analysis, Int. J. Approximate Reasoning 1 (1987) 221–242.

Ismat Beg (begismat@yahoo.com)
Centre for Mathematics and Statistical Sciences, Lahore School of Economics, Lahore, Pakistan.

Samina Ashraf (saminamazhar@ciitlahore.edu.pk)
Department of Mathematics, COMSATS Institute of Information Technology, Lahore, Pakistan.