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Characterizations of semirings by their anti fuzzy ideals

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Abstract. In this paper we define anti fuzzy ideals, anti fuzzy (generalized) bi-ideals and anti fuzzy quasi-ideals in semirings. We characterize different classes of semirings by the properties of their anti fuzzy ideals, anti fuzzy quasi-ideals and anti fuzzy bi-ideals.

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1. Introduction

The fundamental concept of a fuzzy set, introduced by Zadeh in his definitive paper [15] of 1965, provides a natural frame work for generalizing several basic notions of algebra. Rosenfeld formulated the rudiments of the theory of fuzzy groups in [13]. Kuroki initiated the theory of fuzzy semigroups in his papers [11, 12]. Ahsan et al. discussed fuzzy semirings in [2] in 1993. In 1998, Ahsan characterized semirings by their fuzzy ideals in his paper [1]. Many researchers worked on fuzzy ideals of semirings, for example [3, 5, 6, 7, 8, 9]. On the other hand Biswas [4] introduced the concept of anti fuzzy subgroup of a group. Hong and Jun [10] modified Biswas's idea and applied it to BCK-algebras. They defined anti fuzzy ideal of a BCK-algebra. In 2010, Shabir and Nawaz characterized semigroups by their anti fuzzy ideals in [14]. In [3], Akram and Dar defined anti fuzzy h-ideals in hemirings. In this paper we introduced the concept of anti fuzzy ideal, anti fuzzy (generalized) bi-ideal and anti fuzzy quasi-ideal in semirings and characterized different classes of semirings by the properties of these ideals.

2. Preliminaries

A semiring R is a non-empty set R equipped with two binary operations addition "$+$" and multiplication "$\cdot$" such that $(R,+)$ is a commutative semigroup, $(R,\cdot)$ is a semigroup, multiplication distributes over addition from both sides and R contains an element 0 such that $a+0=0+a=a$ and $a0=0a=0$ for all $a\in R$. A non-empty subset A of a semiring R is called a subsemiring of R if it is closed under addition and multiplication. A non-empty subset L of R is called a left (right) ideal of R if it is closed under addition and $ab\in L$ $(ba\in L)$, for all $a\in R$ and $b\in L$. A non-empty subset L of R is called an ideal of R if it is both a left and a right ideal of R. A non-empty subset B of R is called a generalized bi-ideal of R if it is closed under addition and $BRB\subseteq B$. A non-empty subset B of R is called a bi-ideal of R if it is a subsemiring of R and $BRB\subseteq B$. A non-empty subset Q of R is called a quasi-ideal of R if it is closed under addition and $QR\cap RQ\subseteq Q$.

It is obvious that every left (right) ideal of a semirng is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal. But the converse is not true.

An element a of a semiring R is called regular if there exists an element $x\in R$ such that $a=axa$. A semiring R is regular if every element of R is regular.

It is well known that:

Theorem 2.1. *For a semiring R the following conditions are equivalent.*

(1) *R is regular.*

(2) *$A\cap L=AL$ for every right ideal A and every left ideal L of R.*

A fuzzy subset λ of a universe X is a function from X to the unit closed interval $[0,1]$, that is $\lambda : X\to[0,1]$. For any two fuzzy subsets λ and μ of R, $\lambda\subseteq\mu$ means that, for all $x\in R$, $\lambda(x)\leq\mu(x)$. The symbols $\lambda\wedge\mu$ and $\lambda\vee\mu$ means the following fuzzy subsets of R.

$$(\lambda\wedge\mu)(x)=\lambda(x)\wedge\mu(x)\text{ and }(\lambda\vee\mu)(x)=\lambda(x)\vee\mu(x)$$

for all $x\in R$.

More generally if $\{\lambda_i\}_{i\in I}$ is a family of fuzzy subsets of X, then their union and intersection is defined as follows:

$$\left(\bigvee_{i\in I}\lambda_i\right)(x)=\bigvee_{i\in I}\left(\lambda_i(x)\right)\text{ and }\left(\bigwedge_{i\in I}\lambda_i\right)(x)=\bigwedge_{i\in I}\left(\lambda_i(x)\right)$$

for all $x\in R$.

Let A be a subset of X. Then the characteristic function of A is defined as:

$$C_A(x)=\begin{cases}1 & \text{if } x\in A\\ 0 & \text{otherwise}\end{cases}$$

3. Anti fuzzy ideals

In this section we define anti fuzzy left (right) ideal, bi-ideal, generalized bi ideal and quasi-ideal of semiring and study some properties of these ideals.

Definition 3.1. Let λ be a fuzzy subset of a semiring R and $x,y\in R$.

(1) $\lambda(x+y)\leq\lambda(x)\vee\lambda(y)$

(2) $\lambda(xy)\leq\lambda(x)\vee\lambda(y)$

(3) $\lambda(xy)\leq\lambda(y)$ $(\lambda(xy)\leq\lambda(x))$

(4) $\lambda(xyz) \leq \lambda(x) \vee \lambda(z)$.

Then λ is called an anti fuzzy subsemiring of R, if it satisfies (1) and (2).

It is called an anti fuzzy left (right) ideal of R, if it satisfies (1) and (3).

A fuzzy subset λ of R is called an anti fuzzy ideal of R if it is both an anti fuzzy left and right ideal of R.

λ is called an anti fuzzy generalized bi-ideal of R, if it satisfies (1) and (4).

λ is called an anti fuzzy bi-ideal of R, if it satisfies (1), (2) and (4).

Definition 3.2. For any fuzzy subset λ of a universe X and $t \in [0,1]$ we define

$$L(\lambda;t) = \{x \in X : \lambda(x) \leq t\}$$

which is called the anti level cut of λ.

Next we characterize anti fuzzy subsemiring (left ideal, right ideal, bi-ideal) by their anti level cuts.

Theorem 3.3. (1) *A fuzzy subset λ of a semiring R is an anti fuzzy subsemiring of R if and only if $L(\lambda;t) \neq \varnothing$ is a subsemiring of R for all $t \in [0,1]$.*

(2) *A fuzzy subset λ of a semiring R is an anti fuzzy generalized bi-ideal of R if and only if $L(\lambda;t) \neq \varnothing$ is a generalized bi-ideal of R for all $t \in [0,1]$.*

(3) *A fuzzy subset λ of a semiring R is an anti fuzzy bi-ideal of R if and only if $L(\lambda;t) \neq \varnothing$ is a bi-ideal of R for all $t \in [0,1]$.*

(4) *A fuzzy subset λ of a semiring R is an anti fuzzy left (right) ideal of R if and only if $L(\lambda;t) \neq \varnothing$ is a left (right) ideal of R for all $t \in [0,1]$.*

Proof. (1) Let λ be an anti fuzzy subsemiring of R and $x, y \in L(\lambda;t)$. Then $\lambda(x) \leq t$ and $\lambda(y) \leq t$. Since $\lambda(x+y) \leq \lambda(x) \vee \lambda(y) \leq t \vee t = t$, we have $\lambda(x+y) \leq t$, that is $x + y \in L(\lambda;t)$. Also $\lambda(xy) \leq \lambda(x) \vee \lambda(y) \leq t \vee t = t$. This implies that $\lambda(xy) \leq t$, that is $xy \in L(\lambda;t)$. Hence $L(\lambda;t)$ is a subsemiring of R.

Conversely, assume that $L(\lambda;t) \neq \varnothing$ is a subsemiring of R for all $t \in [0,1]$. Suppose there exist $x, y \in R$ such that $\lambda(x+y) > \lambda(x) \vee \lambda(y)$. Choose $t \in [0,1]$ such that $\lambda(x+y) > t \geq \lambda(x) \vee \lambda(y)$. Then $x, y \in L(\lambda;t)$ but $x + y \notin L(\lambda;t)$, which is a contradiction. Hence $\lambda(x+y) \leq \lambda(x) \vee \lambda(y)$. Similarly, we can prove that $\lambda(xy) \leq \lambda(x) \vee \lambda(y)$. This proves that λ is an anti fuzzy subsemiring of R.

The proofs of (2), (3) and (4) are similar to the proof of part (1). □

Example 3.4. Consider the semiring $R = \{0, 1, a, b, c\}$ defined by the following tables

$+$	0	1	a	b	c
0	0	1	a	b	c
1	1	b	1	a	1
a	a	1	a	b	a
b	b	a	b	1	b
c	c	1	a	b	c

$\cdot$	0	1	a	b	c
0	0	0	0	0	0
1	0	1	a	b	c
a	0	a	a	a	c
b	0	b	a	1	c
c	0	c	c	c	0

Then $(R, +, \cdot)$ is a semiring. Define $\lambda : R \to [0,1]$ by $\lambda(0) = 0.1$, $\lambda(c) = 0.3$, $\lambda(a) = 0.6$, $\lambda(b) = \lambda(1) = 0.7$.

$$L(\lambda; t) = \begin{cases} R & 0.7 \leq t < 1 \\ \{0, a, c\} & 0.6 \leq t < 0.7 \\ \{0, c\} & 0.3 \leq t < 0.6 \\ \{0\} & 0.1 \leq t < 0.3 \\ \varnothing & t < 0.1 \end{cases}$$

Then by Theorem 3.3, λ is an anti fuzzy ideal of R.

Theorem 3.5. *Let A be a non-empty subset of a semiring R. Define the fuzzy subset λ of R by*

$$\lambda(x) = \begin{cases} t & \text{if } x \notin A \\ r & \text{if } x \in A \end{cases}$$

where $t, r \in [0,1]$ such that $t \geq r$. Then

(1) *A is a left (right) ideal of R if and only if λ is an anti fuzzy left (right) ideal of R.*

(2) *A is a subsemiring of R if and only if λ is an anti fuzzy subsemiring of R.*

(3) *A is a (generalized) bi-ideal of R if and only if λ is an anti fuzzy (generalized) bi-ideal of R.*

Proof. Straightforward. □

Remark 3.6. From above theorems we conclude that a non-empty subset A of a semiring R is a left (right, bi, generalized bi) ideal of R if and only if the characteristic function of the complement of A, that is C_{A^c} is an anti fuzzy left (right, bi, generalized bi) ideal of R.

Lemma 3.7. *The union of any family of anti fuzzy left (right) ideals of a semiring R is an anti fuzzy left (right) ideal of R.*

Proof. Let $\{\lambda_i : i \in I\}$ be a family of anti fuzzy left ideals of R and $x, y \in R$. Then

$$\left(\bigvee_{i \in I} \lambda_i\right)(x + y) = \bigvee_{i \in I} (\lambda_i (x + y))$$

(Since each λ_i is an anti fuzzy left ideal of R, so $\lambda_i(x + y) \leq \lambda_i(x) \vee \lambda_i(y)$ for all $i \in I$). Thus

$$\begin{aligned} \left(\bigvee_{i \in I} \lambda_i\right)(x + y) &= \bigvee_{i \in I} (\lambda_i (x + y)) \\ &\leq \bigvee_{i \in I} (\lambda_i(x) \vee \lambda_i(y)) \\ &= \left(\left(\bigvee_{i \in I} \lambda_i\right)(x)\right) \vee \left(\left(\bigvee_{i \in I} \lambda_i\right)(y)\right). \end{aligned}$$

Now

$$\left(\bigvee_{i \in I} \lambda_i\right)(xy) = \bigvee_{i \in I} (\lambda_i (xy))$$

$\lambda_i(xy) \leq \lambda_i\,(y)$ for all $i \in I$). Thus

$$\left(\bigvee_{i\in I}\lambda_i\right)(xy) = \bigvee_{i\in I}\left(\lambda_i\,(xy)\right) \leq \bigvee_{i\in I}\left(\lambda_i\,(y)\right) = \left(\bigvee_{i\in I}\lambda_i\right)(y)\,.$$

Hence $\bigvee\limits_{i\in I}\lambda_i$ is an anti fuzzy left ideal of R. □

Definition 3.8. Let λ, μ be fuzzy subsets of R. Then the anti product of λ and μ is defined by

$$(\lambda * \mu)(x) = \bigwedge_{x=\sum_{i=1}^{p} y_i z_i}\left[\bigvee_{1\leq i\leq p}[\lambda(y_i) \vee \mu(z_i)]\right]$$

for $x \in R$ and $p \in \mathbb{N}$.

Proposition 3.9. *If λ, μ are anti fuzzy left (right) ideals of R, then $\lambda * \mu$ is an anti fuzzy left (right) ideal of R.*

Proof. Let λ and μ be anti fuzzy left ideals of R and $a, b, r \in R$. Then

$$(\lambda * \mu)(a) = \bigwedge_{a=\sum_{i=1}^{p} y_i z_i}\left[\bigvee_{1\leq i\leq p}[\lambda(y_i) \vee \mu(z_i)]\right]$$

and

$$(\lambda * \mu)(b) = \bigwedge_{b=\sum_{j=1}^{q} y'_j z'_j}\left[\bigvee_{1\leq j\leq q}\left[\lambda(y'_j) \vee \mu(z'_j)\right]\right].$$

Thus

$$\begin{aligned}
(\lambda * \mu)(a) \vee (\lambda * \mu)(b) &= \left[\bigwedge_{a=\sum_{i=1}^{p} y_i z_i}\left[\bigvee_{1\leq i\leq p}[\lambda(y_i) \vee \mu(z_i)]\right]\right] \\
&\quad \vee \left[\bigwedge_{b=\sum_{j=1}^{q} y'_j z'_j}\left[\bigvee_{1\leq j\leq q}[\lambda(\acute{y}_i) \vee \mu(\acute{z}_i)]\right]\right] \\
&= \bigwedge_{a=\sum_{i=1}^{p} y_i z_i}\bigwedge_{b=\sum_{j=1}^{q} y'_j z'_j}\left[\bigvee_{1\leq i\leq p}[\lambda(y_i) \vee \mu(z_i)]\right] \\
&\quad \vee \left[\bigvee_{1\leq j\leq q}[\lambda(\acute{y}_j) \vee \mu(\acute{z}_j)]\right] \\
&\geq \bigwedge_{a+b=\sum_{k=1}^{s} y''_k z''_k}\left[\bigvee_{1\leq k\leq s}\left[\lambda(y''_k) \vee \mu(z''_k)\right]\right] \\
&= (\lambda * \mu)(a + b).
\end{aligned}$$

Now

$$\begin{aligned}(\lambda*\mu)(a) &= \bigwedge_{a=\sum_{j=1}^{q}\acute{y}_j\acute{z}_j}\left[\bigvee_{1\le j\le q}[\lambda(\acute{y}_j)\vee\mu(\acute{z}_j)]\right]\\ &\geq \bigwedge_{a=\sum_{j=1}^{q}\acute{y}_j\acute{z}_j}\left[\bigvee_{1\le j\le q}[\lambda(r\acute{y}_j)\vee\mu(r\acute{z}_j)]\right]\\ &= \bigwedge_{ra=\sum_{j=1}^{q}y_n^{''}z_n^{''}}\left[\bigvee_{1\le n\le t}\left[\lambda(y_n^{''})\vee\mu(z_n^{''})\right]\right]\\ &= (\lambda*\mu)(ra).\end{aligned}$$

Thus $\lambda*\mu$ is an anti fuzzy left ideal of R. □

Theorem 3.10. *Let λ be an anti fuzzy right ideal and μ be an anti fuzzy left ideal of a semiring R. Then $\lambda*\mu\geq\lambda\vee\mu$.*

Proof. Let λ be an anti fuzzy right ideal and μ be an anti fuzzy left ideal of a semiring R and $x\in R$. Then

$$\begin{aligned}(\lambda*\mu)(x) &= \bigwedge_{x=\sum_{i=1}^{p}y_iz_i}\left[\bigvee_{1\le i\le p}(\lambda(y_i)\vee\mu(z_i))\right]\\ &\geq \bigwedge_{x=\sum_{i=1}^{p}y_iz_i}\left[\bigvee_{1\le i\le p}(\lambda(y_iz_i)\vee\mu(y_iz_i))\right]\\ &= \bigwedge_{x=\sum_{i=1}^{p}y_iz_i}\left[\left[\bigvee_{1\le i\le p}\lambda(y_iz_i)\right]\vee\left[\bigvee_{1\le i\le p}\mu(y_iz_i)\right]\right]\\ &\geq \bigwedge_{x=\sum_{i=1}^{p}y_iz_i}\left[\lambda\left(\sum_{i=1}^{p}y_iz_i\right)\vee\mu\left(\sum_{i=1}^{p}y_iz_i\right)\right]\\ &= \bigwedge_{x=\sum_{i=1}^{p}y_iz_i}\left[\bigvee_{1\le i\le p}(\lambda(x)\vee\mu(x))\right]\\ &= (\lambda\vee\mu)(x)\end{aligned}$$

Thus $\lambda*\mu\geq\lambda\vee\mu$. □

Now we show that if λ and μ are anti fuzzy ideals of a semiring R, then $\lambda*\mu\nleq\lambda\vee\mu$ and $\lambda*\mu\nleq\lambda\wedge\mu$.

Example 3.11. Consider the semiring $R=\{0,1,a,b,c\}$ defined by the following tables

$+$	0	1	a	b	c
0	0	1	a	b	c
1	1	b	1	a	1
a	a	1	a	b	a
b	b	a	b	1	b
c	c	1	a	b	c

$\cdot$	0	1	a	b	c
0	0	0	0	0	0
1	0	1	a	b	c
a	0	a	a	a	c
b	0	b	a	1	c
c	0	c	c	c	0

The (crisp) ideals of R are $\{0\}$, $\{0, c\}$, $\{0, a, c\}$, and R. Let $A = \{0, a, c\}$ and $B = \{0, c\}$ then $(C_{A^c} * C_{B^c})(c) = 1$ but $(C_{A^c} \vee C_{B^c})(c) = 0$ and $(C_{A^c} \wedge C_{B^c})(c) = 0$. Hence $\lambda * \mu \nleq \lambda \vee \mu$ and $\lambda * \mu \nleq \lambda \wedge \mu$.

Definition 3.12. Let λ and μ be fuzzy subsets of R. Then their anti sum $\lambda \oplus \mu$ is defined as

$$(\lambda \oplus \mu)(x) = \bigwedge_{x=y+z} [\lambda(y) \vee \mu(z)]$$

for all $x \in R$.

Proposition 3.13. *Let λ and μ be anti fuzzy left (right) ideals of R. Then $\lambda \oplus \mu$ is an anti fuzzy left (right) ideal of R.*

Proof. Let λ and μ be anti fuzzy left ideals of R and $x, \acute{x} \in R$. Then

$$(\lambda \oplus \mu)(x) = \bigwedge_{x=y+z} [\lambda(y) \vee \mu(z)]$$

and

$$(\lambda \oplus \mu)(\acute{x}) = \bigwedge_{\acute{x}=\acute{y}+\acute{z}} [\lambda(\acute{y}) \vee \mu(\acute{z})]$$

Thus

$$\begin{aligned}
(\lambda \oplus \mu)(x) \vee (\lambda \oplus \mu)(\acute{x}) &= \left[\bigwedge_{x=y+z} [\lambda(y) \vee \mu(z)]\right] \vee \left[\bigwedge_{\acute{x}=\acute{y}+\acute{z}} [\lambda(\acute{y}) \vee \mu(\acute{z})]\right] \\
&= \bigwedge_{x=y+z \ \acute{x}=\acute{y}+\acute{z}} [[\lambda(y) \vee \mu(z)] \vee [\lambda(\acute{y}) \vee \mu(\acute{z})]] \\
&= \bigwedge_{x=y+z \ \acute{x}=\acute{y}+\acute{z}} [[\lambda(y) \vee \lambda(\acute{y})] \vee [\mu(z) \vee \mu(\acute{z})]] \\
&\geq \bigwedge_{x=y+z \ \acute{x}=\acute{y}+\acute{z}} [\lambda(y+\acute{y}) \vee \mu(z+\acute{z})] \\
&\geq \bigwedge_{x+\acute{x}=a+b} [\lambda(a) \vee \mu(b)] \\
&= (\lambda \oplus \mu)(x+\acute{x}).
\end{aligned}$$

Again, let $x, a \in R$. Then

$$\begin{aligned}(\lambda \oplus \mu)(x) &= \bigwedge_{x=y+z} [\lambda(y) \vee \mu(z)] \\ &\geq \bigwedge_{x=y+z} [\lambda(ay) \vee \mu(az)] \\ &= \bigwedge_{ax=\acute{y}+\acute{z}} [\lambda(\acute{y}) \vee \mu(\acute{z})] \\ &= (\lambda \oplus \mu)(ax).\end{aligned}$$

Hence $\lambda \oplus \mu$ is an anti fuzzy left ideal of R. □

Lemma 3.14. *A fuzzy subset λ of a semiring R is an anti fuzzy subsemiring of R if and only if $\lambda \oplus \lambda \supseteq \lambda$ and $\lambda^2 = \lambda * \lambda \supseteq \lambda$.*

Proof. Let λ be an anti fuzzy subsemiring of R and $x \in R$. Then

$$\begin{aligned}(\lambda \oplus \lambda)(x) &= \bigwedge_{x=y+z} [\lambda(y) \vee \lambda(z)] \\ &\geq \bigwedge_{x=y+z} [\lambda(y+z)] \\ &= \bigwedge_{x=y+z} \lambda(x) \\ &= \lambda(x) \\ &\Longrightarrow (\lambda \oplus \lambda)(x) \geq \lambda(x).\end{aligned}$$

Thus $\lambda \oplus \lambda \supseteq \lambda$. Now

$$\begin{aligned}\lambda^2(x) &= (\lambda * \lambda)(x) \\ &= \bigwedge_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1 \leq i \leq p} [\lambda(y_i) \vee \lambda(z_i)] \right] \\ &\geq \bigwedge_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1 \leq i \leq p} [\lambda(y_i z_i) \vee \lambda(y_i z_i)] \right] \\ &= \bigwedge_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1 \leq i \leq p} \lambda(y_i z_i) \right] \\ &\geq \bigwedge_{x=\sum_{i=1}^{p} y_i z_i} \left[\lambda(\sum_{i=1}^{p} y_i z_i) \right]\end{aligned}$$

$$\begin{aligned} &= \bigwedge_{x=\sum\limits_{i=1}^{p} y_i z_i} \lambda(x) = \lambda(x) \\ &\Longrightarrow \lambda^2(x) \geq \lambda(x). \end{aligned}$$

Thus $\lambda^2 \supseteq \lambda$.

Conversely, assume that λ is a fuzzy subset of R such that $\lambda \oplus \lambda \supseteq \lambda$ and $\lambda^2 \supseteq \lambda$. Then for $x, y \in R$, we have

$$\begin{aligned} \lambda(x+y) &\leq (\lambda \oplus \lambda)(x+y) \\ &= \bigwedge_{x+y=a+b} [\lambda(a) \vee \lambda(b)] \\ &\leq \lambda(x) \vee \lambda(y) \\ &\Longrightarrow \lambda(x+y) \leq \lambda(x) \vee \lambda(y) \end{aligned}$$

and

$$\begin{aligned} \lambda(xy) &\leq \lambda^2(xy) \\ &= (\lambda * \lambda)(xy) \\ &= \bigwedge_{xy=\sum\limits_{i=1}^{p} y_i z_i} \left[\bigvee_{1 \leq i \leq p} [\lambda(y_i) \vee \lambda(z_i)] \right] \\ &\leq \lambda(x) \vee \lambda(y) \\ &\Longrightarrow \lambda(xy) \leq \lambda(x) \vee \lambda(y) \end{aligned}$$

Thus λ is an anti fuzzy subsemiring of R. □

Lemma 3.15. *A fuzzy subset λ of a semiring R is an anti fuzzy left (right) ideal of R if and only if $\lambda \oplus \lambda \supseteq \lambda$ and $\tilde{O} * \lambda \supseteq \lambda$ $(\lambda * \tilde{O} \supseteq \lambda)$, where $\tilde{O}$ is the fuzzy subset of R mapping every element of R on 0.*

Proof. Let λ be an anti fuzzy left ideal of R and $x \in R$. Then

$$\begin{aligned} (\lambda \oplus \lambda)(x) &= \bigwedge_{x=y+z} [\lambda(y) \vee \lambda(z)] \\ &\geq \bigwedge_{x=y+z} [\lambda(y+z)] \\ &= \bigwedge_{x=y+z} [\lambda(x)] \\ &= \lambda(x) \\ &\Longrightarrow (\lambda \oplus \lambda)(x) \geq \lambda(x). \end{aligned}$$

Thus $\lambda \oplus \lambda \supseteq \lambda$. Now

$$\begin{aligned}
\left(\tilde{O} * \lambda\right)(x) &= \bigwedge_{x=\sum\limits_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} \left[\tilde{O}(y_i) \vee \lambda(z_i)\right]\right] \\
&\geq \bigwedge_{x=\sum\limits_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} \left[\tilde{O}(y_i) \vee \lambda(y_i z_i)\right]\right] \\
&= \bigwedge_{x=\sum\limits_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} [\lambda(y_i z_i)]\right] \\
&\geq \bigwedge_{x=\sum\limits_{i=1}^{p} y_i z_i} \left[\lambda(\sum_{i=1}^{p} y_i z_i)\right] \\
&= \bigwedge_{x=\sum\limits_{i=1}^{p} y_i z_i} \lambda(x) \\
&= \lambda(x) \\
&\Longrightarrow \left(\tilde{O} * \lambda\right)(x) \geq \lambda(x).
\end{aligned}$$

Thus $\tilde{O} * \lambda \supseteq \lambda$.

Conversely, assume that λ is a fuzzy subset of R such that $\lambda \oplus \lambda \supseteq \lambda$ and $\tilde{O} * \lambda \supseteq \lambda$. Then, for $x, y \in R$ we have

$$\begin{aligned}
\lambda(x+y) &\leq (\lambda \oplus \lambda)(x+y) \\
&= \bigwedge_{x+y=a+b} [\lambda(a) \vee \lambda(b)] \\
&\leq \lambda(x) \vee \lambda(y) \\
&\Longrightarrow \lambda(x+y) \leq \lambda(x) \vee \lambda(y).
\end{aligned}$$

And

$$\begin{aligned}
\lambda(xy) &\leq (\tilde{O} * \lambda)(xy) \\
&= \bigwedge_{xy=\sum\limits_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} \left[\tilde{O}(y_i) \vee \lambda(z_i)\right]\right] \\
&\leq \tilde{O}(x) \vee \lambda(y) \\
&= \lambda(y) \\
&\Longrightarrow \lambda(xy) \leq \lambda(y).
\end{aligned}$$

Thus λ is an anti fuzzy left ideal of R. □

Definition 3.16. A fuzzy subset λ of a semiring R is called an anti fuzzy quasi-ideal of R, if it satisfies,

(1) $\lambda(x+y) \leq \lambda(x) \vee \lambda(y)$

(2) $\lambda(x) \leq \max\left\{\left(\lambda * \tilde{O}\right)(x), \left(\tilde{O} * \lambda\right)(x)\right\}$

for all $x, y \in R$.

Where $\tilde{O}$ is the fuzzy subset of R mapping every element of R on 0.

Theorem 3.17. *A fuzzy subset λ of a semiring R is an anti fuzzy quasi-ideal of R if and only if $L(\lambda; t) \neq \varnothing$ is a quasi-ideal of R for all $t \in [0,1]$.*

Proof. Suppose λ is an anti fuzzy quasi-ideal of R and $x, y \in L(\lambda; t)$. Then $\lambda(x) \leq t$ and $\lambda(y) \leq t$. Since $\lambda(x+y) \leq \lambda(x) \vee \lambda(y) \leq t$, we have $\lambda(x+y) \leq t$. Thus $x + y \in L(\lambda; t)$. Now let $a \in (RL(\lambda; t)) \cap (L(\lambda; t) R)$. Then $a = \sum_{i=1}^{p} y_i z_i$ and $a = \sum_{j=1}^{q} s_j t_j$, where $y_i, t_j \in R$ and $z_i, s_j \in L(\lambda; t)$ $(1 \leq i \leq p,\ 1 \leq j \leq q)$. Thus $\lambda(z_i) \leq t$ for $1 \leq i \leq p$ and $\lambda(s_j) \leq t$ for $1 \leq j \leq q$. Since

$$\begin{aligned}
\left(\tilde{O} * \lambda\right)(a) &= \bigwedge_{a=\sum_{k=1}^{r} a_k b_k} \left[\bigvee_{1\leq k\leq r} \left[\tilde{O}(a_k) \vee \lambda(b_k)\right]\right] \\
&\leq \bigvee_{1\leq i\leq p} \left[\tilde{O}(y_i) \vee \lambda((z_i)\right] \\
&\leq t
\end{aligned}$$

and

$$\begin{aligned}
\left(\lambda * \tilde{O}\right)(a) &= \bigwedge_{a=\sum\limits_{l=1}^{a} u_l v_l} \left[\bigvee_{1\leq l\leq a} \left[\lambda(u_l) \vee \tilde{O}(v_l)\right]\right] \\
&\leq \bigvee_{1\leq j\leq q} \left[\lambda(s_j) \vee \tilde{O}(t_j)\right] \\
&\leq t.
\end{aligned}$$

We have $\lambda(a) \leq \max\left\{\left(\lambda * \tilde{O}\right)(a), \left(\tilde{O} * \lambda\right)(a)\right\} \leq t$. Thus $a \in L(\lambda; t)$. This shows that $L(\lambda; t)$ is a quasi-ideal of R.

Conversely, assume that $L(\lambda; t) \neq \varnothing$ is a quasi-ideal of R for all $t \in [0,1]$. Let $x, y \in R$ be such that $\lambda(x+y) \geq \lambda(x) \vee \lambda(y)$. Choose $t \in [0,1]$ such that $\lambda(x+y) > t \geq \lambda(x) \vee \lambda(y) \implies x, y \in L(\lambda; t)$ but $x + y \notin L(\lambda; t)$ which is a contradiction. Hence $\lambda(x+y) \leq \lambda(x) \vee \lambda(y)$.

Now let $a \in R$ be such that $\max\left\{\left(\lambda * \tilde{O}\right)(a), \left(\tilde{O} * \lambda\right)(a)\right\} < \lambda(a)$. Choose $t \in [0,1]$ such that $\max\left\{\left(\lambda * \tilde{O}\right)(a), \left(\tilde{O} * \lambda\right)(a)\right\} \leq t < \lambda(a)$. Then $\left(\lambda * \tilde{O}\right)(a) \leq t \implies a \in L(\lambda; t) R$ and $\left(\tilde{O} * \lambda\right)(a) \leq t \implies a \in RL(\lambda; t)$. Hence $a \in (RL(\lambda; t)) \cap (L(\lambda; t) R) \subseteq L(\lambda; t)$, this implies that $a \in L(\lambda; t)$, that is $\lambda(a) \leq t$, which is a contradiction. Hence $\lambda(a) \leq \max\left\{\left(\lambda * \tilde{O}\right)(a), \left(\tilde{O} * \lambda\right)(a)\right\}$. Thus λ is an anti fuzzy quasi-ideal of R. $\square$

Theorem 3.18. *A non-empty subset Q of a semiring R is a quasi-ideal of R if and only if the fuzzy subset λ of R defined as*

$$\lambda(x) = \begin{cases} t & \text{if } x \notin Q \\ r & \text{if } x \in Q \end{cases}$$

is an anti fuzzy quasi-ideal of R, where $t, r \in [0,1]$ such that $t \geq r$.

Proof. The proof is similar to the proof of Theorem 3.5. □

Remark 3.19. From above theorem we conclude that a non-empty subset Q of a semiring R is a quasi-ideal of R if and only if the characteristic function of the complement of Q, that is C_{Q^c} is an anti fuzzy quasi-ideal of R.

Theorem 3.20. *Every anti fuzzy left ideal of R is an anti fuzzy quasi-ideal of R.*

Proof. Let λ be an anti fuzzy left ideal of R. Then by Theorem 3.15 $\tilde{O} * \lambda \geq \lambda$. Thus

$$\max\left\{\left(\lambda * \tilde{O}\right)(x), \left(\tilde{O} * \lambda\right)(x)\right\} \geq \left(\tilde{O} * \lambda\right)(x) \geq \lambda(x).$$

Hence λ is an anti fuzzy quasi-ideal of R. □

Theorem 3.21. *Every anti fuzzy quasi-ideal of R is an anti fuzzy bi-ideal of R.*

Proof. Suppose λ is an anti fuzzy quasi-ideal of R and $x, y \in R$. Then

$$\begin{aligned}
\lambda(xy) &\leq \left(\lambda * \tilde{O}\right)(xy) \vee \left(\tilde{O} * \lambda\right)(xy) \\
&= \left[\bigwedge_{xy=\sum_{i=1}^{p} a_i b_i} \left[\bigvee_{1 \leq i \leq p} \left[\lambda(a_i) \vee \tilde{O}(b_i)\right]\right]\right] \vee \\
&\qquad \left[\bigwedge_{xy=\sum_{j=1}^{q} s_j t_j} \left[\bigvee_{1 \leq j \leq q} \left[\tilde{O}(s_j) \vee \tilde{O}(t_j)\right]\right]\right] \\
&\leq \left[\lambda(x) \vee \tilde{O}(y)\right] \vee \left[\tilde{O}(x) \vee \lambda(y)\right] \\
&= [\lambda(x) \vee 0] \vee [0 \vee \lambda(y)] \\
&= \lambda(x) \vee \lambda(y).
\end{aligned}$$

So $\lambda(xy) \leq \lambda(x) \vee \lambda(y)$. Also,

$$\begin{aligned}
\lambda(xyz) &\leq \left(\lambda * \tilde{O}\right)(xyz) \vee \left(\tilde{O} * \lambda\right)(xyz) \\
&= \left[\bigwedge_{xyz=\sum\limits_{i=1}^{p} a_i b_i} \left[\bigvee_{1\leq i\leq p} \left[\lambda(a_i) \vee \tilde{O}(b_i)\right]\right]\right] \vee \\
&\qquad\qquad \left[\bigwedge_{xyz=\sum\limits_{j=1}^{q} s_j t_j} \left[\bigvee_{1\leq j\leq q} \left[\tilde{O}(s_j) \vee \tilde{O}(t_j)\right]\right]\right] \\
&\leq \left[\lambda(x) \vee \tilde{O}(yz)\right] \vee \left[\tilde{O}(xy) \vee \lambda(z)\right] \\
&= [\lambda(x) \vee 0] \vee [0 \vee \lambda(z)] \\
&= \lambda(x) \vee \lambda(z).
\end{aligned}$$

So $\lambda(xyz) \leq \lambda(x) \vee \lambda(z)$. □

4. Regular and intra-regular semirings

In this section we characterize regular and intra-regular semirings by the properties of their anti fuzzy ideal, anti fuzzy bi-ideals and anti fuzzy generalized by-ideals.

Lemma 4.1. *Let A, B be subsets of a semiring R and C_{A^c}, C_{B^c} be the characteristic functions of the complements of A and B, respectively. Then*
(1) $C_{A^c} * C_{B^c} = C_{(AB)^c}$
(2) $C_A \vee C_B = C_{A\cup B}$.

Proof. (1) Let $x \in (AB)^c$. Then $C_{(AB)^c}(x) = 1$. Now

$$\left(C_{A^c} * C_{B^c}\right)(x) = \bigwedge_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} [C_{A^c}(y_i) \vee C_{B^c}(z_i)]\right].$$

As $x \in (AB)^c$, so $x \notin AB$. Thus there does not exist $y_i \in A$, $z_i \in B$ such that $x = \sum_{i=1}^{p} y_i z_i$. Hence whenever $x = \sum_{i=1}^{p} y_i z_i$, then $y_i \in A^c$ or $z_i \in B^c$. Thus

$$\left(C_{A^c} * C_{B^c}\right)(x) = \bigwedge_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} [C_{A^c}(y_i) \vee C_{B^c}(z_i)]\right] = 1.$$

If $x \notin (AB)^c$, then $x \in AB$. Thus there exist at least one expression form of x as $\sum_{i=1}^{p} y_i z_i$ where $y_i \in A$ and $z_i \in B$. For this expression form

$$\bigvee_{1\leq i\leq p} [C_{A^c}(y_i) \vee C_{B^c}(z_i)] = 0.$$

Thus

$$\left(C_{A^c} * C_{B^c}\right)(x) = \bigwedge_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} [C_{A^c}(y_i) \vee C_{B^c}(z_i)]\right] = 0$$

Hence $C_{A^c} * C_{B^c} = C_{(AB)^c}$.

(2) Straightforward. $\square$

Theorem 4.2. *For a semiring R, the following conditions are equivalent.*

(1) *R is regular.*

(2) *$(\lambda \vee \mu) = (\lambda * \mu)$ for every anti fuzzy right ideal λ and every anti fuzzy left ideal μ of R.*

Proof. (1) $\implies$ (2) Let λ be an anti fuzzy right ideal and μ be an anti fuzzy left ideal of R. Then by Theorem 3.10, $\lambda * \mu \geq \lambda \vee \mu$. Let $a \in R$. Then there exists $x \in R$ such that $a = axa$. Thus we have

$$\begin{aligned}
(\lambda * \mu)(a) &= \bigwedge_{a=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1 \leq i \leq p} [\lambda(y_i) \vee \mu(z_i)] \right] \\
&\leq \lambda(ax) \vee \mu(a) \\
&\leq \lambda(a) \vee \mu(a) \\
&= (\lambda \vee \mu)(a).
\end{aligned}$$

Which implies that $\lambda * \mu \leq \lambda \vee \mu$. Hence $(\lambda \vee \mu) = (\lambda * \mu)$.

(2) $\implies$ (1) Let A be a right ideal and L be a left ideal of R. Then by Remark 3.6, C_{A^c} is an anti fuzzy right and C_{L^c} is an anti fuzzy left ideal of R. Thus by hypothesis $C_{A^c} \vee C_{L^c} = C_{A^c} * C_{L^c}$. By Lemma 4.1, $C_{A^c \cup L^c} = C_{(AL)^c}$. This implies $A^c \cup L^c = (AL)^c \implies (A \cap L)^c = (AL)^c \implies A \cap L = AL$. So by Theorem 2.1, R is regular. $\square$

Theorem 4.3. *For a semiring R the following conditions are equivalent.*

(1) *R is regular.*

(2) *$(\zeta \vee \lambda \vee \mu) \geq (\zeta * \lambda * \mu)$ for every anti fuzzy right ideal ζ, every anti fuzzy generalized bi-ideal λ, and every anti fuzzy left ideal μ of R.*

(3) *$(\zeta \vee \lambda \vee \mu) \geq (\zeta * \lambda * \mu)$ for every anti fuzzy right ideal ζ, every anti fuzzy bi-ideal λ, and every anti fuzzy left ideal μ of R.*

(4) *$(\zeta \vee \lambda \vee \mu) \geq (\zeta * \lambda * \mu)$ for every anti fuzzy right ideal ζ, every anti fuzzy quasi-ideal λ, and every anti fuzzy left ideal μ of R.*

Proof. (1) $\implies$ (2) Let ζ, λ and μ be any anti fuzzy right ideal, anti fuzzy generalized bi-ideal, and anti fuzzy left ideal of R, respectively. Let a be any element of R. Since R is regular, so there exists $x \in R$ such that $a = axa$. Hence we have

$$\begin{aligned}
(\zeta * \lambda * \mu)(a) &= \bigwedge_{a=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1 \leq i \leq p} [\zeta(y_i) \vee (\lambda * \mu)(z_i)] \right] \\
&\leq \zeta(ax) \vee (\lambda * \mu)(a) \\
&\leq \zeta(a) \vee \bigwedge_{a=\sum_{j=1}^{q} s_j t_j} \left[\bigvee_{1 \leq j \leq q} [\lambda(s_j) \vee \mu(t_j)] \right]
\end{aligned}$$

$$\begin{aligned} &\leq \zeta(a) \vee [\lambda(a) \vee \mu(xa)] \\ &\leq \zeta(a) \vee [\lambda(a) \vee \mu(a)] \\ &= (\zeta \vee \lambda \vee \mu)(a). \end{aligned}$$

Thus $(\zeta \vee \lambda \vee \mu) \geq (\zeta * \lambda * \mu)$.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) straightforward.

(4) $\Longrightarrow$ (1) Let ζ and μ be any anti fuzzy right and any anti fuzzy left ideal of R, respectively. Since $\tilde{O}$ is an anti fuzzy quasi-ideal of R, by assumption we have

$$\begin{aligned} (\zeta \vee \mu)(a) &= \left(\zeta \vee \tilde{O} \vee \mu\right)(a) \geq \left(\zeta * \tilde{O} * \mu\right)(a) \\ &= \bigwedge_{a=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} \left[\left(\zeta * \tilde{O}\right)(y_i) \vee \mu(z_i) \right] \right] \\ &= \bigwedge_{a=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} \left[\left[\bigwedge_{y_i=\sum_{j=1}^{q} a_j b_j} \left[\bigvee_{1\leq j\leq q} \left[\zeta(a_j) \vee \tilde{O}(b_j) \right] \right] \right] \vee \mu(z_i) \right] \right] \\ &= \bigwedge_{a=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} \left[\left[\bigwedge_{y_i=\sum_{j=1}^{q} a_j b_j} \left[\bigvee_{1\leq j\leq q} [\zeta(a_j)] \right] \right] \vee \mu(z_i) \right] \right] \\ &\geq \bigwedge_{a=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} \left[\left[\bigwedge_{y_i=\sum_{j=1}^{q} a_j b_j} \left[\bigvee_{1\leq j\leq q} [\zeta(a_j b_j)] \right] \right] \vee \mu(z_i) \right] \right] \\ &\geq \bigwedge_{a=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} \left[\left[\bigwedge_{y_i=\sum_{j=1}^{q} a_j b_j} \left[\bigvee_{1\leq j\leq q} \left[\zeta(\sum_{i=1}^{p} a_j b_j) \right] \right] \right] \vee \mu(z_i) \right] \right] \\ &= \bigwedge_{a=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} \left[\left[\bigwedge_{y_i=\sum_{j=1}^{q} a_j b_j} [\zeta(y_i)] \right] \vee \mu(z_i) \right] \right] \\ &= \bigwedge_{a=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} [\zeta(y_i) \vee \mu(z_i)] \right] = (\zeta * \mu)(a). \end{aligned}$$

Thus it follows that $(\zeta * \mu) \leq (\zeta \vee \mu)$ for every anti fuzzy right ideal ζ and every anti fuzzy left ideal μ of R. But $(\zeta * \mu) \geq (\zeta \vee \mu)$ by Theorem.3.10. So $(\zeta * \mu) = (\zeta \vee \mu)$. Hence it follows from Theorem 4.2 that R is regular. □

Recall that a semiring R is intra-regular if for each $a \in R$ there exist $x_i, y_i \in R$ such that $a =_{i=1}^{p} x_i a^2 y_i$.

It is well known that

Theorem 4.4. *A semiring R is intra-regular if and only if $A \cap L \subseteq LA$ for every right ideal A and left ideal L of R.*

Theorem 4.5. *The following assertions are equivalent for a semiring R*

(1) *R is both regular and intra-regular.*

(2) *$B = BB$ for each bi-ideal B of R.*

(3) *$Q = QQ$ for each quasi-ideal Q of R.*

Theorem 4.6. *For a semiring R the following conditions are equivalent.*

(1) *R is intra regular.*

(2) *$\lambda \vee \mu \geq \lambda * \mu$ for every anti fuzzy left ideal λ and every anti fuzzy right ideal μ of R.*

Proof. (1) $\Longrightarrow$ (2) Let λ and μ be anti fuzzy left and right ideals of R, respectively and $a \in R$. Then there exist $x_i, y_i \in R$ such that $a = \sum_{i=1}^{p} x_i a^2 y_i$. Thus

$$\begin{aligned}
(\lambda * \mu)(a) &= \bigwedge_{a=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1 \leq i \leq p} [\lambda(y_i) \vee \mu(z_i)] \right] \\
&\leq \bigvee_{1 \leq i \leq p} [\lambda(x_i a) \vee \mu(a y_i)] \\
&\leq \bigvee_{1 \leq i \leq p} [\lambda(a) \vee \mu(a)] \\
&= (\lambda \vee \mu)(a).
\end{aligned}$$

Thus $\lambda * \mu \leq \lambda \vee \mu$.

(2) $\Longrightarrow$ (1) Let L be a left ideal and A a right ideal of R. Then C_{L^c} is anti fuzzy left ideal and C_{A^c} is an anti fuzzy right ideal of R. By hypothesis $C_{L^c} * C_{A^c} \leq C_{L^c} \vee C_{A^c} \Longrightarrow C_{(LA)^c} \leq C_{L^c \cup A^c} \Longrightarrow (LA)^c \subseteq L^c \cup A^{c.} \Longrightarrow L \cap A \subseteq LA$. Thus by Theorem 4.4, R is intra-regular. □

Theorem 4.7. *For a semiring R the following conditions are equivalent.*

(1) *R is both regular and intra-regular.*

(2) *$\lambda * \lambda = \lambda$ for every anti fuzzy quasi-ideal λ of R.*

(3) *$\lambda * \lambda = \lambda$ for every anti fuzzy bi-ideal λ of R.*

(4) *$\lambda * \mu \leq \lambda \vee \mu$ for every anti fuzzy quasi-ideals λ and μ of R.*

(5) *$\lambda * \mu \leq \lambda \vee \mu$ for every anti fuzzy quasi-ideal λ of R and for all anti fuzzy bi-ideals μ of R.*

(6) *$\lambda * \mu \leq \lambda \vee \mu$ for every anti fuzzy bi-ideals λ and μ of R.*

Proof. (1) $\implies$ (6) Let λ and μ be any anti fuzzy bi-ideals of R and $a \in R$. Then, since R is regular, there exists $x \in R$ such that $a = axa$. Since$R$ is intra-regular, there exist $y_i, z_i \in R$ such that $a = \sum_{i=1}^{n} y_i a^2 z_i$. Thus $a = axa = axaxa = ax\left(\sum_{i=1}^{n} y_i a^2 z_i\right) xa = \sum_{i=1}^{n} (axy_ia)(az_ixa)$. Hence

$$
\begin{aligned}
(\lambda * \mu)(a) &= \bigwedge_{a=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} [\lambda(y_i) \vee \mu(z_i)] \right] \\
&\leq \bigvee_{1\leq i\leq n} [\lambda(axy_ia) \vee \mu(az_ixa)] \\
&\leq \bigvee_{1\leq i\leq n} [\lambda(a) \vee \mu(a)] \\
&= (\lambda \vee \mu)(a)
\end{aligned}
$$

and so $\lambda * \mu \leq \lambda \vee \mu$.

It is clear that (6) $\implies$ (5) $\implies$ (4).

(4) $\implies$ (2) Taking $\lambda = \mu$ in (4), we get $\lambda * \lambda \leq \lambda \vee \lambda = \lambda$. Since every anti fuzzy quasi-ideal of R is an anti fuzzy subsemiring of R, so by Theorem 3.14, we have $\lambda * \lambda \geq \lambda$. Thus $\lambda * \lambda = \lambda$.

(6) $\implies$ (3) Taking $\lambda = \mu$ in (6), we get $\lambda * \lambda \leq \lambda \vee \lambda = \lambda$. Since every anti fuzzy quasi-ideal of R is an anti fuzzy subsemiring of R, so by Theorem 3.14 we have $\lambda * \lambda \geq \lambda$. Thus $\lambda * \lambda = \lambda$.

(3) $\implies$ (2) Obvious.

(2) $\implies$ (1) Suppose Q is a quasi-ideal of R. Then C_{Q^c} is an anti fuzzy quasi-ideal of R, thus by hypothesis $C_{Q^c} * C_{Q^c} = C_{Q^c} \implies (QQ)^c = Q^c \implies QQ = Q$ and hence by Theorem 4.5, R is regular and intra regular. □

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