Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 6, No. 1, (July 2013), pp. 77–94
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Soft metric

Sujoy Das, S. K. Samanta

Received 15 July 2012; Revised 1 October 2012; Accepted 10 October 2012

Abstract. Beginning with the concept of soft point in soft sets, in the present paper, a notion of soft metric is introduced and some basic properties of soft metric spaces are investigated. Some topological structures such as soft open sets, soft closed sets, soft closures of soft sets etc. have been defined and their properties are studied. Completeness of soft metric spaces is also investigated and Cantor's Intersection Theorem is established in soft metric space settings.

2010 AMS Classification: 03E72, 08A72

Keywords: Soft sets, soft points, Soft metric spaces, Soft open sets, Soft closed sets, Soft topology.

Corresponding Author: S. K. Samanta (syamal_123@yahoo.co.in)

1. Introduction

In the year 1999, Molodtsov [15] initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties. He has shown several applications of this theory in solving many practical problems in economics, engineering, social science, medical science, etc. Research works in soft set theory and its applications in various fields have been progressing rapidly since Maji et al. ([12],[13]) introduced several operations on soft sets and applied it to decision making problems. In the line of reduction and addition of parameters of soft sets some works have been done by Chen [4], Pei and Miao [16] , Kong et al. [11] , Zou and Xiao [19]. Aktas and Cagman [1] introduced the notion of soft group and discussed various properties. Jun ([9],[10]) investigated soft BCK/BCI – algebras and its application in ideal theory. Feng et al. [7] worked on soft semirings, soft ideals and idealistic soft semirings. Ali et al. [3] and Shabir and Irfan Ali ([3],[17]) studied soft semigroups and soft ideals over a semi group which characterize generalized fuzzy ideals and fuzzy ideals with thresholds of a semigroup. The idea of soft topological spaces was first given by M. Shabir, M. Naz [18] and mappings between soft sets were described by P. Majumdar, S. K. Samanta

[14]. Feng et al. [8] worked on soft sets combined with fuzzy sets and rough sets. A. Ayg*ü*no*ğ*lu and H. Ayg*ü*n [2] studied soft topological spaces and also considered the concept of soft point. Recently in ([5],[6]) we have introduced a notion of soft real sets, soft real numbers, soft complex sets, soft complex numbers and some of their basic properties have been investigated. Some applications of soft real sets and soft real numbers have been presented in real life problems.

In the present paper we introduce a notion of soft metric space which is based on 'soft point' of soft sets. In section 2, some preliminary results are given. In section 3, a notion of 'soft point' is given and some basic properties of soft points have been studied. In section 4, a definition of soft metric is given and some properties of soft metrics have been investigated in details with examples and counter examples. It has been shown that every 'soft metric space' is also a 'soft topological space' [18]. A notion of convergence of sequence of soft points is introduced in section 5. In that section, completeness of soft metric spaces is studied and Cantor's Intersection Theorem is established in soft metric space settings. Section 6 concludes the paper.

2. Preliminaries

Definition 2.1 ([15])**.** Let U be an universe and E be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of U and A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to \mathcal{P}(U)$. In other words, a soft set over U is a parametrized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε − approximate elements of the soft set (F, A).

Definition 2.2 ([8])**.** For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if

(1) $A \subseteq B$ and
(2) for all $e \in A$, $F(e) \subseteq G(e)$. We write $(F, A)\tilde{\subset}\ (G, B)$.

(F, A) is said to be a soft superset of (G, B), if (G, B) is a soft subset of (F, A). We denote it by$(F, A)\tilde{\supset}\ (G, B)$.

Definition 2.3 ([8])**.** Two soft sets (F, A) and (G, B) over a common universe U are said to be equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.4 ([8])**.** The complement of a soft set (F, A) is denoted by $(F, A)^c = (F^c, A)$, where $F^c : A \to \mathcal{P}(U)$ is a mapping given by $F^c(\alpha) = U - F(\alpha)$, for all $\alpha \in A$.

Definition 2.5 ([13])**.** A soft set (F, E) over U is said to be an *absolute* soft set denoted by $\check{U}$ if for all $\varepsilon \in E$, $\ F(\varepsilon) = U$.

Definition 2.6 ([13])**.** A soft set (F, E) over U is said to be a *null* soft set denoted by Φ if for all $\varepsilon \in E$, $F(\varepsilon) = \varnothing$.

Definition 2.7 ([13])**.** The union of two soft sets (F, A) and (G, B) over the common universe U is the soft set

(H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ G(e) & if\ e \in B - A \\ F(e) \cup G(e) & if\ e \in A \cap B. \end{cases}$$

We express it as $(F, A) \widetilde{\cup} (G, B) = (H, C)$.

The following definition of intersection of two soft sets is given as that of the bi-intersection in [7].

Definition 2.8 ([7])**.** The intersection of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$. We write $(F, A) \widetilde{\cap} (G, B) = (H, C)$.

Let X be an initial universal set and E be the non-empty set of parameters.

Definition 2.9 ([18])**.** The difference (H, E) of two soft sets (F, E) and (G, E) over X, denoted by $(F, E) \setminus (G, E)$, is defined by $H(e) = F(e) \setminus G(e)$ for all $e \in E$.

Proposition 2.10 ([18])**.** *Let (F, E) and (G, E) be two soft sets over X. Then*

(i). $((F, E) \widetilde{\cup} (G, E))^c = (F, E)^c \widetilde{\cap} (G, E)^c$

(ii). $((F, E) \widetilde{\cap} (G, E))^c = (F, E)^c \widetilde{\cup} (G, E)^c$.

Definition 2.11 ([5])**.** Let X *be* a non-empty set and E be a non-empty parameter set. Then a function
$\varepsilon : E \to X$ is said to be a soft element of X. A soft element ε of X is said to belongs to a soft set A of X, which is denoted by $\varepsilon \widetilde{\in} A$, if $\varepsilon(e) \in A(e)$, $\forall e \in E$. Thus for a soft set A of X with respect to the index set E, we have $A(e) = \{\varepsilon(e), \varepsilon \widetilde{\in} A\}$, $e \in E$.

It is to be noted that every singleton soft set (a soft set (F, E) for which $F(e)$ is a singleton set, $\forall e \in E$) can be identified with a soft element by simply identifying the singleton set with the element that it contains $\forall e \in E$.

Definition 2.12 ([5])**.** Let R be the set of real numbers and $\mathfrak{B}(R)$ the collection of all non-empty bounded subsets of R and A taken as a set of parameters. Then a mapping $F : A \to \mathfrak{B}(R)$ is called a *soft real set*. It is denoted by (F, A). If specifically (F, A) is a singleton soft set, then after identifying (F, A) with the corresponding soft element, it will be called a *soft real number*.

We use notations $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ to denote soft real numbers whereas $\overline{r}$, $\overline{s}$, $\overline{t}$ will denote a particular type of soft real numbers such that $\overline{r}(\lambda) = r$, for all $\lambda \in A$ etc. For example $\overline{0}$ is the soft real number where $\overline{0}(\lambda) = 0$, for all $\lambda \in A$.

3. Soft point in soft sets

Let X be an initial universal set and A be the non-empty set of parameters. In this section proofs of some straight forward propositions are omitted. We define soft point as the following:

Definition 3.1. A soft set (P, A) over X is said to be a *soft point* if there is exactly one $\lambda \in A$, such that $P(\lambda) = \{x\}$ for some $x \in X$ and $P(\mu) = \varnothing$, $\forall\ \mu \in A \setminus \{\lambda\}$. It will be denoted by P_λ^x.

Example 3.2. Let $X = \{x, y, z\}$, $A = \{\lambda, \mu\}$, then P_μ^z is a soft point where $P(\mu) = \{z\}$ and $P(\lambda) = \varnothing$.

Definition 3.3. A soft point P_λ^x is said to belongs to a soft set (F, A) if $\lambda \in A$ and $P(\lambda) = \{x\} \subset F(\lambda)$.
We write $P_\lambda^x \tilde{\in} (F, A)$.

Definition 3.4. Two soft points P_λ^x, P_μ^y are said to be equal if $\lambda = \mu$ and $P(\lambda) = P(\mu)$ i.e., $x = y$. Thus $P_\lambda^x \neq P_\mu^y \Leftrightarrow x \neq y$ or $\lambda \neq \mu$.

Proposition 3.5. *The union of any collection of soft points can be considered as a soft set and every soft set can be expressed as union of all soft points belonging to it; i.e.,* $(F, A) = \bigcup_{P_\lambda^x \tilde{\in} (F,A)} P_\lambda^x$.

Proposition 3.6. *For two soft sets* (F, A) *and* (G, A), $(F, A) \tilde{\subset} (G, A) \Leftrightarrow P_\lambda^x \tilde{\in} (F, A) \Rightarrow P_\lambda^x \tilde{\in} (G, A)$ *and hence* $(F, A) = (G, A)$ *if and only if* $P_\lambda^x \tilde{\in} (F, A) \Leftrightarrow P_\lambda^x \tilde{\in} (G, A)$.

Proposition 3.7. *For a soft point* P_λ^x,

(i) $P_\lambda^x \tilde{\in} (F, A) \Leftrightarrow P_\lambda^x \tilde{\notin} (F, A)^c$;
(ii) $P_\lambda^x \tilde{\in} (F, A) \tilde{\cup} (G, A) \Leftrightarrow P_\lambda^x \tilde{\in} (F, A)$ *or* $P_\lambda^x \tilde{\in} (G, A)$; *and*
(iii) $P_\lambda^x \tilde{\in} (F, A) \tilde{\cap} (G, A) \Leftrightarrow P_\lambda^x \tilde{\in} (F, A)$ *and* $P_\lambda^x \tilde{\in} (G, A)$.

Remark 3.8. If $\mathfrak{B}$ be a collection of soft points then the soft set generated by taking all the soft points of $\mathfrak{B}$ will be denoted by $SS(\mathfrak{B})$; whereas the collection of all soft points of a soft set (F, A) will be expressed by $SP(F, A)$.

Proposition 3.9. *For the collections* $\mathfrak{B}$, $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *of soft points, and for the soft sets* (F, A), (G, A);

(i) $SP(SS(\mathfrak{B})) = \mathfrak{B}$; $SS(SP(F, A)) = (F, A)$;
(ii) $SP\left((F, A) \tilde{\cup} (G, A)\right) = SP(F, A) \cup SP(G, A)$ *and* $SP\left((F, A) \tilde{\cap} (G, A)\right) = SP(F, A) \cap SP(G, A)$;
(iii) $SS(\mathfrak{B}_1 \cup \mathfrak{B}_2) = SS(\mathfrak{B}_1) \tilde{\cup} SS(\mathfrak{B}_2)$ *and* $SS(\mathfrak{B}_1 \cap \mathfrak{B}_2) = SS(\mathfrak{B}_1) \tilde{\cap} SS(\mathfrak{B}_2)$.

Before the introduction of soft metric we define the inequalities of soft real numbers as the following;

Definition 3.10. For two soft real numbers $\tilde{r}, \tilde{s}$ we define

(i) $\tilde{r} \tilde{\leq} \tilde{s}$ if $\tilde{r}(\lambda) \leq \tilde{s}(\lambda)$, for all $\lambda \in A$;
(ii) $\tilde{r} \tilde{\geq} \tilde{s}$ if $\tilde{r}(\lambda) \geq \tilde{s}(\lambda)$, for all $\lambda \in A$;
(iii) $\tilde{r} \tilde{<} \tilde{s}$ if $\tilde{r}(\lambda) < \tilde{s}(\lambda)$, for all $\lambda \in A$;
(iv) $\tilde{r} \tilde{>} \tilde{s}$ if $\tilde{r}(\lambda) > \tilde{s}(\lambda)$, for all $\lambda \in A$.

4. Soft metric and soft metric spaces

Let X be an initial universal set and A be the non-empty set of parameters. Let $\check{X}$ be the absolute soft set i.e., $F(\lambda) = X$, $\forall \lambda \in A$, where $(F, A) = \check{X}$. Let $SP(\check{X})$ be the collection of all soft points of $\check{X}$. Let $\mathbb{R}(A)^*$ denote the set of all non-negative soft real numbers. We define soft metric using soft points as the following;

4.1. **Definitions and examples of soft metric and soft metric spaces.**

Definition 4.1. A mapping $d : SP(\check{X}) \times SP(\check{X}) \to \mathbb{R}(A)^*$, is said to be a *soft metric* on the soft set $\check{X}$ if d satisfies the following conditions:
(M1). $d\left(P_\lambda^x, P_\mu^y\right) \tilde{\geq} \overline{0}$, for all $P_\lambda^x, P_\mu^y \tilde{\in} \check{X}$.
(M2). $d\left(P_\lambda^x, P_\mu^y\right) = \overline{0}$ if and only if $P_\lambda^x = P_\mu^y$.
(M3). $d\left(P_\lambda^x, P_\mu^y\right) = d\left(P_\mu^y, P_\lambda^x\right)$ for all $P_\lambda^x, P_\mu^y \tilde{\in} \check{X}$.
(M4). For all $P_\lambda^x, P_\mu^y, P_\gamma^z \tilde{\in} \check{X}$, $d\left(P_\lambda^x, P_\gamma^z\right) \tilde{\leq} d\left(P_\lambda^x, P_\mu^y\right) + d\left(P_\mu^y, P_\gamma^z\right)$.

The soft set $\check{X}$ with a soft metric d on $\check{X}$ is called a *soft metric space* and denoted by $(\check{X}, d, A)$ or $(\check{X}, d)$. The conditions (M1), (M2), (M3) and (M4) are said to be soft metric axioms.

Example 4.2. Let X be a non-empty set and A be the non empty set of parameters. Let $\check{X}$ be the absolute soft set i.e., $F(\lambda) = X$, $\forall \lambda \in A$, where $(F, A) = \check{X}$. We define $d : SP(\check{X}) \times SP(\check{X}) \to \mathbb{R}(A)^*$ by,
$d\left(P_\lambda^x, P_\mu^y\right) = \overline{0}$, if $P_\lambda^x = P_\mu^y$ and $d\left(P_\lambda^x, P_\mu^y\right) = \overline{1}$, if $P_\lambda^x \neq P_\mu^y$, for all $P_\lambda^x, P_\mu^y \tilde{\in} \check{X}$. Normally d satisfies all the soft metric axioms. So, d is a soft metric on the soft set $\check{X}$. d is called the discrete soft metric on the soft set $\check{X}$ and $(\check{X}, d)$ is said to be the discrete soft metric space.

Example 4.3. Let $X \subset$ R be a non-empty set and $A \subset$ R be the non empty set of parameters. Let $\check{X}$ be the absolute soft set i.e., $F(\lambda) = X$, $\forall \lambda \in A$, where $(F, A) = \check{X}$. Let $\overline{x}$ denote the soft real number such that $\overline{x}(\lambda) = x$, $\forall \lambda \in A$. We define $d : SP(\check{X}) \times SP(\check{X}) \to \mathbb{R}(A)^*$ by,
$d\left(P_\lambda^x, P_\mu^y\right) = |\overline{x} - \overline{y}| + |\overline{\lambda} - \overline{\mu}|$, for all $P_\lambda^x, P_\mu^y \tilde{\in} \check{X}$; ; where '$|.|'$ denotes the modulus of soft real numbers. Then d is a soft metric on $\check{X}$. Let us verify (M1), (M2), (M3) and (M4) for soft metric.
(M1). It is obvious from the above definition that $d\left(P_\lambda^x, P_\mu^y\right) \tilde{\geq} \overline{0}$, for all $P_\lambda^x, P_\mu^y \tilde{\in} \check{X}$.
(M2). We have, $d\left(P_\lambda^x, P_\mu^y\right) = \overline{0} \Leftrightarrow |\overline{x} - \overline{y}| + |\overline{\lambda} - \overline{\mu}| = \overline{0} \Leftrightarrow |\overline{x} - \overline{y}| = \overline{0}$ and $|\overline{\lambda} - \overline{\mu}|$ $= \overline{0} \Leftrightarrow x = y$ and $\lambda = \mu \Leftrightarrow P_\lambda^x = P_\mu^y$.
(M3). We have, $d\left(P_\lambda^x, P_\mu^y\right) = |\overline{x} - \overline{y}| + \left|\overline{\lambda} - \overline{\mu}\right| = |\overline{y} - \overline{x}| + \left|\overline{\mu} - \overline{\lambda}\right| = d\left(P_\mu^y, P_\lambda^x\right)$ for all $P_\lambda^x, P_\mu^y \tilde{\in} \check{X}$.
(M4). We have for all $P_\lambda^x, P_\mu^y, P_\gamma^z \tilde{\in} \check{X}$,
$\left(P_\lambda^x, P_\gamma^z\right) = |\overline{x} - \overline{z}| + \left|\overline{\lambda} - \overline{\gamma}\right| = |\overline{x} - \overline{y} + \overline{y} - \overline{z}| + \left|\overline{\lambda} - \overline{\mu} + \overline{\mu} - \overline{\gamma}\right|$
$\tilde{\leq} \left(|\overline{x} - \overline{y}| + \left|\overline{\lambda} - \overline{\mu}\right|\right) + \left(|\overline{y} - \overline{z}| + |\overline{\mu} - \overline{\gamma}|\right) \tilde{\leq} d\left(P_\lambda^x, P_\mu^y\right) + d\left(P_\mu^y, P_\gamma^z\right)$.
Thus, d is a soft metric on $\check{X}$.

Definition 4.4. Let $(\check{X}, d)$ be a soft metric space. If there exists a positive soft real number $\tilde{k}$ such that $d\left(P_\lambda^x, P_\mu^y\right) \tilde{\leq} \tilde{k}$, $\forall P_\lambda^x, P_\mu^y \tilde{\in} \check{X}$, then $(\check{X}, d)$ is called as a bounded soft metric space. Otherwise we consider it unbounded.

Definition 4.5. A mapping $d : SP(\check{X}) \times SP(\check{X}) \to \mathbb{R}(A)^*$, is said to be a *soft pseudo-metric* on the soft set $\check{X}$ if d satisfies the following conditions:
(ρ1). $d\left(P_\lambda^x, P_\mu^y\right) \tilde{\geq} \overline{0}$, for all $P_\lambda^x, P_\mu^y \tilde{\in} \check{X}$.
(ρ2). $d\left(P_\lambda^x, P_\mu^y\right) = \overline{0}$ if $P_\lambda^x = P_\mu^y$.
(ρ3). $d\left(P_\lambda^x, P_\mu^y\right) = d\left(P_\mu^y, P_\lambda^x\right)$ for all $P_\lambda^x, P_\mu^y \tilde{\in} \check{X}$.

(ρ4). For all $P_\lambda^x, P_\mu^y, P_\gamma^z \tilde{\in} \check{X}$, $d\left(P_\lambda^x, P_\gamma^z\right) \tilde{\leq} d\left(P_\lambda^x, P_\mu^y\right) + d\left(P_\mu^y, P_\gamma^z\right)$
The soft set $\check{X}$ with a soft pseudo-metric d on $\check{X}$ is said to be a *soft pseudo-metric space* and is denoted by $(\check{X}, d, A)$ or $(\check{X}, d)$.

Example 4.6. Let X be a crisp set containing at least two elements. Let $\check{X}$ be the absolute soft set over the non-empty set of parameters A. Let $d : SP(\check{X}) \times SP(\check{X}) \to \mathbb{R}(A)^*$ be defined by $d\left(P_\lambda^x, P_\mu^y\right) = \overline{0}$, for all $P_\lambda^x, P_\mu^y \tilde{\in} \check{X}$. Then it can be easily verified that d is a soft pseudo-metric on $\check{X}$ and not a soft metric on $\check{X}$.

4.2. **Subspace, diameter and distance of soft sets.**

Definition 4.7. Let $(\check{X}, d)$ be a soft metric space and (Y, A) be a non-null soft subset of $\check{X}$. Then the mapping $d_Y : SP(Y, A) \times SP(Y, A) \to \mathbb{R}(A)^*$ given by $d_Y\left(P_\lambda^x, P_\mu^y\right) = d\left(P_\lambda^x, P_\mu^y\right)$ for all $P_\lambda^x, P_\mu^y \tilde{\in} (Y, A)$ is a soft metric on (Y, A). This metric d_Y is known as the relative metric induced on (Y, A) by d. The soft metric space (Y, d_Y, A) is called a metric subspace or simply a subspace of the soft metric space $(\check{X}, d, A)$.

Definition 4.8. Let $(\check{X}, d)$ be a soft metric space and (Y, A) be a non-null soft subset of $\check{X}$. Then the diameter of (Y, A) is denoted by $\delta((Y, A))$ and is defined by $\delta\left((Y, A)\right)(\gamma) = sup\left\{\, d\left(P_\lambda^x, P_\mu^y\right)(\gamma)\,;\; P_\lambda^x, P_\mu^y \tilde{\in} (Y, A)\right\}$, $\forall \gamma \in A$.

In case, the supremum does not exists finitely for any $\lambda \in A$, we consider that (Y, A) is a soft set of infinite diameter.

It is obvious that for any non-null soft set (Y, A) of $\check{X}$, $\delta((Y, A)) \tilde{\geq} \overline{0}$.

Theorem 4.9. *Let $(\check{X}, d)$ be a soft metric space. Then*
(i) $\delta((Y, A))=\overline{0}$ *if and only if* (Y, A) *consists of a single soft element,*
(ii) *For every soft subsets* $(Y, A), (Z, A)$ *of* $\check{X}$, $(Y, A)\tilde{\subset}(Z, A)$
$\Longrightarrow \delta\left((Y, A)\right) \tilde{\leq} \delta((Z, A))$
(iii) *For every soft subsets* $(Y, A), (Z, A)$ *of* $\check{X}$, *with* $(Y, A)\tilde{\cap}(Z, A) \neq \Phi$,
$\delta((Y, A)\tilde{\cup}(Z, A)) \tilde{\leq} \delta\left((Y, A)\right) + \delta((Z, A))$.

Proof. We prove only (iii). In case $(Y, A)\tilde{\subset}(Z, A)$, then $(Y, A)\tilde{\cup}(Z, A) = (Z, A)$, then $\delta\left((Y, A)\tilde{\cup}(Z, A)\right) = \delta\left((Z, A)\right)$, so that $\delta\left((Y, A)\tilde{\cup}(Z, A)\right) \tilde{\leq} \delta\left((Y, A)\right) + \delta((Z, A))$, since $\delta\left((Y, A)\right) \tilde{\geq} \overline{0}$. Similarly when $(Z, A)\tilde{\subset} (Y, A)$, this proposition can be proved.
Next let us suppose that neither $(Y, A)\tilde{\subset}(Z, A)$ nor $(Z, A)\tilde{\subset} (Y, A)$.
Let $P_\lambda^x, P_\mu^y \tilde{\in}(Y, A)\tilde{\cup}(Z, A)$, then $P_\lambda^x, P_\mu^y \tilde{\in}(Y, A)$ or $P_\lambda^x, P_\mu^y \tilde{\in}(Z, A)$. We consider the following cases:
Case-I. Suppose P_λ^x, P_μ^y belongs to any one of the soft sets (Y, A) and (Z, A).
If $P_\lambda^x, P_\mu^y \tilde{\in}(Y, A)$, then $d\left(P_\lambda^x, P_\mu^y\right) \tilde{\leq} \delta\left((Y, A)\right)$; if $P_\lambda^x, P_\mu^y \tilde{\in}(Z, A)$, then
$d\left(P_\lambda^x, P_\mu^y\right) \tilde{\leq} \delta\left((Z, A)\right)$.
In either case $d\left(P_\lambda^x, P_\mu^y\right) \tilde{\leq} \delta\left((Y, A)\right) + \delta((Z, A))$.
Case-II. Next we consider the case when any one of the soft points belongs to (Y, A) and another to the soft set (Z, A). Without the loss of generality we can assume that $P_\lambda^x \tilde{\in}(Y, A)$, $P_\mu^y \tilde{\in}(Z, A)$.
Since $(Y, A)\tilde{\cap}(Z, A) \neq \Phi$, $\exists\, P_\gamma^z \tilde{\in}(Y, A)\tilde{\cap}(Z, A)$.

Then by triangle inequality of d, $d\left(P^x_\lambda, P^y_\mu\right) \widetilde{\leq} d\left(P^x_\lambda, P^z_\gamma\right) + d\left(P^z_\gamma, P^y_\mu\right)$
So, $d\left(P^x_\lambda, P^y_\mu\right)(\gamma) \leq d\left(P^x_\lambda, P^z_\gamma\right)(\gamma) + d\left(P^z_\gamma, P^y_\mu\right)(\gamma)$
$\leq sup\ \left\{d\left(P^x_\lambda, P^z_\gamma\right)(\gamma) ; P^x_\lambda, P^z_\gamma\ \widetilde{\in}(Y,A)\right\} + sup\ \left\{d\left(P^z_\gamma, P^y_\mu\right)(\gamma) ; P^z_\gamma, P^y_\mu\ \widetilde{\in}(Z,A)\right\}$
$=\ \delta\left((Y,A)\right)(\gamma) + \delta\left((Z,A)\right)(\gamma) = \left[\delta\left((Y,A)\right) + \delta((Z,A))\right](\gamma),\ \forall\gamma\in A.$
Thus $d\left(P^x_\lambda, P^y_\mu\right) \widetilde{\leq} \delta\left((Y,A)\right) + \delta((Z,A))$.
Thus we find for $P^x_\lambda, P^y_\mu\ \widetilde{\in}(Y,A)\widetilde{\cup}(Z,A)$, $d\left(P^x_\lambda, P^y_\mu\right) \widetilde{\leq} \delta\left((Y,A)\right) + \delta((Z,A))$.
So, $\sup\left\{d\left(P^x_\lambda, P^y_\mu\right)(\gamma) ; P^x_\lambda, P^y_\mu\ \widetilde{\in}(Y,A)\widetilde{\cup}(Z,A)\right\} \leq \left[\delta\left((Y,A)\right) + \delta((Z,A))\right](\gamma),\ \forall\gamma\in A$,
i.e., $\delta\left((Y,A)\widetilde{\cup}(Z,A)\right)(\gamma) \leq \left[\delta\left((Y,A)\right) + \delta((Z,A))\right](\gamma),\ \forall\gamma\in A$,
i.e., $\delta\left((Y,A)\widetilde{\cup}(Z,A)\right) \widetilde{\leq} \delta\left((Y,A)\right) + \delta((Z,A))$. □

Definition 4.10. Let $(\check{X}, d)$ be a soft metric space. Let P^a_e be a fixed soft point of $\check{X}$ and (S,A) be a non-null soft subset of $\check{X}$. Then the distance of the soft point P^a_e from the soft set (S,A) is denoted by $\delta\left(P^a_e, (S,A)\right)$ and defined by $\delta\left(P^a_e, (S,A)\right)(\gamma) = inf\ \left\{\ d\left(P^a_e, P^x_\lambda\right)(\gamma) ; P^x_\lambda\ \widetilde{\in}(S,A)\right\},\ \forall\gamma\in A$.

In case P^a_e be a soft point of (S,A), we get,
$\delta\left(P^a_e, (S,A)\right)(\gamma) = inf\ \left\{\ d\left(P^a_e, P^x_\lambda\right)(\gamma) ; P^x_\lambda\ \widetilde{\in}(S,A)\right\},\ \forall\gamma\in A.$
$= d\left(P^a_e, P^a_e\right)(\gamma) = 0 = \overline{0}(\gamma),\ \forall\gamma\in A. \therefore\ \delta\left(P^a_e, (S,A)\right) = \overline{0}.$

Definition 4.11. Let $(\check{X}, d)$ be a soft metric space and $(Y,A), (Z,A)$ be two non-null soft subsets of $\check{X}$. The distance between the soft sets $(Y,A), (Z,A)$ is denoted by $\delta\left((Y,A), (Z,A)\right)$ and is defined by
$\delta\left((Y,A), (Z,A)\right)(e) = inf\left\{\ d\left(P^x_\lambda, P^y_\mu\right)(e) ; P^x_\lambda\ \widetilde{\in}\ (Y,A), P^y_\mu\ \widetilde{\in}(Z,A)\right\},\ \forall e\in A.$

For soft subsets $(Y,A), (Z,A)$, we have
$\delta\left((Y,A), (Z,A)\right)(e) = \inf\ \left\{\ d\left(P^x_\lambda, P^y_\mu\right)(e) ; P^x_\lambda\ \widetilde{\in}\ (Y,A), P^y_\mu\ \widetilde{\in}(Z,A)\right\},\ \forall e\in A.$
$= \inf\ \left\{\ d\left(P^y_\mu, P^x_\lambda\right)(e) ; P^x_\lambda\ \widetilde{\in}\ (Y,A), P^y_\mu\ \widetilde{\in}(Z,A)\right\}$, $\forall e\in A$, since d is symmetric.
$= \delta\left((Z,A), (Y,A)\right)(e), \forall e\in A. \therefore\ \delta\left((Y,A), (Z,A)\right) = \delta\left((Z,A), (Y,A)\right).$
Let $\left((Y,A)\widetilde{\cap}(Z,A)\right) \neq \Phi$, then $\exists\ P^z_\gamma \widetilde{\in} \left((Y,A)\widetilde{\cap}(Z,A)\right)$, and then
$\delta\left((Y,A), (Z,A)\right)(e) = \inf\ \left\{d\left(P^x_\lambda, P^y_\mu\right)(e) ; P^x_\lambda\ \widetilde{\in}\ (Y,A), P^y_\mu\ \widetilde{\in}(Z,A)\right\}$
$= d\left(P^z_\gamma, P^z_\gamma\right)(e) = 0,\ \forall e\in A. \therefore\ \delta\left((Y,A), (Z,A)\right) = \overline{0}.$
But the converse is not necessarily true. It may so happen that $\delta\left((Y,A), (Z,A)\right) = \overline{0}$, but $\left((Y,A)\widetilde{\cap}(Z,A)\right) = \Phi$, it is shown by the following example.

Example 4.12. Consider the soft metric space $\left(\check{X}, d\right)$ or $(\check{X}, d, A)$ as in Example 4.3
Let $(Y,A), (Z,A)$ be two non-null soft subsets of $\check{X}$, in such way that $Y(\lambda) = (u,v)$, $Z(\lambda) = (v,w)$ are open intervals in real line and $Y(e) = \varnothing,\ Z(e) = \varnothing, \forall e\in A\backslash\{\lambda\}$.
Then $\delta\left((Y,A), (Z,A)\right)(e) = \inf\ \left\{\ \left(|\overline{x}-\overline{y}| + \left|\overline{\lambda}-\overline{\lambda}\right|\right)(e) ; P^x_\lambda\ \widetilde{\in}\ (Y,A), P^y_\lambda\ \widetilde{\in}(Z,A)\right\}$
$= \inf\left\{|\overline{x}-\overline{y}|(e) ; u<x<v, v<y<w\right\} = 0 = \overline{0}(\lambda),\ \forall e\in A.$
$\therefore\ \delta\left((Y,A), (Z,A)\right) = \overline{0}$. But $\left((Y,A)\widetilde{\cap}(Z,A)\right) = \Phi$.

4.3. **Soft open and soft closed balls.**

Definition 4.13. Let $(\check{X}, d)$ be a soft metric space and $\tilde{r}$ be a non-negative soft real number. For any $P^a_e \widetilde{\in} \check{X}$, by an *open ball* with centre P^a_e and radius $\tilde{r}$, we mean the collection of soft points of $\check{X}$ satisfying $d(P^x_\lambda, P^a_e) \widetilde{<} \tilde{r}$.
The open ball with centre P^a_e and radius $\tilde{r}$ is denoted by $B(P^a_e, \tilde{r})$.

Thus $B\left(P_e^a, \tilde{r}\right) = \{P_\lambda^x \tilde{\in} \check{X}; d(P_\lambda^x, P_e^a) \tilde{<} \tilde{r}\} \subset SP(\check{X})$.
$SS(B\left(P_e^a, \tilde{r}\right))$ will be called a *soft open ball* with centre P_e^a and radius $\tilde{r}$.

Definition 4.14. Let $(\check{X}, d)$ be a soft metric space and $\tilde{r}$ be a non-negative soft real number. For any $P_e^a \tilde{\in} \check{X}$, by a *closed ball* with centre P_e^a and radius $\tilde{r}$, we mean the collection of soft points of $\check{X}$ satisfying $d(P_\lambda^x, P_e^a) \tilde{\leq} \tilde{r}$.
The closed ball with centre P_e^a and radius $\tilde{r}$ is denoted by $B[P_e^a, \tilde{r}]$.
Thus $B\left[P_e^a, \tilde{r}\right] = \left\{P_\lambda^x \tilde{\in} \check{X}; d\left(P_\lambda^x, P_e^a\right) \tilde{\leq} \tilde{r}\right\} \subset SP(\check{X})$.
$SS(B\left[P_e^a, \tilde{r}\right])$ will be called a *soft closed ball* with centre P_e^a and radius $\tilde{r}$.

Example 4.15. Consider the discrete soft metric space $\left(\check{X}, d\right)$ as in Example 4.3 Then for any $P_e^a \tilde{\in} \check{X}$, $B\left(P_e^a, \tilde{r}\right) = SP(\check{X})$, if $\overline{1} \tilde{<} \tilde{r}$ and $B\left(P_e^a, \tilde{r}\right) = \{P_e^a\}$, if $\tilde{r} \tilde{\leq} \overline{1}$. Also, $B\left[P_e^a, \tilde{r}\right] = SP(\check{X})$, if $\overline{1} \tilde{\leq} \tilde{r}$ and $B\left[P_e^a, \tilde{r}\right] = \{P_e^a\}$, if $\tilde{r} \tilde{<} \overline{1}$.

Definition 4.16. Let $(\check{X}, d)$ be a soft metric space having at least two soft points. Then $(\check{X}, d)$ is said to possess Hausdorff property, If P_e^a, P_f^b are two soft points in $\check{X}$ in the way that $d\left(P_e^a,\ P_f^b\right) \tilde{>} \overline{0}$, then there are two soft open balls $SS(B\left(P_e^a, \tilde{r}\right))$ and $SS(B\left(P_f^b, \tilde{r}\right))$ with centre P_e^a and P_f^b respectively and radius $\tilde{r} \tilde{>} \overline{0}$ such that $SS(B\left(P_e^a, \tilde{r}\right)) \tilde{\cap} SS(B\left(P_f^b, \tilde{r}\right)) = \Phi$.

Theorem 4.17. *Every soft metric space is Hausdorff.*

Proof. Let $(\check{X}, d)$ be a soft metric space having at least two soft points. Let P_e^a, P_f^b be two soft points in $\check{X}$ in the manner that $d\left(P_e^a,\ P_f^b\right) \tilde{>} \overline{0}$. Let us consider any soft real number $\tilde{r}$ satisfying $\overline{0} \tilde{<} \tilde{r} \tilde{<} \frac{1}{2}\, d\left(P_e^a,\ P_f^b\right)$.
Then $\tilde{r} \tilde{\in} \mathbb{\check{R}}$ and the soft open balls of radius $\tilde{r}$ with centres P_e^a, P_f^b are $SS(B\left(P_e^a, \tilde{r}\right)) = SS(\left\{P_\lambda^x \tilde{\in} \check{X}; d\left(P_\lambda^x, P_e^a\right) \tilde{<} \tilde{r}\right\}), SS(B\left(P_f^b, \tilde{r}\right)) = SS(\left\{P_\lambda^x \tilde{\in} \check{X}; d\left(P_\lambda^x, P_f^b\right) \tilde{<} \tilde{r}\right\})$.
We now prove that these two soft open balls have void intersection.
If not, then there is some $P_g^z \tilde{\in} SS(B\left(P_e^a, \tilde{r}\right)) \tilde{\cap} SS(B\left(P_f^b, \tilde{r}\right))$.
Now,

$$P_g^z \tilde{\in} SS(B\left(P_e^a, \tilde{r}\right)) \Longrightarrow d(P_e^a, P_g^z)) \tilde{<} \tilde{r}$$

and

$$P_g^z \tilde{\in} SS(B\left(P_f^b, \tilde{r}\right)) \Longrightarrow d(P_f^b, P_g^z)) \tilde{<} \tilde{r}.$$

By (M4), $d\left(P_e^a,\ P_f^b\right) \tilde{\leq} d\left(P_e^a, P_g^z\right) + d(P_f^b, P_g^z) \tilde{<} \tilde{r} + \tilde{r} = 2\tilde{r} \Longrightarrow \tilde{r} \tilde{>} \frac{1}{2}\, d\left(P_e^a,\ P_f^b\right)$.
This contradicts the hypothesis, So we must have $SS(B\left(P_e^a, \tilde{r}\right)) \tilde{\cap} SS(B\left(P_f^b, \tilde{r}\right)) = \Phi$. □

Definition 4.18. Let $(\check{X}, d)$ be a soft metric space and $P_e^a \tilde{\in} \check{X}$. A collection $N(P_e^a)$ of soft points containing the soft point P_e^a is said to be *neighbourhood* of the soft point P_e^a, if there exists a positive soft real number $\tilde{r}$ such that $P_e^a \in B\left(P_e^a, \tilde{r}\right) \subset N(P_e^a)$.
$SS(N(P_e^a))$ will be called a *soft neighbourhood* of the soft point P_e^a.

Theorem 4.19. *Let* $(\check{X}, d)$ *be a soft metric space and* $P_e^a \tilde{\in} \check{X}$. *Let* N_1 *and* N_2 *be neighbourhoods of* P_e^a *in* $(\check{X}, d)$. *Then* $SS(N_1) \tilde{\cap} SS(N_2)$ *is a soft neighbourhood of* P_e^a *in* $(\check{X}, d)$.

Proof. The proof is straight forward. □

Theorem 4.20. *Every soft open ball is a soft neighbourhood of each of its soft points.*

Proof. Let $(\check{X}, d)$ be a soft metric space. Let us consider any soft open ball $SS(B(P_e^a, \tilde{r}))$ with centre P_e^a and radius $\tilde{r}$. From the definition of soft neighbourhood it follows that $SS(B(P_e^a, \tilde{r}))$ is a soft neighbourhood of P_e^a in $(\check{X}, d)$.
Let us consider any soft point $P_\lambda^x \tilde{\in} SS(B(P_e^a, \tilde{r}))$, other than P_e^a.
Then we must have, $\overline{0} \neq d(P_\lambda^x, P_e^a) \tilde{<} \tilde{r}$. Choose $\tilde{r}^/$ with $\overline{0} \tilde{<} \tilde{r}^/ \tilde{<} \tilde{r} - d(P_\lambda^x, P_e^a)$.
Then $\tilde{r}^/$is a positive soft real number. Now for any $P_\mu^y \tilde{\in} SS(B(P_\lambda^x, \tilde{r}^/))$, we have $d(P_\lambda^x, P_\mu^y) \tilde{<} \tilde{r}^/$.
Now by M4, we get, $d(P_e^a, P_\mu^y) \tilde{\leq} d(P_e^a, P_\lambda^x) + d(P_\lambda^x, P_\mu^y) \tilde{<} d(P_e^a, P_\lambda^x) + \tilde{r}^/ \tilde{<} \tilde{r}$
$\therefore P_\mu^y \tilde{\in} SS(B(P_e^a, \tilde{r}))$ $\therefore P_\lambda^x \tilde{\in} SS(B(P_\lambda^x), \tilde{r}^/)) \tilde{\subset} SS(B(P_e^a, \tilde{r}))$.
$\Longrightarrow SS(B(P_e^a, \tilde{r}))$ is a soft neighbourhood of each of its soft points. □

Definition 4.21. Let (Y, A) be a soft subset in a soft metric space $(\check{X}, d)$. Then a soft point P_e^a is said to be a *interior point* of the soft set (Y, A) if $\exists$ a positive soft real number $\tilde{r}$ such that $P_e^a \in B(P_e^a, \tilde{r}) \subset SP(Y, A)$.

Definition 4.22. Let (Y, A) be a soft subset in a soft metric space $(\check{X}, d)$. Then the *interior* of the soft set (Y, A) is defined to be the set consisting of all interior points of (Y, A).

The interior of the soft set (Y, A) is denoted by $(Y, A)^o$.
Thus $(Y, A)^o = \{P_\lambda^x \tilde{\in} (Y, A); P_\lambda^x \in B(P_\lambda^x, \tilde{r}) \subset SP(Y, A)$, for some positive soft real number $\tilde{r}\}$.
$SS((Y, A)^o)$ is said to be the *soft interior* of (Y, A).

Theorem 4.23. *Let* (Y, A), (Z, A) *be two non-null soft subsets in a soft metric space* $(\check{X}, d)$. *Then*

(i) $SS((Y, A)^o) \tilde{\subset} (Y, A)$,
(ii) $(Y, A) \tilde{\subset} (Z, A) \Longrightarrow SS((Y, A)^o) \tilde{\subset} SS((Z, A)^o)$,
(iii) $SS((Y, A)^o) \tilde{\cap} SS((Z, A)^o) = SS(((Y, A) \tilde{\cap} (Z, A))^o)$,
(iv) $SS(((Y, A) \tilde{\cup} (Z, A))^o) \tilde{\supset} SS((Y, A)^o) \tilde{\cup} SS((Z, A)^o)$.

Proof. We prove only (iii). Let $P_\lambda^x \tilde{\in} SS(((Y, A) \tilde{\cap} (Z, A))^o)$, $\exists$ a positive soft real number $\tilde{r}$ such that
$P_\lambda^x \in B(P_\lambda^x, \tilde{r}) \subset SP((Y, A) \tilde{\cap} (Z, A)) = SP(Y, A) \cap SP(Z, A)$.
$\Longrightarrow P_\lambda^x \in B(P_\lambda^x, \tilde{r}) \subset SP(Y, A)$ and $P_\lambda^x \in B(P_\lambda^x, \tilde{r}) \subset SP(Z, A)$
$\Longrightarrow P_\lambda^x \tilde{\in} SS((Y, A)^o)$ and $P_\lambda^x \tilde{\in} SS((Z, A)^o) \Longrightarrow P_\lambda^x \tilde{\in} SS((Y, A)^o) \tilde{\cap} SS((Z, A)^o)$.

$$\therefore SS((Y, A)^o) \tilde{\cap} SS((Z, A)^o) \tilde{\supset} SS(((Y, A) \tilde{\cap} (Z, A))^o) \tag{4.1}$$

Conversely, let $P_\mu^y \tilde{\in} SS((Y, A)^o) \tilde{\cap} SS((Z, A)^o)$

$\Longrightarrow P_{\mu}^{y} \tilde{\in} SS((Y,A)^{o})$ and $P_{\mu}^{y} \tilde{\in} SS((Z,A)^{o})$
$\exists$ a positive soft real numbers $\tilde{r}_1,\ \tilde{r}_2$ such that $P_{\mu}^{y} \in B\left(P_{\mu}^{y},\tilde{r}_1\right) \subset SP(Y,A)$ and $P_{\mu}^{y} \in B\left(P_{\mu}^{y},\tilde{r}_2\right) \subset SP(Z,A)$
Let $\tilde{r}(\lambda) = \min\{\tilde{r}_1(\lambda),\ \tilde{r}_2(\lambda)\},\ \forall\lambda \in A$. Then $\tilde{r}$ is a positive soft real number.
Now $P_{\mu}^{y} \in B\left(P_{\mu}^{y},\tilde{r}\right) \subset SP(Y,A)$ and $P_{\mu}^{y} \in B\left(P_{\mu}^{y},\tilde{r}\right) \subset SP(Z,A)$
$\Longrightarrow P_{\mu}^{y} \in B\left(P_{\mu}^{y},\tilde{r}\right) \subset SP(Y,A) \cap SP(Z,A) = SP(((Y,A)\tilde{\cap}(Z,A)))$
$\Longrightarrow P_{\mu}^{y} \tilde{\in} SS(((Y,A)\tilde{\cap}(Z,A))^{o})$

$$\therefore\ SS((Y,A)^{o})\tilde{\cap}SS((Z,A)^{o})\ \tilde{\subset}\ SS(((Y,A)\tilde{\cap}(Z,A))^{o}) \tag{4.2}$$

From (4.1) and (4.2) we get,

$$SS((Y,A)^{o})\tilde{\cap}SS((Z,A)^{o}) = SS(((Y,A)\tilde{\cap}(Z,A))^{o}).$$

□

4.4. **Soft open and soft closed sets.**

Definition 4.24. Let $(\check{X},d)$ be a soft metric space and (Y,A) be a non-null soft subset of $\check{X}$ in $(\check{X},d)$. Then (Y,A) is said to be '*soft open in $\check{X}$ with respect to d*' if and only if all soft points of (Y,A) be interior points of (Y,A).

Theorem 4.25. *In a soft metric space every soft open ball is a soft open set.*

Proof. Let $SS(B(P_e^a,\tilde{r}))$ be a soft open ball with centre P_e^a and radius $\tilde{r}$ in a soft metric space $(\check{X},d)$. We are to prove that $SS(B(P_e^a,\tilde{r}))$ is a soft open set. Clearly, $SS(B(P_e^a,\tilde{r}))$ is generated by the set of all soft points of $B(P_e^a,\tilde{r})$ and all of them are interior points of $SS(B(P_e^a,\tilde{r}))$.
Thus $SS(B(P_e^a,\tilde{r}))$ is a soft open set in the soft metric space $(\check{X},d)$. □

Theorem 4.26. *In any soft metric space $(\check{X},d)$,*

(i) *the null soft set Φ is soft open;*
(ii) *the absolute soft set $\check{X}$ itself is soft open;*
(iii) *arbitrary union of soft open sets is soft open;*
(iv) *intersection of a finite number of soft open sets is soft open.*

Proof. (i) & (ii) are obvious.
(iii) Let Λ be an arbitrary index set such that $\forall\ \alpha \in \Lambda$ the soft set $(Y_\alpha,A)\tilde{\subset}\check{X}$ are soft open in $(\check{X},d)$. We have to prove that $(Y,A) = \tilde{\cup}_{\alpha\in\Lambda}(Y_\alpha,A)$ is also soft open in $(\check{X},d)$.
In case $\Lambda=\varnothing$, we have $(Y,A) = \tilde{\cup}_{\alpha\in\Lambda}(Y_\alpha,A) = \Phi$, which is soft open.
Let $\Lambda \neq \varnothing$, if $(Y_\alpha,A) = \Phi,\ \forall\ \alpha \in \Lambda$ then $(Y,A) = \tilde{\cup}_{\alpha\in\Lambda}(Y_\alpha,A) = \Phi$, which is soft open.
Finally let $\Lambda \neq \varnothing$ and $(Y_\alpha,A) \neq \Phi$, for at least one $\alpha \in \Lambda$.
Then $(Y,A) = \tilde{\cup}_{\alpha\in\Lambda}(Y_\alpha,A)$. Let $P_\lambda^x\tilde{\in}(Y,A) = \tilde{\cup}_{\alpha\in\Lambda}(Y_\alpha,A)$. Then $P_\lambda^x\tilde{\in}(Y_\alpha,A)$, for some $\alpha \in \Lambda$.
Since each (Y_α,A) is soft open in $(\check{X},d)$, there is a soft real number $\tilde{r}\ \tilde{>}\ \overline{0}$ such that $P_\lambda^x \in B(P_\lambda^x,\tilde{r}) \subset SP(Y_\alpha,A) \subset SP(\tilde{\cup}_{\alpha\in\Lambda}(Y_\alpha,A)) = SP(Y,A)$. $\Longrightarrow$ (Y,A) is soft open.
(iv) Proof of (iv) is similar to that of (iii). □

Remark 4.27. From the above theorem, it follows that, the collection τ of all soft open sets in a soft metric space $(\check{X}, d)$ form a '*soft topology on* X' as defined by M. Shabir, et al. [18]. From this point of view, it also follows that every soft metric on a soft set $\check{X}$ gives a 'soft topology' on X. Hence every soft metric space $(\check{X}, d, A)$ is also a 'soft topological space' (X, τ, A).

Definition 4.28. Let $(\check{X}, d)$ be a soft metric space. A soft set $(Y, A)\, \tilde{\subset} \check{X}$, is said to be '*soft closed in* $\check{X}$ *with respect to* d' if its complement $(Y, A)^c$ is soft open in $(\check{X}, d)$.

Theorem 4.29. *In any soft metric space* $\left(\check{X}, d\right)$,

(i) *the null soft set* Φ *is soft closed;*
(ii) *the absolute soft set* $\check{X}$ *itself is soft closed;*
(iii) *arbitrary Intersection of soft closed sets is soft closed;*
(iv) *union of a finite number of soft closed sets is soft closed.*

Proof. Proof follows by Proposition 2.10 and Theorem 4.26. □

Theorem 4.30. *(First countability of soft metric space). Let* $\left(\check{X}, d\right)$ *be a soft metric space with a finite set of parameters and* $P_\lambda^x \tilde{\in} \check{X}$ *be arbitrary. Then there exists a countable collection* $\{SS(B_n); n \in \mathbb{N}\}$ *of soft open neighbourhood of* P_λ^x *(particularly, soft open balls each having centre at* P_λ^x *) such that for any soft neighbourhood* $SS(N_*)$ *of* P_λ^x*,* $\exists\ m \in \mathbb{N}$ *with* $SS\left(B_m\right) \tilde{\subset} SS(N_*)$.

Proof. Let us consider $B_n = B\left(P_\lambda^x,\ \frac{1}{\overline{n}}\right)$. Then $\{SS(B_n)\}$ is a countable family of soft open balls with centre at P_λ^x and hence soft open neighbourhoods of P_λ^x. Now for any soft neighbourhood $SS(N_*)$ of P_λ^x, $\exists$ a soft real number $\tilde{r} \tilde{>} \overline{0}$, such that $P_\lambda^x \tilde{\in}\ SS\left(B\left(P_\lambda^x, \tilde{r}\right)\right) \tilde{\subset} SS(N_*)$. Let $\tilde{r}\left(\lambda\right) = r_\lambda$, $\forall \lambda \in A$. Then $r_\lambda > 0$ and by Archimedean property of $\mathbb{R}$ $\exists\ m_\lambda \in \mathbb{N}$ such that $\frac{1}{m_\lambda} < r_\lambda$, $\forall \lambda \in A$.
Let m be a positive integer such that $m > m_\lambda$, $\forall \lambda \in A$, (such an m exists because A is a finite set) then $\frac{1}{m} < \frac{1}{m_\lambda} < r_\lambda$, $\forall \lambda \in A$.
Then $SS(B_m) = SS(B\left(P_\lambda^x,\ \frac{1}{\overline{m}}\right)) \tilde{\subset}\ SS(B\left(P_\lambda^x, \tilde{r}\right)) \tilde{\subset}\ SS(N_*)$. This completes the proof. □

Remark 4.31. However for a soft metric space having an infinite set of parameters the above result does not hold. For an infinite set A of parameters and for $m_\lambda \in \mathbb{N}$, for each $\lambda \in A$ the choice $m > m_\lambda$, $\forall \lambda \in A$ is not necessarily a finite number.

Definition 4.32. (Second countable space). A soft metric space $\left(\check{X}, d\right)$ is said to be second countable if there exists a countable family of soft open sets in $\left(\check{X}, d\right)$ such that every soft open set of $\left(\check{X}, d\right)$ is expressible as a union of some soft sets of that family.

Definition 4.33. A sub-collection of the collection of all soft open sets in $\left(\check{X}, d\right)$ is said to be a base for $\left(\check{X}, d\right)$ if every soft open set of $\left(\check{X}, d\right)$ is expressible as a union of some soft sets of that sub-collection.

4.5. **Soft limit point and soft closure.**

Definition 4.34. Let $(\check{X}, d)$ be a soft metric space and $(Y, A) \tilde{\subset} \check{X}$. A soft point $P_e^a \tilde{\in} \check{X}$ is said to be a *soft limit point* of (Y, A), if and only if every soft open ball $SS(B(P_e^a, \tilde{r}))$ containing P_e^a in $(\check{X}, d)$ contains at least one soft point of (Y, A) other than P_e^a.

A soft limit point of a soft set (Y, A) may or may not belong to the soft set (Y, A). The set of all soft limit points of (Y, A) is said to be the derived set of (Y, A) and is denoted by $(Y, A)^d$.
Clearly a soft point $P_e^a \tilde{\in} \check{X}$ is a soft limit point of (Y, A), if and only if every soft neighbourhood $SS(N(P_e^a))$ containing P_e^a in $(\check{X}, d)$ contains at least one soft point of (Y, A) other than P_e^a.

Example 4.35. Let us consider the soft metric space $(\check{X}, d)$ or $(\check{X}, d, A)$ as in Example 4.3. Let $(Y, A) \tilde{\subset} \check{X}$ be such that $Y(\lambda) = (a, b)$, for some $a, b \in X$ with $a < b$, and $Y(e) = \varnothing$, $\forall e \in A \backslash \{ \lambda \}$. Now for a soft point P_λ^x, we have for any positive soft real number $\tilde{r}$ with $\tilde{r}(\lambda) = r_\lambda$.
$SS(B(P_\lambda^x, \tilde{r}))(\lambda) \cap [(Y, A) \backslash \{P_\lambda^x\}](\lambda) = (x - r_\lambda, x + r_\lambda) \cap [(a, b) \backslash \{x\}] = (\alpha_\lambda, \beta_\lambda) \backslash \{x\} \neq \varnothing$, where $\alpha_\lambda = \max\{x - r_\lambda, a\}$, $\beta_\lambda = \min\{x + r_\lambda, b\}$, whatever may be the value of $r_\lambda > 0$. So $SS(B(P_\lambda^x, \tilde{r}))$ contains a soft point of (Y, A) other than P_λ^x. Thus any soft point P_λ^x with $x \in [a, b]$, is a soft limit point of (Y, A) in $(\check{X}, d)$.

Theorem 4.36. *Let $(\check{X}, d)$ be a soft metric space and $(Y, A) \tilde{\subset} \check{X}$. Then (Y, A) is soft closed in $(\check{X}, d)$, if and only if no soft point of $(Y, A)^c$ is a soft limit point of (Y, A).*

Proof. Let (Y, A) be soft closed in $(\check{X}, d)$. Let $P_e^a \tilde{\in} (Y, A)^c$, then $(Y, A)^c$ being soft open in $(\check{X}, d)$, P_e^a is an interior point of $(Y, A)^c$. Then $\exists$ a positive soft real number $\tilde{r}$ such that $P_e^a \in B(P_e^a, \tilde{r}) \subset SP((Y, A)^c)$, so $B(P_e^a, \tilde{r})$ and hence $SS(B(P_e^a, \tilde{r}))$ contains no soft point of (Y, A); proving that P_e^a cannot be a soft limit point of (Y, A). Since $P_e^a \tilde{\in} (Y, A)^c$ is arbitrary, it follows that no soft point of $(Y, A)^c$ is a soft limit point of (Y, A).

Conversely, let the given condition be satisfied. Then for $P_e^a \tilde{\in} (Y, A)^c$, $\exists$ a positive soft real number $\tilde{r}$ such that $SS(B(P_e^a, \tilde{r}))$ and hence $B(P_e^a, \tilde{r})$ contains no soft point of (Y, A); i.e., $P_e^a \in B(P_e^a, \tilde{r}) \subset SP((Y, A)^c)$. Then $\cup \{P_e^a\} \subset \cup B(P_e^a, \tilde{r}) \subset SP((Y, A)^c)$, where the union is taken over all soft points of of $(Y, A)^c$. Now $(Y, A)^c = \tilde{\cup} SS(B(\tilde{a}, \tilde{r}))$, so $(Y, A)^c$ is soft open and hence (Y, A) is soft closed in $(\check{X}, d)$. □

Using Theorem 4.36, it follows that, every soft set consisting of a single soft element in a soft metric space $(\check{X}, d)$ is soft closed. Hence every soft metric space is a 'soft T_1- space' as defined by M. Shabir, et al. in [18].

Remark 4.37. We can write $\check{X} = (Y, A) \tilde{\cup} (Y, A)^c$ and with this point of view we can say as a consequence of the Theorem 4.36 that, (Y, A) is soft closed in $(\check{X}, d)$, if and only if it contains all of its soft limit points.

Definition 4.38. A soft set (F, A) is said to be a finite (or countable) soft set if $F(\lambda)$ is finite (or countable) for all $\lambda \in A$. A soft set is said to be an infinite soft set if it is not finite.

Theorem 4.39. *A finite soft set (Y,A) has no soft limit point.*

Proof. Let (Y,A) be a finite soft set. Let $P_\lambda^x \tilde{\in} \check{X} = (Y,A) \tilde{\cup} (Y,A)^c$. Then either $P_\lambda^x \tilde{\in} (Y,A)$ or $P_\lambda^x \tilde{\in} (Y,A)^c$. If $P_\lambda^x \tilde{\in} (Y,A)$, then let us consider a soft real number $\tilde{r}$ with $\tilde{r}(\lambda) = r_\lambda$, for all $\lambda \in A$, where $r_\lambda = \min\{d(P_\mu^y, P_\gamma^z)(\lambda); P_\mu^y, P_\gamma^z \tilde{\in} (Y,A), P_\mu^y \neq P_\gamma^z\}$. The minimum operation is well defined since $d(P_\mu^y, P_\gamma^z)(\lambda)$ is a finite set for all $\lambda \in A$.
Then $SS(B(P_\mu^y, \tilde{r}))$ contains no soft point of (Y,A) other than P_μ^y, for all $P_\mu^y \tilde{\in} (Y,A)$.
$\therefore$ No soft point of (Y,A) can be a soft limit point of (Y,A).
Next, if $P_\lambda^x \tilde{\in} (Y,A)^c$, then we consider a soft real number $\tilde{r}$ with $\tilde{r}(\lambda) = r_\lambda$, for all $\lambda \in A$, where $r_\lambda = \frac{1}{2}\min\{d(P_\lambda^x, P_\mu^y)(\lambda); P_\mu^y \tilde{\in} (Y,A)\}$ we find $SS(B(P_\lambda^x, \tilde{r}))$ contains no soft point of (Y,A). $\Longrightarrow$ No soft point $P_\lambda^x \tilde{\in} \check{X}$ can be a soft limit point of (Y,A).
$\therefore$ The soft set (Y,A) has no soft limit point. □

Theorem 4.40. *Let $(\check{X}, d)$ be a soft metric space and $(Y,A), (Z,A) \tilde{\subset} \check{X}$. Then*

(i) $\Phi^d = \Phi$ *and* $\check{X}^d = \check{X}$;
(ii) $(Y,A) \tilde{\subset} (Z,A) \Longrightarrow (Y,A)^d \tilde{\subset} (Z,A)^d$;
(iii) $((Y,A)^d)^d \tilde{\subset} (Y,A)^d$;
(iv) $(Y,A)^d \tilde{\cup} (Z,A)^d = ((Y,A) \tilde{\cup} (Z,A))^d$;
(v) $((Y,A) \tilde{\cap} (Z,A))^d \tilde{\subset} (Y,A)^d \tilde{\cap} (Z,A)^d$.

Proof. We prove only (iii). Let $P_\lambda^x \tilde{\in} ((Y,A)^d)^d$, then P_λ^x is a soft limit point of $(Y,A)^d$. Then for any soft real number $\tilde{r} \tilde{>} \overline{0}$, $SS(B(P_\lambda^x, \tilde{r}))$ contains at least one soft point of $(Y,A)^d$ other than P_λ^x. Let $P_\lambda^x \neq P_\mu^y \tilde{\in} SS(B(P_\lambda^x, \tilde{r})) \tilde{\cap} (Y,A)^d$. Then P_μ^y is a soft limit point of (Y,A). Let $d(P_\lambda^x, P_\mu^y) = \tilde{r}'$, then $SS(B(P_\mu^y, \tilde{r}'/2))$ is a soft open ball containing P_μ^y and $P_\lambda^x \tilde{\notin} SS(B(P_\mu^y, \tilde{r}'/2))$. Then $SS(B(P_\mu^y, \tilde{r}'/2)) \tilde{\cap} (Y,A)$ contains at least one soft point of (Y,A) other than P_μ^y and P_λ^x. Then P_λ^x is a soft limit point of (Y,A). $\therefore P_\lambda^x \tilde{\in} (Y,A)^d$. $\therefore ((Y,A)^d)^d \tilde{\subset} (Y,A)^d$. □

Definition 4.41. Let $(\check{X}, d)$ be a soft metric space and $(Y,A) \tilde{\subset} \check{X}$. Then the soft set generated by the collection of all soft points of (Y,A) and soft limit points of (Y,A) in $(\check{X}, d)$ is said to be the *soft closure* of (Y,A) in $(\check{X}, d)$. It is denoted by $\overline{(Y,A)}$.

Theorem 4.42. *Let $(\check{X}, d)$ be a soft metric space and $(Y,A), (Z,A) \tilde{\subset} \check{X}$. Then*

(i) $\overline{\Phi} = \Phi$ *and* $\overline{\check{X}} = \check{X}$;
(ii) $(Y,A) \tilde{\subset} \overline{(Y,A)}$;
(iii) $(Y,A)^d \tilde{\subset} \overline{(Y,A)}$;
(iv) $\overline{(Y,A)} = \overline{\overline{(Y,A)}}$;
(v) (Y,A) *is a soft closed set if and only if* $(Y,A) = \overline{(Y,A)}$;
(vi) $(Y,A) \tilde{\subset} (Z,A) \Longrightarrow \overline{(Y,A)} \tilde{\subset} \overline{(Z,A)}$;
(vii) $\overline{(Y,A) \tilde{\cup} (Z,A)} = \overline{(Y,A)} \tilde{\cup} \overline{(Z,A)}$;
(viii) $\overline{(Y,A) \tilde{\cap} (Z,A)} \tilde{\subset} \overline{(Y,A)} \tilde{\cap} \overline{(Z,A)}$.

Proof. The proof follows from Theorem 4.40 and Definition 4.41 □

Theorem 4.43. *Let (Y, d_Y, A) be a metric subspace of a soft metric space $(\check{X}, d)$ and $P_\mu^y \tilde{\in}(Y, A)$. Then for any soft open ball $SS(B(P_\mu^y, \tilde{r}))$ in $\check{X}$, $SS(B(P_\mu^y, \tilde{r}))\tilde{\cap}(Y, A)$ is a soft open ball in (Y, d_Y, A), and also any soft open ball in (Y, d_Y, A) is obtained as the intersection of a soft open ball in $\check{X}$ with (Y, A).*

Proof. The theorem will be completely proved if it is proved that for any $P_\mu^y \tilde{\in} (Y, A)$ and any $\tilde{r} \tilde{>} \bar{0}$,
$SS(B_Y(P_\mu^y, \tilde{r})) = SS(B(P_\mu^y, \tilde{r}))\tilde{\cap}(Y, A)$.
$SS(B(P_\mu^y, \tilde{r}))\tilde{\cap}(Y, A) = SS(\{P_\lambda^x \tilde{\in} \check{X}; d(P_\lambda^x, P_\mu^y) \tilde{<} \tilde{r}\})\tilde{\cap}(Y, A)$
$= SS(\{P_\gamma^z \tilde{\in} (Y, A); d(P_\gamma^z, P_\mu^y) \tilde{<} \tilde{r}\}) = SS(\{P_\gamma^z \tilde{\in} (Y, A); d_Y(P_\gamma^z, P_\mu^y)) \tilde{<} \tilde{r}\})$
[since $P_\gamma^z, P_\mu^y \tilde{\in}(Y, A)$, $d(P_\gamma^z, P_\mu^y) = d_Y(P_\gamma^z, P_\mu^y)$]
$= SS(B_Y(P_\mu^y, \tilde{r}))$. □

Theorem 4.44. *Let (Y, d_Y, A) be a metric subspace of a soft metric space $(\check{X}, d)$. Then*

(i) *$(U, A)\tilde{\subset}(Y, A)$ is soft open in the metric subspace $(Y, d_Y, A) \Leftrightarrow (U, A) = (V, A)\tilde{\cap}(Y, A)$, for some soft open set (V, A) in $(\check{X}, d)$.*
(ii) *$(F, A)\tilde{\subset}(Y, A)$ is soft closed in the metric subspace $(Y, d_Y, A) \Leftrightarrow (F, A) = (G, A)\tilde{\cap}(Y, A)$, for some soft closed set (G, A) in $(\check{X}, d)$.*

Proof. (i) Let (U, A) be soft open in (Y, d_Y, A) and let $P_\mu^y \tilde{\in} (U, A)$. Then for some $\tilde{r}_y \tilde{>} \bar{0}$, $SS(B_Y(P_\mu^y, \tilde{r}_y))\tilde{\subset}(U, A)$.
Then $(U, A) = \tilde{\cup}_{P_\mu^y \tilde{\in}(U,A)} SS(B_Y(P_\mu^y, \tilde{r}_y)) = \tilde{\cup}_{P_\mu^y \tilde{\in}(U,A)}[SS(B(P_\mu^y, \tilde{r}_y))\tilde{\cap}(Y, A)]$, [by Theorem 4.43]

$$= [\tilde{\cup}_{P_\mu^y \tilde{\in}(U,A)} SS(B(P_\mu^y, \tilde{r}_y))]\tilde{\cap}(Y, A) = (V, A)\tilde{\cap}(Y, A),$$

where $(V, A) = [\tilde{\cup}_{P_\mu^y \tilde{\in}(U,A)} SS(B(P_\mu^y, \tilde{r}_y))]$, being a union of soft open sets is soft open in $(\check{X}, d)$.
Conversely, let $(U, A) = (V, A)\tilde{\cap}(Y, A)$, where (V, A) is soft open in $(\check{X}, d)$. Then for any $P_\mu^y \tilde{\in} (U, A)$, $P_\mu^y \tilde{\in} (V, A)$ and $P_\mu^y \tilde{\in} (Y, A)$. Since (V, A) is soft open in $(\check{X}, d)$, $\exists\ \tilde{r} \tilde{>} \tilde{0}$, such that SS$(B(P_\mu^y, \tilde{r}))\tilde{\subset}(V, A)$
i.e., $SS(B_Y(P_\mu^y, \tilde{r})) = SS(B(P_\mu^y, \tilde{r}))\tilde{\cap}(Y, A) \tilde{\subset} (V, A)\tilde{\cap}(Y, A) = (U, A)$. Thus (U, A) is soft open in the metric subspace (Y, d_Y, A).
(ii) (F, A) is soft closed in $(Y, d_Y, A) \Leftrightarrow (Y, A) \setminus (F, A)$ is soft open in (Y, d_Y, A)
$\Leftrightarrow (Y, A) \setminus (F, A) = (V, A)\tilde{\cap}(Y, A)$, for some soft open set (V, A) in $(\check{X}, d)$. [by (i)]
$\Leftrightarrow (F, A) = (Y, A) \setminus [(V, A)\tilde{\cap}(Y, A)] = [\check{X}\tilde{\cap}(Y, A)] \setminus [(V, A)\tilde{\cap}(Y, A)]$
$= [\check{X} \setminus (V, A)]\tilde{\cap}(Y, A) = (G, A)\tilde{\cap}(Y, A)$, where $(G, A) = [\check{X} \setminus (V, A)]$ is soft closed in $(\check{X}, d)$. □

Definition 4.45. (Separable Metric space). A soft metric space $(\check{X}, d)$ is said to be separable if there exists a countable soft subset (Y, A) of $\check{X}$ such that $\overline{(Y, A)} = \check{X}$.

Theorem 4.46. *Every separable metric space with countable set of parameters is second countable.*

Proof. Let $(\check{X}, d)$ be a separable metric space. Then there exists a countable soft subset (Y, A) of $\check{X}$ such that $\overline{(Y, A)} = \check{X}$. Let us consider the collection $\mathfrak{B} = \{SS(B(P_\lambda^x, \overline{r})); P_\lambda^x \widetilde{\in} (Y, A), r \in \mathbb{Q}, r > 0\}$. Then $\mathfrak{B}$ is a countable collection of soft open sets in $(\check{X}, d)$. To show $\mathfrak{B}$ be a base, let U be a soft open set in $(\check{X}, d)$ and $P_\lambda^x \widetilde{\in}\, U$. Then there exists $r \in \mathbb{Q}$, $r > 0$ such that $SS(B(P_\lambda^x, \overline{r})) \widetilde{\subset} U$. As $\overline{(Y, A)} = \check{X}$, $\exists P_\mu^y \widetilde{\in} (Y, A)$ such that $P_\mu^y \widetilde{\in}\, SS(B(P_\lambda^x, \frac{\overline{r}}{2}))$. We put $B_{P_\lambda^x} = SS(B(P_\mu^y, \frac{\overline{r}}{2}))$. Then $B_{P_\lambda^x} \in \mathfrak{B}$ and we prove that $P_\lambda^x \widetilde{\in}\, B_{P_\lambda^x} \widetilde{\subset}\, U$. In fact, $P_\mu^y \widetilde{\in}\, SS(B(P_\lambda^x, \frac{\overline{r}}{2})) \Rightarrow d(P_\lambda^x, P_\mu^y) \widetilde{<} \frac{\overline{r}}{2} \Rightarrow P_\lambda^x \widetilde{\in}\, SS(B(P_\mu^y, \frac{\overline{r}}{2})) = B_{P_\lambda^x}$ and $P_\gamma^z \widetilde{\in}\, B_{P_\lambda^x} \Rightarrow d(P_\gamma^z, P_\mu^y) \widetilde{<} \frac{\overline{r}}{2} \Rightarrow d(P_\lambda^x, P_\gamma^z) \widetilde{\leq} d(P_\lambda^x, P_\mu^y) + d(P_\mu^y, P_\gamma^z) \widetilde{<} \frac{\overline{r}}{2} + \frac{\overline{r}}{2} = \overline{r} \Rightarrow P_\gamma^z \widetilde{\in}\, SS(B(P_\lambda^x, \overline{r})) \widetilde{\subset}\, U$.
Thus for a given soft open set U and for any $P_\lambda^x \widetilde{\in}\, U$, $\exists\, B_{P_\lambda^x}$ (say) $\in \mathfrak{B}$ such that $P_\lambda^x \widetilde{\in}\, B_{P_\lambda^x} \widetilde{\subset}\, U$. We have $U = \bigcup_{P_\lambda^x \in U} B_{P_\lambda^x}$. Hence $\mathfrak{B}$ is a countable open base for $(\check{X}, d)$, proving $\check{X}$ to be second countable. □

Remark 4.47. However a soft metric space having an uncountable set of parameters the collection $\mathfrak{B} = \{SS(B(P_\lambda^x, \overline{r})); P_\lambda^x \widetilde{\in} (Y, A), r \in \mathbb{Q}, r > 0\}$, as considered above is not countable.

5. Completeness of soft metric spaces

In this section proofs of some results are omitted as they are straight forward and similar to the case of crisp sets.

Definition 5.1. Let $\{P_{\lambda,n}^x\}_n$ be a sequence of soft points in a soft metric space $(\check{X}, d)$. The sequence $\{P_{\lambda,n}^x\}_n$ is said to be convergent in $(\check{X}, d)$ if there is a soft point $P_\mu^y \widetilde{\in}\, \check{X}$ such that $d(P_{\lambda,n}^x, P_\mu^y) \to \overline{0}$ as $n \to \infty$.
This means for every $\widetilde{\varepsilon} \widetilde{>} \overline{0}$, chosen arbitrarily, $\exists$ a natural number $N = N(\widetilde{\varepsilon})$, such that $\overline{0} \widetilde{\leq} d(P_{\lambda,n}^x, P_\mu^y) \widetilde{<} \widetilde{\varepsilon}$, whenever $n > N$.
i.e., $n > N \Longrightarrow P_{\lambda,n}^x \widetilde{\in}\, SS(B(P_\mu^y, \widetilde{\varepsilon}))$. We denote this by $P_{\lambda,n}^x \to P_\mu^y$ as $n \to \infty$ or by $\lim_{n\to\infty} P_{\lambda,n}^x = P_\mu^y$. P_μ^y is said to be the limit of the sequence $\{P_{\lambda,n}^x\}_n$ as $n \to \infty$.

Example 5.2. Consider the soft metric space $(\check{X}, d)$ or $(\check{X}, d, A)$ as in Example 4.3. Let $(Y, A) \widetilde{\subset} \check{X}$ be such that $Y(\lambda) = (0, 1]$ in the real line, $Y(\lambda) = \varnothing$, $\forall e \in A \backslash \{\lambda\}$. Let us choose a sequence $\{P_{\lambda,n}^x\}_n$ of soft points of (Y, A) where $P_{\lambda,n}^x(\lambda) = \frac{1}{n}$. Then there is no $P_\lambda^y \widetilde{\in} (Y, A)$ such that $P_{\lambda,n}^x \to P_\lambda^y$ in (Y, d_Y, A). However the sequence $\{P_{\lambda,n}^y\}$ of soft points of (Y, A) where $P_{\lambda,n}^y(\lambda) = \frac{1}{2}$, $\forall n \in \mathbb{N}$ is convergent in (Y, d_Y, A) and converges to $P_\lambda^{0.5}$.

Theorem 5.3. *Limit of a sequence in a soft metric space, if exists is unique.*

Theorem 5.4. *Let (Y, A) be a soft subset in a soft metric space $(\check{X}, d)$. Then $P_\mu^y \widetilde{\in} \check{X}$ is a soft limit point of (Y, A) if and only if there is a sequence $\{P_{\lambda,n}^x\}_n$ of soft points in (Y, A) other than $\{P_\mu^y\}_n$ which converges to P_μ^y.*

Theorem 5.5. *Let (Y, A) be a soft subset in a soft metric space $(\check{X}, d)$. Then (Y, A) is soft closed if and only if every sequence $\{P_{\lambda,n}^x\}_n$ in (Y, A) which converges in $\check{X}$ cannot converges to a soft point of $(Y, A)^c$.*

Definition 5.6. (Bounded sequences). A sequence $\{P^x_{\lambda,n}\}_n$ of soft points in $(\check{X}, d)$ is said to be bounded if the set $\{d\left(P^x_{\lambda,m}, P^x_{\lambda,n}\right); m, n \in N\}$ of soft real numbers is bounded, i.e., $\exists$ a positive soft real number $\tilde{M} \tilde{>} \overline{0}$ such that $d\left(P^x_{\lambda,m}, P^x_{\lambda,n}\right) \tilde{\leq} \tilde{M}$, $\forall m, n \in N$.

Definition 5.7. (Cauchy sequence). A sequence $\{P^x_{\lambda,n}\}_n$ of soft points in $(\check{X}, d)$ is considered as a Cauchy sequence in $\check{X}$ if corresponding to every $\tilde{\varepsilon} \tilde{>} \overline{0}$, $\exists\, m \in N$ such that $d\left(P^x_{\lambda,i}, P^x_{\lambda,j}\right) \tilde{\leq} \tilde{\varepsilon}$, $\forall\, i, j \geq m$ i.e., $d\left(P^x_{\lambda,i}, P^x_{\lambda,j}\right) \to \overline{0}$ as $i, j \to \infty$.

Theorem 5.8. *Every convergent sequence in a soft metric space is Cauchy and every Cauchy sequence is bounded.*

Definition 5.9. (Complete Metric Space). A soft metric space $(\check{X}, d)$ is called complete if every Cauchy sequence in $\check{X}$ converges to some soft point of $\check{X}$. The soft metric space $(\check{X}, d)$ is called incomplete if it is not complete.

Example 5.10. Consider the soft metric space $(\check{X}, d)$ or $(\check{X}, d, A)$ as in Example 4.3. Let (Y, d_Y, A) be a subspace with $Y(\lambda) = (0,\ 1]$, $Y(\lambda) = \varnothing$, $\forall e \in A \backslash \{\ \lambda\ \}$. Then the sequence $\{P^x_{\lambda,n}\}_n$ in $\check{X}$ with $P^x_{\lambda,n}(\lambda) = \frac{1}{n}$, $\forall n \in N$, is clearly a Cauchy sequence in (Y, A) and also in $(\check{X}, d, A)$. In fact for any $\varepsilon > 0$, choose $m \in N$ with $m > \frac{1}{\varepsilon}$, then for $i \geq j \geq m$, $\left|\frac{1}{i} - \frac{1}{j}\right| = \left|\frac{i-j}{ij}\right| \leq \frac{i}{ij} = \frac{1}{j} \leq \frac{1}{m} < \varepsilon$, then for $i \geq j \geq m$, $d\left(P^x_{\lambda,i}, P^x_{\lambda,j}\right) \tilde{<} \overline{\varepsilon}$, Also the sequence obviously converges to $P^0_{\lambda,}$ in $\check{X}$, where $P^0_\lambda \tilde{\in} \check{X} \backslash (Y, A)$, it cannot converge in (Y, d_Y, A). Thus $\{P^x_{\lambda,n}\}_n$ is a Cauchy sequence in (Y, d_Y, A) which is not convergent in (Y, d_Y, A). $\therefore$ (Y, d_Y, A) cannot be a complete soft metric space.

Proposition 5.11. *In a soft metric space* $(\check{X}, d)$ *for any soft set* (Y, A), $\delta((Y, A)) = \delta\left(\overline{(Y, A)}\right)$.

Proof. Since $(Y, A) \tilde{\subset} \overline{(Y, A)}$, we have $\delta((Y, A)) \tilde{\leq} \delta\left(\overline{(Y, A)}\right)$. Let $\tilde{\varepsilon} \tilde{>} \overline{0}$ be arbitrary and $P^x_{\lambda,}$, $P^y_{\mu,} \tilde{\in} \overline{(Y, A)}$ be arbitrary. Since $P^x_{\lambda,} \tilde{\in} \overline{(Y, A)}$, $\exists\, P^z_\gamma \tilde{\in} SS(B\left(P^x_{\lambda,},\ \frac{\tilde{\varepsilon}}{2}\right)) \tilde{\cap} (Y, A)$ and similarly $\exists\, P^w_\beta \tilde{\in} SS(B\left(P^y_{\mu,},\ \frac{\tilde{\varepsilon}}{2}\right)) \tilde{\cap} (Y, A)$.
Then $P^z_\gamma, P^w_\beta \tilde{\in} (Y, A)$ so that $d(P^x_{\lambda,}, P^z_\gamma) \tilde{<} \frac{\tilde{\varepsilon}}{2}$ and $d(P^x_{\lambda,}, P^w_\beta) \tilde{<} \frac{\tilde{\varepsilon}}{2}$.
Thus $d\left(P^x_{\lambda,}, P^y_\mu\right) \tilde{\leq} d\left(P^x_{\lambda,}, P^z_\gamma\right) + d\left(P^z_\gamma, P^w_\beta\right) + d\left(P^w_\beta, P^y_{\mu,}\right) \tilde{<} \frac{\tilde{\varepsilon}}{2} + \frac{\tilde{\varepsilon}}{2} + d\left(P^z_\gamma, P^w_\beta\right) = \tilde{\varepsilon} + d\left(P^z_\gamma, P^w_\beta\right) \tilde{<} \tilde{\varepsilon} + \delta((Y, A))$.
Since $P^x_{\lambda,}$, $P^y_{\mu,} \tilde{\in} \overline{(Y, A)}$ are arbitrary soft point, we have
$\delta\left(\overline{(Y, A)}\right) \tilde{\leq} \tilde{\varepsilon} + \delta((Y, A))$, which gives $\delta\left(\overline{(Y, A)}\right) \tilde{\leq} \delta((Y, A))$, as $\tilde{\varepsilon} \tilde{>} \overline{0}$ is arbitrary.
$\therefore$ $\delta((Y, A)) = \delta\left(\overline{(Y, A)}\right)$. □

Theorem 5.12. *(Cantor's Intersection Theorem for soft metric spaces). A soft metric space* $(\check{X}, d)$ *is complete if and only if for any sequence* $\{(Y, A)_n\}$ *of closed soft* *sets with* $(Y,A)_1 \,\widetilde{\supset}(Y,A)_2\widetilde{\supset}\ldots\ldots\ldots\widetilde{\supset}\,(Y,A)_n\widetilde{\supset}\ldots\ldots$ *such that* $\delta((Y,A)_n)\to\ \overline{0}$ *as* $n\to\infty$ *the intersection* $(Y,A)=\widetilde{\cap}_{n=1}^{\infty}(Y,A)_n$ *consists of exactly one soft point.*

Proof. Let $(\check{X},d)$ be a complete soft metric space and $\{(Y,A)_n\}$ be a sequence of soft closed sets with $(Y,A)_1 \,\widetilde{\supset}(Y,A)_2\widetilde{\supset}\ldots\ldots\ldots\widetilde{\supset}\,(Y,A)_n\widetilde{\supset}\ldots\ldots$ such that $\delta((Y,A)_n)\to$ $\overline{0}$ as $n\to\infty$. Since $\{(Y,A)_n\}\neq\Phi$, $\forall\lambda\in A$, $\forall n\in N$, choose a soft point $P^x_{\lambda,n}\widetilde{\in}$ $(Y,A)_n$, for each $n\in N$. We claim that the sequence $\left\{P^x_{\lambda,n}\right\}_n$ of soft points is Cauchy in $(\check{X},d)$. In fact, $m>n\Longrightarrow(Y,A)_m\widetilde{\subset}(Y,A)_n\Longrightarrow P^x_{\lambda,m},\ P^x_{\lambda,n}\widetilde{\in}(Y,A)_n\Longrightarrow$ $d(\,P^x_{\lambda,m},\ P^x_{\lambda,n})\,\widetilde{\leq}\,\delta((Y,A)_n)\to\ \overline{0}$ as $n\to\infty$ (and hence $m\to\infty$), i.e., $d(\,P^x_{\lambda,m},\ P^x_{\lambda,n})$ $\to\ \overline{0}$ as $m,n\to\infty$. $\Longrightarrow\left\{P^x_{\lambda,n}\right\}_n$ is Cauchy in $(\check{X},d)$.
Then by hypothesis $\exists$ a soft point $P^y_\mu\widetilde{\in}\check{X}$ such that $\lim_{n\to\infty}P^x_{\lambda,n}\ =P^y_\mu$. We show that $P^y_\mu\ \widetilde{\in}\ \widetilde{\cap}_{n=1}^{\infty}(Y,A)_n$. As $P^x_{\lambda,n}\to P^y_\mu$, every soft neighbourhood of P^y_μ contains all but finitely many terms of the sequence and for each $n\in N$, $(Y,A)_n$ contains all these terms excepting finitely many possibly (for $P^x_{\lambda,m}\widetilde{\in}(Y,A)_n,\forall\ m\geq n$ as $P^x_{\lambda,m}\widetilde{\in}(Y,A)_m\ \widetilde{\subset}(Y,A)_n,\ \forall\ m\geq n$). Then each soft neighbourhood of P^y_μ intersects $(Y,A)_n$, so that $P^x_{\lambda,n}\widetilde{\in}\ \overline{(Y,A)_n}\ =(Y,A)_n$[By hypothesis$(Y,A)_n$ is soft closed,$\forall\ n$]. Hence $\widetilde{\cap}_{n=1}^{\infty}(Y,A)_n$ contains at least one soft point. Now, if $P^z_\gamma\ \widetilde{\in}\ \widetilde{\cap}_{n=1}^{\infty}(Y,A)_n$ with $P^y_\mu\neq P^z_\gamma$, then at least for one $\lambda\in A$, $d\left(P^y_\mu,P^z_\gamma\right)(\lambda)=r(say)>0$. Then $\delta((Y,A)_n(\lambda))\geq r>0,\ \forall\ n\in N$. This contradicts that $\delta((Y,A)_n)\to\ \overline{0}$ as $n\to\infty$. Thus (Y,A) contains exactly one soft point.
Conversely, let the given conditions hold and $\left\{P^x_{\lambda,n}\right\}_n$ be a Cauchy sequence of soft points in $(\check{X},d)$.
For each $n\in N$, let $(Y,A)_n=\{P^x_{\lambda,n},\ P^x_{\lambda,n+1},\ \ldots\ldots\}$.
Obviously $(Y,A)_1\,\widetilde{\supset}(Y,A)_2\widetilde{\supset}\ldots\ldots\ldots\widetilde{\supset}\,(Y,A)_n\widetilde{\supset}\ldots\ldots$ and hence
$\overline{(Y,A)_1}\,\widetilde{\supset}\overline{(Y,A)_2}\widetilde{\supset}\ldots\ldots\widetilde{\supset}\overline{(Y,A)_n}\widetilde{\supset}\ldots\ldots$. Since $\left\{P^x_{\lambda,n}\right\}_n$ is Cauchy, $\delta((Y,A)_n)\to$ $\overline{0}$ and hence $\delta(\overline{(Y,A)_n})\to\ \overline{0}$ as $n\to\infty$. Then by hypothesis, $\widetilde{\cap}_{n=1}^{\infty}\overline{(Y,A)_n}$ consists of a single soft point P^y_μ (say). Thus $d(P^y_\mu,P^x_{\lambda,n})\widetilde{\leq}\ \delta(\overline{(Y,A)_n})\to\ \overline{0}$ as $n\to\infty$, i.e., $d(P^y_\mu,P^x_{\lambda,n})\to\ \overline{0}$ as $n\to\infty$. Hence $\left\{P^x_{\lambda,n}\right\}_n$ converges to P^y_μ in $(\check{X},d)$, proving $(\check{X},d)$ to be complete. □

6. Conclusions

In this paper we have introduced a concept of soft points in soft sets and using this concept we have given a definition of soft metric. Soft metric topology is developed and properties of soft open sets and soft closed sets are studied. The soft metric topology induced from the soft metric space is of the type of Shabir et al. [18]. A notion of convergence in such spaces is investigated and an analogue of Cantor's Intersection Theorem in this setting is furnished. There is an ample scope for further research on soft metric spaces and also on norm and inner product structures in soft set settings.

Acknowledgements. The authors express their sincere thanks to the Referees for their valuable suggestions in rewriting the paper in the present form. The authors are also very much thankful to the Editors-in-Chief and the Managing Editors for their valuable advice.
This work is partially supported by the Department of Mathematics, Visva Bharati, under the project UGC SAP (DRS) Phase – II.

References

[1] H. Aktas and N. Cagman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2226–2735.
[2] A. Ayg*ü*no*ğ*lu and H. Ayg*ü*n, Some notes on soft topological spaces, Neural Comput & Applic 21 (2012) 113-119.
[3] M.I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (2009) 1547–1553.
[4] D. Chen, The parametrization reduction of soft sets and its applications, Comput. Math. Appl. 49 (2005) 757–763.
[5] Sujoy Das and S. K. Samanta, Soft real sets, soft real numbers and their properties, J. Fuzzy Math. 20(3) (2012) 551–576.
[6] Sujoy Das and S. K. Samanta, On soft complex sets and soft complex numbers, J. Fuzzy Math. 21(1) (2013) 195–216.
[7] F. Feng, Y. B. Jun and X. Zhao, Soft semirings, Comput. Math. Appl. 56 (2008) 2621–2628.
[8] F. Feng, C. X. Li, B. Davvaz and M. I. Ali, Soft sets combined with fuzzy sets and rough sets: a tentative approach, Soft Computing 14 (2010) 8999–9911.
[9] Y. B. Jun, Soft BCK/BCI-algebras, Comput. Math. Appl. 56 (2008) 1408–1413.
[10] Y. B. Jun and C. H. Park, Applications of soft sets in ideal theory of BCK/BCI-algebras, Inform. Sci. 178 (2008) 2466–2475.
[11] Z. Kong, L. Gao, L. Wong and S. Li, The normal parameter reduction of soft sets and its algorithm, Comput. Math. Appl. 56 (2008) 3029–3037.
[12] P. K. Maji, R. Biswas and A. R. Roy, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077–1083.
[13] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[14] P. Majumdar and S. K. Samanta, On soft mappings, Comput. Math. Appl. 60 (2010) 2666–2672.
[15] D. Molodtsov, Soft set theory first results, Comput. Math. Appl. 37 (1999) 19–31.
[16] D. Pie and D. Miao, From soft sets to information systems, Granular Computing, 2005, IEEE International Conference, Volume 2, 617–622.
[17] M. Shabir and M. Irfan Ali, Soft ideals and generalized fuzzy ideals in semigroups, New Math. Nat. Comput. 5 (2009) 599–615.
[18] M. Shabir and M. Naz, On soft topological Spaces, Comput. Math. Appl. 61 (2011) 1786–1799.
[19] Y. Zou and Z. Xiao, Data analysis approaches of soft sets under incomplete information, Knowledge-Based Systems 21 (2008) 941–945.

Sujoy Das (sujoy_math@yahoo.co.in)
Department of Mathematics, Bidhan Chandra College, Asansol-4, West Bengal, India

S. K. Samanta (syamal_123@yahoo.co.in)
Department of Mathematics, Visva Bharati, Santiniketan, West Bengal, India