Characterizations of semirings by their anti fuzzy ideals

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Abstract. In this paper we define anti fuzzy ideals, anti fuzzy (generalized) bi-ideals and anti fuzzy quasi-ideals in semirings. We characterize different classes of semirings by the properties of their anti fuzzy ideals, anti fuzzy quasi-ideals and anti fuzzy bi-ideals.

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1. Introduction

2. Preliminaries

A semiring $R$ is a non-empty set $R$ equipped with two binary operations addition "+" and multiplication "." such that $(R, +)$ is a commutative semigroup, $(R, ·)$ is a semigroup, multiplication distributes over addition from both sides and $R$ contains an element 0 such that $a + 0 = 0 + a = a$ and $a0 = 0a = 0$ for all $a \in R$. A non-empty subset $A$ of a semiring $R$ is called a subsemiring of $R$ if it is closed under addition and multiplication. A non-empty subset $L$ of $R$ is called a left (right) ideal of $R$ if it is closed under addition and multiplication and $a \in L$ then $aL \subseteq L$ (or $La \subseteq L$), for all $a \in R$. A non-empty subset $L$ of $R$ is called a subsemiring of $R$ if it is both a left and a right ideal of $R$. A non-empty subset $B$ of $R$ is called a subsemiring of $R$ if it is closed under addition and $BRB \subseteq B$. A non-empty subset $B$ of $R$ is called a bi-ideal if it is a subsemiring of $R$ and $BRB \subseteq B$. A non-empty subset $Q$ of $R$ is called a quasi-ideal of $R$ if it is closed under addition and $QR \subseteq Q$. A semiring $R$ is regular if every element of $R$ is regular. It is obvious that every left (right) ideal of a semiring is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal. But the converse is not true.

An element $a$ of a semiring $R$ is called regular if there exists an element $x \in R$ such that $a = axa$. A semiring $R$ is regular if every element of $R$ is regular.

It is well known that:

**Theorem 2.1.** For a semiring $R$ the following conditions are equivalent.

1. $R$ is regular.
2. $A \cap L = AL$ for every right ideal $A$ and every left ideal $L$ of $R$.

A fuzzy subset $\lambda$ of a universe $X$ is a function from $X$ to the unit closed interval $[0, 1]$, that is $\lambda : X \rightarrow [0, 1]$. For any two fuzzy subsets $\lambda$ and $\mu$ of $R$, $\lambda \subseteq \mu$ means that, for all $x \in R$, $\lambda(x) \leq \mu(x)$. The symbols $\lambda \wedge \mu$ and $\lambda \vee \mu$ means the following fuzzy subsets of $R$.

$$(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x) \text{ and } (\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x)$$

for all $x \in R$.

More generally if $\{\lambda_i\}_{i \in I}$ is a family of fuzzy subsets of $X$, then their union and intersection is defined as follows:

$$(\bigvee_{i \in I} \lambda_i)(x) = \bigvee_{i \in I} (\lambda_i(x)) \text{ and } (\bigwedge_{i \in I} \lambda_i)(x) = \bigwedge_{i \in I} (\lambda_i(x))$$

for all $x \in R$.

Let $A$ be a subset of $X$. Then the characteristic function of $A$ is defined as:

$$C_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{otherwise}
\end{cases}$$

3. Anti fuzzy ideals

In this section we define anti fuzzy left (right) ideal, bi-ideal, generalized bi ideal and quasi-ideal of semiring and study some properties of these ideals.

**Definition 3.1.** Let $\lambda$ be a fuzzy subset of a semiring $R$ and $x, y \in R$.

1. $\lambda(x + y) \leq \lambda(x) \vee \lambda(y)$
2. $\lambda(xy) \leq \lambda(x) \vee \lambda(y)$
3. $\lambda(xy) \leq \lambda(y)$ (if $\lambda(xy) \leq \lambda(x)$)
(4) \( \lambda(xyz) \leq \lambda(x) \lor \lambda(z) \).

Then \( \lambda \) is called an anti fuzzy subsemiring of \( R \), if it satisfies (1) and (2).

It is called an anti fuzzy left (right) ideal of \( R \), if it satisfies (1) and (3).

A fuzzy subset \( \lambda \) of \( R \) is called an anti fuzzy ideal of \( R \) if it is both an anti fuzzy left and right ideal of \( R \).

\( \lambda \) is called an anti fuzzy generalized bi-ideal of \( R \), if it satisfies (1) and (4).

\( \lambda \) is called an anti fuzzy bi-ideal of \( R \), if it satisfies (1), (2) and (4).

Definition 3.2. For any fuzzy subset \( \lambda \) of a universe \( X \) and \( t \in [0,1] \) we define

\[ L(\lambda; t) = \{ x \in X : \lambda(x) \leq t \} \]

which is called the anti level cut of \( \lambda \).

Next we characterize anti fuzzy subsemiring (left ideal, right ideal, bi-ideal) by their anti level cuts.

**Theorem 3.3.** (1) A fuzzy subset \( \lambda \) of a semiring \( R \) is an anti fuzzy subsemiring of \( R \) if and only if \( L(\lambda; t) \neq \emptyset \) is a subsemiring of \( R \) for all \( t \in [0,1] \).

(2) A fuzzy subset \( \lambda \) of a semiring \( R \) is an anti fuzzy generalized bi-ideal of \( R \) if and only if \( L(\lambda; t) \neq \emptyset \) is a generalized bi-ideal of \( R \) for all \( t \in [0,1] \).

(3) A fuzzy subset \( \lambda \) of a semiring \( R \) is an anti fuzzy bi-ideal of \( R \) if and only if \( L(\lambda; t) \neq \emptyset \) is a bi-ideal of \( R \) for all \( t \in [0,1] \).

(4) A fuzzy subset \( \lambda \) of a semiring \( R \) is an anti fuzzy left (right) ideal of \( R \) if and only if \( L(\lambda; t) \neq \emptyset \) is a left (right) ideal of \( R \) for all \( t \in [0,1] \).

**Proof.** (1) Let \( \lambda \) be an anti fuzzy subsemiring of \( R \) and \( x, y \in L(\lambda; t) \). Then \( \lambda(x) \leq t \) and \( \lambda(y) \leq t \). Since \( \lambda(x+y) \leq \lambda(x) \lor \lambda(y) \leq t \lor t = t \), we have \( \lambda(x+y) \leq t \), that is \( x+y \in L(\lambda; t) \). Also \( \lambda(xy) \leq \lambda(x) \lor \lambda(y) \leq t \lor t = t \). This implies that \( \lambda(xy) \leq t \), that is \( xy \in L(\lambda; t) \). Hence \( L(\lambda; t) \) is a subsemiring of \( R \).

Conversely, assume that \( L(\lambda; t) \neq \emptyset \) is a subsemiring of \( R \) for all \( t \in [0,1] \). Suppose there exist \( x, y \in R \) such that \( \lambda(x+y) > \lambda(x) \lor \lambda(y) \). Choose \( t \in [0,1] \) such that \( \lambda(x+y) > t \geq \lambda(x) \lor \lambda(y) \). Then \( x+y \in L(\lambda; t) \) but \( x+y \notin L(\lambda; t) \), which is a contradiction. Hence \( \lambda(x+y) \leq \lambda(x) \lor \lambda(y) \). Similarly, we can prove that \( \lambda(xy) \leq \lambda(x) \lor \lambda(y) \). This proves that \( \lambda \) is an anti fuzzy subsemiring of \( R \).

The proofs of (2), (3) and (4) are similar to the proof of part (1).

**Example 3.4.** Consider the semiring \( R = \{0,1,a,b,c\} \) defined by the following tables

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<tr>
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<table>
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</table>
Then \((R, +, \cdot)\) is a semiring. Define \(\lambda : R \to [0, 1]\) by \(\lambda(0) = 0.1, \lambda(c) = 0.3, \lambda(a) = 0.6, \lambda(b) = \lambda(1) = 0.7\).

\[
L(\lambda; t) = \begin{cases} 
R & 0.7 \leq t < 1 \\
\{0, a, c\} & 0.6 \leq t < 0.7 \\
\{0, c\} & 0.3 \leq t < 0.6 \\
\{0\} & 0.1 \leq t < 0.3 \\
\varnothing & t < 0.1 
\end{cases}
\]

Then by Theorem 3.3, \(\lambda\) is an anti fuzzy ideal of \(R\).

**Theorem 3.5.** Let \(A\) be a non-empty subset of a semiring \(R\). Define the fuzzy subset \(\lambda\) of \(R\) by

\[
\lambda(x) = \begin{cases} 
t & \text{if } x \notin A \\
r & \text{if } x \in A 
\end{cases}
\]

where \(t, r \in [0, 1]\) such that \(t \geq r\). Then

1. \(A\) is a left (right) ideal of \(R\) if and only if \(\lambda\) is an anti fuzzy left (right) ideal of \(R\).
2. \(A\) is a subsemiring of \(R\) if and only if \(\lambda\) is an anti fuzzy subsemiring of \(R\).
3. \(A\) is a (generalized) bi-ideal of \(R\) if and only if \(\lambda\) is an anti fuzzy (generalized) bi-ideal of \(R\).

**Proof.** Straightforward. \(\square\)

**Remark 3.6.** From above theorems we conclude that a non-empty subset \(A\) of a semiring \(R\) is a left (right, bi, generalized bi) ideal of \(R\) if and only if the characteristic function of the complement of \(A\), that is \(C_A\), is an anti fuzzy left (right, bi, generalized bi) ideal of \(R\).

**Lemma 3.7.** The union of any family of anti fuzzy left (right) ideals of a semiring \(R\) is an anti fuzzy left (right) ideal of \(R\).

**Proof.** Let \(\{\lambda_i : i \in I\}\) be a family of anti fuzzy left ideals of \(R\) and \(x, y \in R\). Then

\[
\left(\bigvee_{i \in I} \lambda_i\right) (x + y) = \bigvee_{i \in I} (\lambda_i (x + y))
\]

(Since each \(\lambda_i\) is an anti fuzzy left ideal of \(R\), so \(\lambda_i (x + y) \leq \lambda_i (x) \lor \lambda_i (y)\) for all \(i \in I\)). Thus

\[
\left(\bigvee_{i \in I} \lambda_i\right) (x + y) = \bigvee_{i \in I} (\lambda_i (x + y)) \\
\leq \bigvee_{i \in I} (\lambda_i (x) \lor \lambda_i (y)) \\
= \left(\left(\bigvee_{i \in I} \lambda_i\right) (x)\right) \lor \left(\left(\bigvee_{i \in I} \lambda_i\right) (y)\right).
\]

Now

\[
\left(\bigvee_{i \in I} \lambda_i\right) (xy) = \bigvee_{i \in I} (\lambda_i (xy))
\]

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\[ \lambda_i(xy) \leq \lambda_i(y) \text{ for all } i \in I. \] Thus
\[ \left( \bigvee_{i \in I} \lambda_i \right)(xy) = \bigvee_{i \in I} \left( \lambda_i(xy) \right) \leq \bigvee_{i \in I} \left( \lambda_i(y) \right) = \left( \bigvee_{i \in I} \lambda_i \right)(y). \]

Hence \( \bigvee_{i \in I} \lambda_i \) is an anti fuzzy left ideal of \( R \). \( \Box \)

**Definition 3.8.** Let \( \lambda, \mu \) be fuzzy subsets of \( R \). Then the anti product of \( \lambda \) and \( \mu \) is defined by
\[ (\lambda \ast \mu)(x) = \bigwedge_{x = \sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} [\lambda(y_i) \vee \mu(z_i)] \right] \]
for \( x \in R \) and \( p \in \mathbb{N} \).

**Proposition 3.9.** If \( \lambda, \mu \) are anti fuzzy left (right) ideals of \( R \), then \( \lambda \ast \mu \) is an anti fuzzy left (right) ideal of \( R \).

**Proof.** Let \( \lambda \) and \( \mu \) be anti fuzzy left ideals of \( R \) and \( a, b, r \in R \). Then
\[ (\lambda \ast \mu)(a) = \bigwedge_{a = \sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} [\lambda(y_i) \vee \mu(z_i)] \right] \]
and
\[ (\lambda \ast \mu)(b) = \bigwedge_{b = \sum_{j=1}^{q} y'_j z'_j} \left[ \bigvee_{1 \leq j \leq q} [\lambda(y'_j) \vee \mu(z'_j)] \right]. \]

Thus
\[ (\lambda \ast \mu)(a) \vee (\lambda \ast \mu)(b) = \bigwedge_{a = \sum_{i=1}^{p} y_i z_i, b = \sum_{j=1}^{q} y'_j z'_j} \left[ \bigvee_{1 \leq i \leq p} [\lambda(y_i) \vee \mu(z_i)] \right] \]
\[ \quad \vee \bigwedge_{b = \sum_{j=1}^{q} y'_j z'_j} \left[ \bigvee_{1 \leq j \leq q} [\lambda(y'_j) \vee \mu(z'_j)] \right] \]
\[ = \bigwedge_{a = \sum_{i=1}^{p} y_i z_i, b = \sum_{j=1}^{q} y'_j z'_j} \left[ \bigvee_{1 \leq i \leq p} [\lambda(y_i) \vee \mu(z_i)] \right] \]
\[ \quad \vee \bigvee_{1 \leq j \leq q} [\lambda(y'_j) \vee \mu(z'_j)] \]
\[ \geq \bigwedge_{a + b = \sum_{k=1}^{s} y''_k z''_k} \left[ \bigvee_{1 \leq k \leq s} [\lambda(y''_k) \vee \mu(z''_k)] \right] \]
\[ = (\lambda \ast \mu)(a + b). \]
Now
\[
(\lambda \ast \mu)(a) = \bigwedge_{a=\sum_{j=1}^{q} \delta_j \varepsilon_j} \left[ \bigvee_{1 \leq j \leq q} [\lambda(\delta_j) \lor \mu(\varepsilon_j)] \right]
\]
\[
\geq \bigwedge_{a=\sum_{j=1}^{q} \delta_j \varepsilon_j} \left[ \bigvee_{1 \leq j \leq q} [\lambda(\delta_j) \lor \mu(\varepsilon_j)] \right]
\]
\[
= \bigwedge_{r_a=\sum_{j=1}^{t} y''_n z''_n} \left[ \bigvee_{1 \leq n \leq t} [\lambda(y''_n) \lor \mu(z''_n)] \right]
\]
\[
= (\lambda \ast \mu)(ra).
\]
Thus \( \lambda \ast \mu \) is an anti fuzzy left ideal of \( R \).

**Theorem 3.10.** Let \( \lambda \) be an anti fuzzy right ideal and \( \mu \) be an anti fuzzy left ideal of a semiring \( R \). Then \( \lambda \ast \mu \geq \lambda \lor \mu \).

**Proof.** Let \( \lambda \) be an anti fuzzy right ideal and \( \mu \) be an anti fuzzy left ideal of a semiring \( R \) and \( x \in R \). Then
\[
(\lambda \ast \mu)(x) = \bigwedge_{x=\sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} [\lambda(y_i) \lor \mu(z_i)] \right]
\]
\[
\geq \bigwedge_{x=\sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} [\lambda(y_i) \lor \mu(z_i)] \right]
\]
\[
= \bigwedge_{x=\sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} \lambda(y_i) \lor \mu(z_i) \right]
\]
\[
\geq \bigwedge_{x=\sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} \lambda(\sum_{i=1}^{p} y_i z_i) \lor \mu(\sum_{i=1}^{p} y_i z_i) \right]
\]
\[
= (\lambda \lor \mu)(x)
\]
Thus \( \lambda \ast \mu \geq \lambda \lor \mu \).

Now we show that if \( \lambda \) and \( \mu \) are anti fuzzy ideals of a semiring \( R \), then \( \lambda \ast \mu \not\leq \lambda \lor \mu \) and \( \lambda \ast \mu \not\leq \lambda \land \mu \).

**Example 3.11.** Consider the semiring \( R = \{0, 1, a, b, c\} \) defined by the following tables
The (crisp) ideals of $R$ are $\{0\}$, $\{0, c\}$, $\{0, a, c\}$, and $R$. Let $A = \{0, a, c\}$ and $B = \{0, c\}$ then $(C_A \ast C_B)(c) = 1$ but $(C_A \lor C_B)(c) = 0$ and $(C_A \land C_B)(c) = 0$. Hence $\lambda \ast \mu \not\geq \lambda \lor \mu$ and $\lambda \ast \mu \not\geq \lambda \land \mu$.

**Definition 3.12.** Let $\lambda$ and $\mu$ be fuzzy subsets of $R$. Then their anti sum $\lambda \oplus \mu$ is defined as

$$(\lambda \oplus \mu)(x) = \bigwedge_{x = y + z} [\lambda(y) \lor \mu(z)]$$

for all $x \in R$.

**Proposition 3.13.** Let $\lambda$ and $\mu$ be anti fuzzy left (right) ideals of $R$. Then $\lambda \oplus \mu$ is an anti fuzzy left (right) ideal of $R$.

**Proof.** Let $\lambda$ and $\mu$ be anti fuzzy left ideals of $R$ and $x, \hat{x} \in R$. Then

$$(\lambda \oplus \mu)(x) = \bigwedge_{x = y + z} [\lambda(y) \lor \mu(z)]$$

and

$$(\lambda \oplus \mu)(\hat{x}) = \bigwedge_{\hat{x} = \hat{y} + \hat{z}} [\lambda(\hat{y}) \lor \mu(\hat{z})]$$

Thus

$$(\lambda \oplus \mu)(x) \lor (\lambda \oplus \mu)(\hat{x}) = \left[ \bigwedge_{x = y + z} [\lambda(y) \lor \mu(z)] \right] \lor \left[ \bigwedge_{\hat{x} = \hat{y} + \hat{z}} [\lambda(\hat{y}) \lor \mu(\hat{z})] \right]$$

$$= \bigwedge_{x = y + z} \left[ [\lambda(y) \lor \mu(z)] \lor [\lambda(\hat{y}) \lor \mu(\hat{z})] \right]$$

$$= \bigwedge_{x = y + z} \left[ [\lambda(y) \lor \lambda(\hat{y})] \lor [\mu(z) \lor \mu(\hat{z})] \right]$$

$$\geq \bigwedge_{x = y + z} \left[ \lambda(y + \hat{y}) \lor \mu(z + \hat{z}) \right]$$

$$\geq \bigwedge_{x + \hat{x} = a + b} \left[ \lambda(a) \lor \mu(b) \right]$$

$$(\lambda \oplus \mu)(x + \hat{x}).$$
Again, let \( x, a \in R \). Then
\[
(\lambda \oplus \mu)(x) = \bigwedge_{x = y + z} [\lambda(y) \vee \mu(z)] \\
\geq \bigwedge_{x = y + z} [\lambda(ay) \vee \mu(az)] \\
= \bigwedge_{ax = \hat{y} + \hat{z}} [\lambda(\hat{y}) \vee \mu(\hat{z})] \\
= (\lambda \oplus \mu)(ax).
\]

Hence \( \lambda \oplus \mu \) is an anti fuzzy left ideal of \( R \). \( \square \)

**Lemma 3.14.** A fuzzy subset \( \lambda \) of a semiring \( R \) is an anti fuzzy subsemiring of \( R \) if and only if \( \lambda \oplus \lambda \supseteq \lambda \) and \( \lambda^2 = \lambda \ast \lambda \supseteq \lambda \).

**Proof.** Let \( \lambda \) be an anti fuzzy subsemiring of \( R \) and \( x \in R \). Then
\[
(\lambda \oplus \lambda)(x) = \bigwedge_{x = y + z} [\lambda(y) \vee \lambda(z)] \\
\geq \bigwedge_{x = y + z} [\lambda(y + z)] \\
= \lambda(x) \\
= \lambda(x) \\
\Rightarrow (\lambda \oplus \lambda)(x) \geq \lambda(x).
\]

Thus \( \lambda \oplus \lambda \supseteq \lambda \). Now
\[
\lambda^2(x) = (\lambda \ast \lambda)(x) \\
= \bigwedge_{x = \sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} [\lambda(y_i) \vee \lambda(z_i)] \right] \\
\geq \bigwedge_{x = \sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} [\lambda(y_i z_i) \vee \lambda(y_i)] \right] \\
= \bigwedge_{x = \sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} \lambda(y_i z_i) \right] \\
\geq \bigwedge_{x = \sum_{i=1}^{p} y_i z_i} \left[ \lambda(\sum_{i=1}^{p} y_i z_i) \right]
\]

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Thus \( \lambda^2 \supseteq \lambda \).

Conversely, assume that \( \lambda \) is a fuzzy subset of \( R \) such that \( \lambda \oplus \lambda \supseteq \lambda \) and \( \lambda^2 \supseteq \lambda \). Then for \( x, y \in R \), we have

\[
\lambda(x + y) \leq (\lambda \oplus \lambda)(x + y) \\
= \bigwedge_{x+y=a+b} [\lambda(a) \lor \lambda(b)] \\
\leq \lambda(x) \lor \lambda(y) \\
\implies \lambda(x + y) \leq \lambda(x) \lor \lambda(y)
\]

and

\[
\lambda(xy) \leq \lambda^2(xy) \\
= (\lambda \ast \lambda)(xy) \\
= \bigwedge_{xy=\sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} [\lambda(y_i) \lor \lambda(z_i)] \right] \\
\leq \lambda(x) \lor \lambda(y) \\
\implies \lambda(xy) \leq \lambda(x) \lor \lambda(y)
\]

Thus \( \lambda \) is an anti fuzzy subsemiring of \( R \). 

\[\square\]

**Lemma 3.15.** A fuzzy subset \( \lambda \) of a semiring \( R \) is an anti fuzzy left (right) ideal of \( R \) if and only if \( \lambda \oplus \lambda \supseteq \lambda \) and \( \tilde{O} \ast \lambda \supseteq \lambda \) (\( \lambda \ast \tilde{O} \supseteq \lambda \)), where \( \tilde{O} \) is the fuzzy subset of \( R \) mapping every element of \( R \) on 0.

**Proof.** Let \( \lambda \) be an anti fuzzy left ideal of \( R \) and \( x \in R \). Then

\[
(\lambda \oplus \lambda)(x) = \bigwedge_{x = y + z} [\lambda(y) \lor \lambda(z)] \\
\geq \bigwedge_{x = y + z} [\lambda(y + z)] \\
= \bigwedge_{x = y + z} [\lambda(x)] \\
= \lambda(x) \\
\implies (\lambda \oplus \lambda)(x) \geq \lambda(x).
\]
Thus $\lambda \oplus \lambda \supseteq \lambda$. Now

$$\left( \tilde{O} \ast \lambda \right)(x) = \bigwedge_{x = \sum_{i=1}^{p} y_{i}z_{i}} \left[ \bigvee_{1 \leq i \leq p} \left( \tilde{O}(y_{i}) \vee \lambda(z_{i}) \right) \right]$$

$$\geq \bigwedge_{x = \sum_{i=1}^{p} y_{i}z_{i}} \left[ \bigvee_{1 \leq i \leq p} \left[ \lambda(y_{i}z_{i}) \right] \right]$$

$$= \bigwedge_{x = \sum_{i=1}^{p} y_{i}z_{i}} \lambda \left( \sum_{i=1}^{p} y_{i}z_{i} \right)$$

$$= \lambda(x)$$

$$\Rightarrow \left( \tilde{O} \ast \lambda \right)(x) \geq \lambda(x).$$

Thus $\tilde{O} \ast \lambda \supseteq \lambda$.

Conversely, assume that $\lambda$ is a fuzzy subset of $R$ such that $\lambda \oplus \lambda \supseteq \lambda$ and $\tilde{O} \ast \lambda \supseteq \lambda$. Then, for $x, y \in R$ we have

$$\lambda(x + y) \leq (\lambda \oplus \lambda)(x + y)$$

$$= \bigwedge_{x + y = a + b} \left[ \lambda(a) \vee \lambda(b) \right]$$

$$\leq \lambda(x) \vee \lambda(y)$$

$$\Rightarrow \lambda(x + y) \leq \lambda(x) \vee \lambda(y).$$

And

$$\lambda(xy) \leq (\tilde{O} \ast \lambda)(xy)$$

$$= \bigwedge_{xy = \sum_{i=1}^{p} y_{i}z_{i}} \left[ \bigvee_{1 \leq i \leq p} \left( \tilde{O}(y_{i}) \vee \lambda(z_{i}) \right) \right]$$

$$\leq \tilde{O}(x) \vee \lambda(y)$$

$$= \lambda(y)$$

$$\Rightarrow \lambda(xy) \leq \lambda(y).$$

Thus $\lambda$ is an anti fuzzy left ideal of $R$. \qed
Definition 3.16. A fuzzy subset $\lambda$ of a semiring $R$ is called an anti fuzzy quasi-ideal of $R$, if it satisfies,

1. $\lambda(x + y) \leq \lambda(x) \lor \lambda(y)$
2. $\lambda(x) \leq \max \left\{ (\lambda \ast \tilde{O})(x), (\tilde{O} \ast \lambda)(x) \right\}$

for all $x, y \in R$.

Where $\tilde{O}$ is the fuzzy subset of $R$ mapping every element of $R$ on 0.

Theorem 3.17. A fuzzy subset $\lambda$ of a semiring $R$ is an anti fuzzy quasi-ideal of $R$ if and only if $L(\lambda; t) \neq \emptyset$ is a quasi-ideal of $R$ for all $t \in [0, 1]$.

Proof. Suppose $\lambda$ is an anti fuzzy quasi-ideal of $R$ and $x, y \in L(\lambda; t)$. Then $\lambda(x) \leq t$ and $\lambda(y) \leq t$. Since $\lambda(x + y) \leq \lambda(x) \lor \lambda(y) \leq t$, we have $\lambda(x + y) \leq t$. Thus $x + y \in L(\lambda; t)$. Now let $a \in (RL(\lambda; t)) \cap (L(\lambda; t) R)$. Then $a = \sum_{i=1}^{p} y_{i}z_{i}$ and $a = \sum_{j=1}^{q} s_{j}t_{j}$, where $y_{i}, t_{j} \in R$ and $z_{i}, s_{j} \in L(\lambda; t) (1 \leq i \leq p, 1 \leq j \leq q)$. Thus $\lambda(z_{i}) \leq t$ for $1 \leq i \leq p$ and $\lambda(s_{j}) \leq t$ for $1 \leq j \leq q$. Since

$$\left( \lambda \ast \tilde{O} \right)(a) = \bigwedge_{\text{a = \sum_{k=1}^{r} a_{k}b_{k}}} \left[ \bigvee_{1 \leq k \leq r} [\tilde{O}(a_{k}) \lor \lambda(b_{k})] \right]$$

$$\leq \bigwedge_{1 \leq i \leq p} \left[ \tilde{O}(y_{i}) \lor \lambda(z_{i}) \right] \leq t$$

and

$$\left( \tilde{O} \ast \lambda \right)(a) = \bigwedge_{a = \sum_{l=1}^{u} u_{l}v_{l}} \left[ \bigvee_{1 \leq l \leq a} [\lambda(u_{l}) \lor \tilde{O}(v_{l})] \right]$$

$$\leq \bigwedge_{1 \leq j \leq q} \left[ \lambda(s_{j}) \lor \tilde{O}(t_{j}) \right] \leq t.$$
Theorem 3.18. A non-empty subset $Q$ of a semiring $R$ is a quasi-ideal of $R$ if and only if the fuzzy subset $\lambda$ of $R$ defined as

$$\lambda(x) = \begin{cases} t & \text{if } x \notin Q \\ r & \text{if } x \in Q \end{cases}$$

is an anti fuzzy quasi-ideal of $R$, where $t, r \in [0, 1]$ such that $t \geq r$.

Proof. The proof is similar to the proof of Theorem 3.5. □

Remark 3.19. From above theorem we conclude that a non-empty subset $Q$ of a semiring $R$ is a quasi-ideal of $R$ if and only if the characteristic function of the complement of $Q$, that is $C_Q$- is an anti fuzzy quasi-ideal of $R$.

Theorem 3.20. Every anti fuzzy left ideal of $R$ is an anti fuzzy quasi-ideal of $R$.

Proof. Let $\lambda$ be an anti fuzzy left ideal of $R$. Then by Theorem 3.15 $\tilde{O} \ast \lambda \geq \lambda$. Thus

$$\max \{ (\lambda \ast \tilde{O})(x), (\tilde{O} \ast \lambda)(x) \} \geq (\tilde{O} \ast \lambda)(x) \geq \lambda(x).$$

Hence $\lambda$ is an anti fuzzy quasi-ideal of $R$. □

Theorem 3.21. Every anti fuzzy quasi-ideal of $R$ is an anti fuzzy bi-ideal of $R$.

Proof. Suppose $\lambda$ is an anti fuzzy quasi-ideal of $R$ and $x, y \in R$. Then

$$\lambda(xy) \leq (\lambda \ast \tilde{O})(xy) \vee (\tilde{O} \ast \lambda)(xy)$$

$$= \bigg[ \bigwedge_{xy = \sum_{i=1}^{p} a_i b_i} \bigg[ \bigvee_{1 \leq i \leq p} \left[ \lambda(a_i) \vee \tilde{O}(b_i) \right] \bigg] \bigg] \vee \bigg[ \bigwedge_{xy = \sum_{j=1}^{q} s_j t_j} \bigg[ \bigvee_{1 \leq j \leq q} \left[ \tilde{O}(s_j) \vee \tilde{O}(t_j) \right] \bigg] \bigg]$$

$$\leq \left[ \lambda(x) \vee \tilde{O}(y) \right] \vee \left[ \tilde{O}(x) \vee \lambda(y) \right]$$

$$= \left[ \lambda(x) \vee 0 \right] \vee \left[ 0 \vee \lambda(y) \right]$$

$$= \lambda(x) \vee \lambda(y).$$

So $\lambda(xy) \leq \lambda(x) \vee \lambda(y)$. Also,
\[
\lambda(xyz) \leq \left( \lambda \ast \tilde{O} \right)(xyz) \lor \left( \tilde{O} \ast \lambda \right)(xyz)
\]
\[
= \left[ \bigwedge_{y \in \sum_{i=1}^{p} a_i, b_i} \left[ \bigvee_{1 \leq i \leq p} [\lambda(a_i) \lor \tilde{O}(b_i)] \right] \right] \lor \left[ \bigwedge_{y \in \sum_{j=1}^{q} s_j, t_j} \left[ \bigvee_{1 \leq j \leq q} [\tilde{O}(s_j) \lor \tilde{O}(t_j)] \right] \right]
\]
\[
\leq \left[ \lambda(x) \lor \tilde{O}(yz) \right] \lor \left[ \tilde{O}(xy) \lor \lambda(z) \right]
\]
\[
= [\lambda(x) \lor 0] \lor [0 \lor \lambda(z)]
\]
\[
= \lambda(x) \lor \lambda(z).
\]
So \(\lambda(xyz) \leq \lambda(x) \lor \lambda(z)\). \qed

4. Regular and intra-regular semirings

In this section we characterize regular and intra-regular semirings by the properties of their anti fuzzy ideal, anti fuzzy bi-ideals and anti fuzzy generalized by-ideals.

**Lemma 4.1.** Let \(A, B\) be subsets of a semiring \(R\) and \(C_{A^c}, C_{B^c}\) be the characteristic functions of the complements of \(A\) and \(B\), respectively. Then

(1) \(C_{A^c} \ast C_{B^c} = C_{(AB)^c}\)

(2) \(C_A \lor C_B = C_{A \cup B}\).

**Proof.** (1) Let \(x \in (AB)^c\). Then \(C_{(AB)^c}(x) = 1\). Now

\[
(C_{A^c} \ast C_{B^c})(x) = \bigwedge_{x' = \sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} [C_{A^c}(y_i) \lor C_{B^c}(z_i)] \right].
\]

As \(x \in (AB)^c\), so \(x \notin AB\). Thus there does not exist \(y_i \in A, z_i \in B\) such that \(x = \sum_{i=1}^{p} y_i z_i\). Hence whenever \(x = \sum_{i=1}^{p} y_i z_i\), then \(y_i \in A^c\) or \(z_i \in B^c\). Thus

\[
(C_{A^c} \ast C_{B^c})(x) = \bigwedge_{x' = \sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} [C_{A^c}(y_i) \lor C_{B^c}(z_i)] \right] = 1.
\]

If \(x \notin (AB)^c\), then \(x \in AB\). Thus there exist at least one expression form of \(x\) as \(\sum_{i=1}^{p} y_i z_i\) where \(y_i \in A\) and \(z_i \in B\). For this expression form

\[
\bigvee_{1 \leq i \leq p} [C_{A^c}(y_i) \lor C_{B^c}(z_i)] = 0.
\]

Thus

\[
(C_{A^c} \ast C_{B^c})(x) = \bigwedge_{x' = \sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} [C_{A^c}(y_i) \lor C_{B^c}(z_i)] \right] = 0
\]
Hence $C_{A^c} \ast C_{B^c} = C_{(AB)^c}$.

(2) Straightforward.

\textbf{Theorem 4.2.} For a semiring $R$, the following conditions are equivalent.

(1) $R$ is regular.

(2) $(\lambda \lor \mu) = (\lambda \ast \mu)$ for every anti fuzzy right ideal $\lambda$ and every anti fuzzy left ideal $\mu$ of $R$.

\textbf{Proof.} (1) $\implies$ (2) Let $\lambda$ be an anti fuzzy right ideal and $\mu$ be an anti fuzzy left ideal of $R$. Then by Theorem 3.10, $\lambda \ast \mu \geq \lambda \lor \mu$. Let $a \in R$. Then there exists $x \in R$ such that $a = axa$. Thus we have

$$ (\lambda \ast \mu)(a) = \bigwedge_{a = \sum \limits_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} [\lambda(y_i) \lor \mu(z_i)] \right] $$

$$ \leq \lambda(ax) \lor \mu(a) $$

$$ \leq \lambda(a) \lor \mu(a) $$

$$ = (\lambda \lor \mu)(a). $$

Which implies that $\lambda \ast \mu \leq \lambda \lor \mu$. Hence $(\lambda \lor \mu) = (\lambda \ast \mu)$.

(2) $\implies$ (1) Let $A$ be a right ideal and $L$ be a left ideal of $R$. Then by Remark 3.6, $C_{A^c}$ is an anti fuzzy right and $C_{L^c}$ is an anti fuzzy left ideal of $R$. Thus by hypothesis $C_{A^c} \lor C_{L^c} = C_{A^c} \ast C_{L^c}$. By Lemma 4.1, $C_{A^c \lor L^c} = C_{(AL)^c}$. This implies $A^c \cup L^c = (AL)^c \implies (A \cap L)^c = (AL)^c \implies A \cap L = AL$. So by Theorem 2.1, $R$ is regular.

\textbf{Theorem 4.3.} For a semiring $R$ the following conditions are equivalent.

(1) $R$ is regular.

(2) $(\zeta \lor \lambda \lor \mu) \geq (\zeta \ast \lambda \ast \mu)$ for every anti fuzzy right ideal $\zeta$, every anti fuzzy generalized bi-ideal $\lambda$, and every anti fuzzy left ideal $\mu$ of $R$.

(3) $(\zeta \lor \lambda \lor \mu) \geq (\zeta \ast \lambda \ast \mu)$ for every anti fuzzy right ideal $\zeta$, every anti fuzzy bi-ideal $\lambda$, and every anti fuzzy left ideal $\mu$ of $R$.

(4) $(\zeta \lor \lambda \lor \mu) \geq (\zeta \ast \lambda \ast \mu)$ for every anti fuzzy right ideal $\zeta$, every anti fuzzy quasi-ideal $\lambda$, and every anti fuzzy left ideal $\mu$ of $R$.

\textbf{Proof.} (1) $\implies$ (2) Let $\zeta, \lambda$ and $\mu$ be any anti fuzzy right ideal, anti fuzzy generalized bi-ideal, and anti fuzzy left ideal of $R$, respectively. Let $a$ be any element of $R$. Since $R$ is regular, so there exists $x \in R$ such that $a = axa$. Hence we have

$$ (\zeta \ast \lambda \ast \mu)(a) = \bigwedge_{a = \sum \limits_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} [\zeta(y_i) \lor (\lambda \ast \mu)(z_i)] \right] $$

$$ \leq \zeta(ax) \lor (\lambda \ast \mu)(a) $$

$$ \leq \zeta(a) \lor \bigwedge_{a = \sum \limits_{i=1}^{p} y_i z_i} \left( \bigvee_{1 \leq j \leq q} [\lambda(s_j) \lor \mu(t_j)] \right) $$

$$ \leq (\zeta \lor \lambda \lor \mu)(a). $$
Thus \( (\zeta \lor \lambda \lor \mu) \geq (\zeta \ast \lambda \ast \mu) \).

(2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) straightforward.

(4) \( \Rightarrow \) (1) Let \( \zeta \) and \( \mu \) be any anti fuzzy right and any anti fuzzy left ideal of \( R \), respectively. Since \( \bar{O} \) is an anti fuzzy quasi-ideal of \( R \), by assumption we have

\[
(\zeta \lor \mu)(a) = \left( \zeta \lor \bar{O} \lor \mu \right)(a) \geq \left( \zeta \ast \bar{O} \ast \mu \right)(a)
\]

\[
= \bigwedge_{a = \sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} \left[ \bigwedge_{j=1}^{p} y_i = \sum_{j=1}^{p} a_j b_j \left[ \bigvee_{1 \leq j \leq q} \left[ \zeta(a_j) \lor \bar{O}(b_j) \right] \right] \right] \lor \mu(z_i) \right]
\]

\[
\geq \bigwedge_{a = \sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} \left[ \bigwedge_{j=1}^{p} y_i = \sum_{j=1}^{p} a_j b_j \left[ \bigvee_{1 \leq j \leq q} \left[ \zeta(a_j b_j) \right] \right] \lor \mu(z_i) \right] \right]
\]

\[
\geq \bigwedge_{a = \sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} \left[ \bigwedge_{j=1}^{p} y_i = \sum_{j=1}^{p} a_j b_j \left[ \bigvee_{1 \leq j \leq q} \left[ \zeta(\sum_{i=1}^{p} a_j b_j) \right] \right] \lor \mu(z_i) \right] \right]
\]

\[
\geq \bigwedge_{a = \sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} \left[ \bigwedge_{j=1}^{p} y_i = \sum_{j=1}^{p} a_j b_j \left[ \zeta(y_i) \right] \lor \mu(z_i) \right] \right]
\]

\[
= \bigwedge_{a = \sum_{i=1}^{p} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} \left[ \zeta(y_i) \lor \mu(z_i) \right] \right] = (\zeta \ast \mu)(a).
\]
Thus it follows that \((\zeta \ast \mu) \leq (\zeta \lor \mu)\) for every anti fuzzy right ideal \(\zeta\) and every anti fuzzy left ideal \(\mu\) of \(R\). But \((\zeta \ast \mu) \geq (\zeta \lor \mu)\) by Theorem 3.10. So \((\zeta \ast \mu) = (\zeta \lor \mu)\). Hence it follows from Theorem 4.2 that \(R\) is regular.

Recall that a semiring \(R\) is intra-regular if for each \(a \in R\) there exist \(x_i, y_i \in R\) such that \(a = \sum_{i=1}^{p} x_i a^2 y_i\).

It is well known that

Theorem 4.4. A semiring \(R\) is intra-regular if and only if \(A \cap L \subseteq LA\) for every right ideal \(A\) and left ideal \(L\) of \(R\).

Theorem 4.5. The following assertions are equivalent for a semiring \(R\).

1. \(R\) is both regular and intra-regular.
2. \(B = BB\) for each bi-ideal \(B\) of \(R\).
3. \(Q = QQ\) for each quasi-ideal \(Q\) of \(R\).

Theorem 4.6. For a semiring \(R\) the following conditions are equivalent.

1. \(R\) is intra-regular.
2. \(\lambda \lor \mu \geq \lambda \ast \mu\) for every anti fuzzy left ideal \(\lambda\) and every anti fuzzy right ideal \(\mu\) of \(R\).

Proof. \((1) \implies (2)\) Let \(\lambda\) and \(\mu\) be anti fuzzy left and right ideals of \(R\), respectively and \(a \in R\). Then there exist \(x_i, y_i \in R\) such that \(a = \sum_{i=1}^{p} x_i a^2 y_i\). Thus

\[
(\lambda \ast \mu) (a) = \bigwedge_{a = \sum_{i=1}^{p} x_i a^2 y_i} \left( \bigvee_{1 \leq i \leq p} [\lambda(y_i) \lor \mu(z_i)] \right)
\]

\[
\leq \bigvee_{1 \leq i \leq p} [\lambda(x_i a) \lor \mu(ay_i)]
\]

\[
\leq \bigvee_{1 \leq i \leq p} [\lambda(a) \lor \mu(a)]
\]

\[
= (\lambda \lor \mu) (a).
\]

Thus \(\lambda \ast \mu \leq \lambda \lor \mu\).

\((2) \implies (1)\) Let \(L\) be a left ideal and \(A\) a right ideal of \(R\). Then \(C_L^e\) is an anti fuzzy left ideal and \(C_A^e\) is an anti fuzzy right ideal of \(R\). By hypothesis \(C_L^e \ast C_A^e \leq C_L^e \lor C_A^e\) \(\implies C_{(LA)^c} \leq C_{L^c \cup A^c}\) \(\implies (LA)^c \subseteq L^c \cup A^c\) \(\implies L \cap A \subseteq LA\). Thus by Theorem 4.4, \(R\) is intra-regular.

Theorem 4.7. For a semiring \(R\) the following conditions are equivalent.

1. \(R\) is both regular and intra-regular.
2. \(\lambda \ast \lambda = \lambda\) for every anti fuzzy quasi-ideal \(\lambda\) of \(R\).
3. \(\lambda \ast \lambda = \lambda\) for every anti fuzzy bi-ideal \(\lambda\) of \(R\).
4. \(\lambda \ast \mu \leq \lambda \lor \mu\) for every anti fuzzy quasi-ideals \(\lambda\) and \(\mu\) of \(R\).
5. \(\lambda \ast \mu \leq \lambda \lor \mu\) for every anti fuzzy quasi-ideal \(\lambda\) of \(R\) and for all anti fuzzy bi-ideals \(\mu\) of \(R\).
6. \(\lambda \ast \mu \leq \lambda \lor \mu\) for every anti fuzzy bi-ideals \(\lambda\) and \(\mu\) of \(R\).
Proof. (1) \Rightarrow (6) Let \( \lambda \) and \( \mu \) be any anti fuzzy bi-ideals of \( R \) and \( a \in R \). Then, since \( R \) is regular, there exists \( x \in R \) such that \( a = axa \). Since \( R \) is intra-regular, there exist \( y_i, z_i \in R \) such that \( a = \sum_{i=1}^{n} y_i a^2 z_i \). Thus \( a = axa = axa = \sum_{i=1}^{n} (axy_ia)(az_ixa) \). Hence

\[
(\lambda \ast \mu)(a) = \bigwedge_{a=\sum_{i=1}^{n} y_i a^2 z_i} \left[ \bigvee_{1 \leq i \leq p} [\lambda(y_i) \vee \mu(z_i)] \right] \\
\leq \bigvee_{1 \leq i \leq n} [\lambda(axy_ia) \vee \mu(az_ixa)] \\
\leq \bigvee_{1 \leq i \leq n} [\lambda(a) \vee \mu(a)] \\
= (\lambda \vee \mu)(a)
\]

and so \( \lambda \ast \mu \leq \lambda \vee \mu \).

It is clear that (6) \Rightarrow (5) \Rightarrow (4).

(4) \Rightarrow (2) Taking \( \lambda = \mu \) in (4), we get \( \lambda \ast \lambda \leq \lambda \vee \lambda = \lambda \). Since every anti fuzzy quasi-ideal of \( R \) is an anti fuzzy subsemiring of \( R \), by Theorem 3.14 we have \( \lambda \ast \lambda \geq \lambda \). Thus \( \lambda \ast \lambda = \lambda \).

(6) \Rightarrow (3) Taking \( \lambda = \mu \) in (6), we get \( \lambda \ast \lambda \leq \lambda \vee \lambda = \lambda \). Since every anti fuzzy quasi-ideal of \( R \) is an anti fuzzy subsemiring of \( R \), by Theorem 3.14 we have \( \lambda \ast \lambda \geq \lambda \). Thus \( \lambda \ast \lambda = \lambda \).

(3) \Rightarrow (2) Obvious.

(2) \Rightarrow (1) Suppose \( Q \) is a quasi-ideal of \( R \). Then \( C_{Q^c} \) is an anti fuzzy quasi-ideal of \( R \), thus by hypothesis \( C_{Q^c} \ast C_{Q^c} = C_{Q^c} \Rightarrow (QQ)^c = Q^c \Rightarrow QQ = Q \) and hence by Theorem 4.5 \( R \) is regular and intra regular.

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