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Uniformity structure in the context of soft set

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Abstract. The main purpose of this study is to contribute to the development of the soft topological structures. For this reason, we deal with the concept of soft uniformity structure. We give the definitions of soft uniformity and soft uniformity base. We also investigate the relations between the soft uniformity and soft topology. Moreover, we study the basic properties of the soft uniformity structure.

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1. Introduction

In 1999, Molodtsov [7] introduced the concept of soft sets, which can be seen as a new mathematical tool for dealing with uncertainties and also established the fundamental results of the new theory. It is a general mathematical tool for dealing with objects which have been defined using a very loose and hence very general set of characteristics. A soft set is a collection of approximate descriptions of an object. Presently, works on soft set theory are progressing rapidly. Molodtsov [7] successfully applied the soft set theory into several directions. Maji et al. [5] defined and studied several basic notions of soft set theory. Aktaş and Çağman [1] introduced the soft group. Shabir and Naz [9] defined the soft topological space. Aygünoğlu and Aygün [2] introduced the soft continuity and studied soft compactness. Zorlutuna et al. [10] investigated soft neighborhood of a point. Pazar Varol and Aygün [8] investigated fundamental properties of soft Hausdorff spaces.

It is well-known that uniformity is a very important concept close to topology and convenient tool for investigating topology. For this reason, we decided to concern the notion of uniformity for the soft sets in order to make a contribution to the development of this new theory. This paper is arranged in the following manner. In section 2, we recall some definitions and notions of soft sets. In section 3, we recall some notions of soft topological spaces such as soft neighborhood and soft Hausdorff space. These definitions will help us in the last section. In the last section, we introduce soft uniformity and soft uniform base structures. We investigate some fundamental properties of soft uniformity structure. Also, we give the relations between soft uniformity and soft topology.

2. Preliminaries

In this chapter, we give some preliminaries about soft sets. We refer to [2, 4, 6, 7] for all the basic definitions and notations. We make some small modifications to some of them in order to make theoretical study in detail. Throughout this study, X refers to an initial universe, $P(X)$ is the power set of X, E is the set of all parameters for X and $A \subseteq E$.

Definition 2.1 ([4, 6, 7])**.** A soft set F_A on the universe X is defined by the set of ordered pairs

$F_A = \{(e, F_A(e)) \mid e \in E, F_A(e) \in P(X)\}$,

where $F_A : E \to P(X)$, such that $F_A(e) \neq \varnothing$, if $e \in A \subseteq E$ and $F_A(e) = \varnothing$ if $e \notin A$.

Since every soft set defined from a subset of the parameter set can be extended to the universal parameter set E, we will use the notation F instead of the notation F_A to denote the soft set, for simplicity. So, throughout this study, a soft set F on X, is a mapping from E into $P(X)$, i.e., $F_e := F(e)$ is a subset of X, for each $e \in E$.

$S(X, E)$ denotes the family of all soft sets defined on X.

Definition 2.2 ([4, 7])**.** Let F and G be two soft sets over X.

(1) We say that F is a soft subset of G and write $F \sqsubseteq G$ if $F_e \subseteq G_e$, for each $e \in E$. F and G are called equal if $F \sqsubseteq G$ and $G \sqsubseteq F$.

(2) Union of two soft sets $F, G \in S(X, E)$ is the soft set $H = F \sqcup G$, where $H_e = F_e \cup G_e$, for each $e \in E$.

(3) Intersection of two soft sets $F, G \in S(X, E)$ is the soft set $H = F \sqcap G$, where $H_e = F_e \cap G_e$, for each $e \in E$.

(4) The complement of a soft set $F \in S(X, E)$ is denoted by F^c, where $F^c : E \longrightarrow P(X)$ is a function given by $F^c_e = X \setminus F_e$, for each $e \in E$. Clearly $(F^c)^c = F$.

(5) (Null soft set) A soft set F over X is called a null soft set and denoted by Φ, if $F_e = \varnothing$, for each $e \in E$.

(6) (Absolute soft set) A soft set F over X is called an absolute soft set and denoted by $\widetilde{E}$, if $F_e = X$, for each $e \in E$. Clearly $(\widetilde{E})^c = \Phi$ and $\Phi^c = \widetilde{E}$.

Proposition 2.3. ([4, 5]) *Let J be an index set and $F, G, H, F_i, G_i \in S(X, E)$, for all $i \in J$, then we have the following properties:*

(1) $F \sqcup (G \sqcup H) = (F \sqcup G) \sqcup H, \quad F \sqcap (G \sqcap H) = (F \sqcap G) \sqcap H.$

(2) $F \sqcap (\sqcup_{i \in J} G_i) = \sqcup_{i \in J} (F \sqcap G_i), \quad F \sqcup (\sqcap_{i \in J} G_i) = \sqcap_{i \in J} (F \sqcup G_i).$

(3) $(\sqcap_{i \in J} F_i)^c = \sqcup_{i \in J} F_i^c, \quad (\sqcup_{i \in J} F_i)^c = \sqcap_{i \in J} F_i^c.$

(4) *If* $F \sqsubseteq G$, *then* $G^c \sqsubseteq F^c$.

Definition 2.4 ([3])**.** Let $F \in S(X_1, E_1)$ and $G \in S(X_2, E_2)$. The cartesian product $F \times G$ is defined as follows:

$(F \times G)(e_1, e_2) = F(e_1) \times G(e_2)$ for each $(e_1, e_2) \in E_1 \times E_2$.

According to this definition the soft set $F \times G$ is a soft set on $X_1 \times X_2$ and its parameter set is $E_1 \times E_2$.

Definition 2.5 ([9]). Let $F \in S(X, E)$ and $x \in X$. $x \widetilde{\in} F$ read as x belongs to the soft set F whenever $x \in F(e)$ for all $e \in E$. For any $x \in X, x \widetilde{\notin} F$ if $x \notin F(e)$ for some $e \in E$.

Definition 2.6 ([9]). Let $x \in X$, then x_E denotes the soft set on X for which $x_E(e) = \{x\}$ for all $e \in E$.

Definition 2.7 ([8]). The soft set $\Delta \in S(X \times X, E)$ is called diagonal soft set on $X \times X$ which is defined by $\Delta(e) = \{(x, x) \mid x \in X\}$, for each $e \in E$.

Definition 2.8. Let $F, G \in S(X \times X, E)$. We define the soft sets $F^{-1}, F \circ G \in S(X \times X, E)$ as follows:

$F^{-1}(e) = \{(x, y) \mid (y, x) \in F(e)\}$ and

$F \circ G(e) = \{(x, y) \mid \exists z \in X$ such that $(x, z) \in G(e)$ and $(z, y) \in F(e)\}$, for all $e \in E$.

$F \in S(X \times X, E)$ is called symmetric if $F = F^{-1}$.

Remark 2.9. Let $F, G, H \in S(X \times X, E)$, then we have the following properties:

(1) If $F \sqsubseteq G$, then $F^{-1} \sqsubseteq G^{-1}$ and $F \circ H \sqsubseteq G \circ H$.

(2) $(F \circ G)^{-1} = G^{-1} \circ F^{-1}$.

(3) $(F \circ G) \circ H = F \circ (G \circ H)$.

Proof. (1) Let $e \in E$ and $(x, y) \in F^{-1}(e)$. Then by the definition, $(y, x) \in F(e) \subseteq G(e)$. So we have $(x, y) \in G^{-1}(e)$. From the arbitrariness of e, we have $F^{-1} \sqsubseteq G^{-1}$.

Let $e \in E$ and $(x, y) \in (F \circ H)(e)$. So, there exists $z \in X$ such that $(x, z) \in H(e)$ and $(x, y) \in F(e)$. Since $F \sqsubseteq G$, $(x, y) \in G(e)$. Hence, by Definition 2.8, $(x, y) \in (G \circ H)(e)$. From the arbitrariness of the parameter e, we obtain $F \circ H \sqsubseteq G \circ H$.

(2) Let $e \in E$. $(x, y) \in (F \circ G)^{-1}(e) \Leftrightarrow (y, x) \in (F \circ G)(e) \Leftrightarrow \exists z \in X : (y, z) \in G(e)$ and $(z, x) \in F(e) \Leftrightarrow \exists z \in X : (z, y) \in G^{-1}(e)$ and $(x, z) \in F^{-1}(e) \Leftrightarrow (x, y) \in (G^{-1} \circ F^{-1})(e)$. So, $(F \circ G)^{-1}(e) = (G^{-1} \circ F^{-1})(e)$, for each $e \in E$. From the arbitrariness of the parameter $e \in E$, we have $(F \circ G)^{-1} = G^{-1} \circ F^{-1}$.

(3) Let $e \in E$. $(x, y) \in ((F \circ G) \circ H)(e) \Leftrightarrow \exists z \in X : (x, z) \in H(e)$ and $(z, y) \in (F \circ G)(e) \Leftrightarrow \exists z, t \in X : (x, z) \in H(e)$ and $(z, t) \in G(e)$ and $(t, y) \in F(e) \Leftrightarrow \exists t \in X : (x, t) \in (G \circ H)(e)$ and $(t, y) \in F(e) \Leftrightarrow (x, y) \in (F \circ (G \circ H))(e)$. So we have $((F \circ G) \circ H)(e) = (F \circ (G \circ H))(e)$, for each $e \in E$. From the arbitrariness of the parameter $e \in E$, we obtain $(F \circ G) \circ H = F \circ (G \circ H)$. □

3. Soft topological spaces

In this section we recall and give some results of soft topological spaces which we need for the next section.

Definition 3.1 ([9]). Let $\mathcal{T}$ be the collection of soft sets on X, then $\mathcal{T}$ is said to be a soft topology on X if

(1) $\Phi, \widetilde{E}$ belongs to $\mathcal{T}$.

(2) the intersection of any two soft sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

(3) the union of any number of soft sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a soft topological space. Every member of $\mathcal{T}$ is called soft open. A soft set G is called soft closed in $(X, \mathcal{T})$ if $G^c \in \mathcal{T}$.

Trivial soft topology, denoted by $\mathcal{T}^0$ contains only Φ and $\widetilde{E}$ while the discrete soft topology, denoted by $\mathcal{T}^1$ contains all soft sets on X.

Example 3.2. (1) Let $X = \{a, b\}$ and E be a nonempty arbitrary set. Consider the family $\mathcal{S} = \{\Phi, \widetilde{E}, a_E\}$, then the pair $(X, \mathcal{S})$ is a soft topological space.

(2) Let $X = E = \mathbb{N}$ and $F \in S(X, E)$ be defined as follows $F_A = \bigcup_{e \in A} F(e)$ with $F(e) = \{1, 2, ..., e\}$. Let us take into account the family $\mathcal{T} = \{F_A \mid A \subseteq E\} \cup \{\Phi, \widetilde{E}\}$, then the pair $(X, \mathcal{T})$ is a soft topological space.

Definition 3.3 ([9])**.** Let $(X, \mathcal{T})$ be a soft topological space, F be a soft set on X and $x \in X$. Then F is called a soft neighborhood of x if there exists a soft open set G such that $x \widetilde{\in} G \sqsubseteq F$.

The neighborhood system of a point x, denoted by $\mathcal{N}_{\mathcal{T}}(x)$, is the family of all its neighborhoods.

Theorem 3.4. *Let $(X, \mathcal{T})$ be a soft topological space. The neighborhood system $\mathcal{N}_{\mathcal{T}}(x)$ in $(X, \mathcal{T})$ has the following properties:*

(1) *If $F \in \mathcal{N}_{\mathcal{T}}(x)$, then $x \widetilde{\in} F$.*

(2) *If $F \in \mathcal{N}_{\mathcal{T}}(x)$ and $F \sqsubseteq G$, then $G \in \mathcal{N}_{\mathcal{T}}(x)$.*

(3) *If $F, G \in \mathcal{N}_{\mathcal{T}}(x)$, then $F \sqcap G \in \mathcal{N}_{\mathcal{T}}(x)$.*

(4) *If $F \in \mathcal{N}_{\mathcal{T}}(x)$, then there exists $G \in \mathcal{N}_{\mathcal{T}}(x)$ such that for all $y \widetilde{\in} G$, we have $F \in \mathcal{N}_{\mathcal{T}}(y)$.*

Conversely, if on a set X a nonempty collection $\mathcal{N}(x)$ of soft subsets of $S(X, E)$ is assigned to each $x \in X$ so as to satisfy (1) *through* (4)*, the result is a soft topology on X, in which the neighborhood system at each $x \in X$ is precisely $\mathcal{N}(x)$.*

Proof. (4) Let $F \in \mathcal{N}_{\mathcal{T}}(x)$, then by the definition of the neighborhood structure, there exists $G \in \mathcal{T}$ such that $x \widetilde{\in} G \sqsubseteq F$. Since $x \widetilde{\in} G$ and $G \in \mathcal{T}$, we have $G \in \mathcal{N}_{\mathcal{T}}(x)$. So, $G \in \mathcal{N}_{\mathcal{T}}(y)$, for all $y \widetilde{\in} G$. Hence by $G \sqsubseteq F$ and by (2), we have $F \in \mathcal{N}_{\mathcal{T}}(y)$.

Conversely, let $\mathcal{N}_{\mathcal{T}}(x) \subseteq S(X, E)$ be a nonempty family of soft sets which satisfies the conditions of Theorem 3.4 for each $x \in X$. Therefore, the family

$$\mathcal{T} = \{G \in S(X, E) \mid \exists F_x \in \mathcal{N}(x) \text{ s. t. } x \widetilde{\in} F_x \sqsubseteq G, \text{ for each } x \widetilde{\in} G\}$$

is a soft topology on X which satisfies the desired properties. It is clear that $\Phi, \widetilde{E} \in \mathcal{T}$.

Let $G, H \in \mathcal{T}$ and $x \widetilde{\in} G \sqcap H$. Then $x \widetilde{\in} G$ or $x \widetilde{\in} H$. By the definition of $\mathcal{T}$ there exist $U_x, V_x \in \mathcal{N}(x)$ such that $x \widetilde{\in} U_x \sqsubseteq G$ and $x \widetilde{\in} V_x \sqsubseteq H$. Then we obtain that $x \widetilde{\in} U_x \sqcap V_x \sqsubseteq G \sqcap H$. Since, by (3), $U_x \sqcap V_x \in \mathcal{N}(x)$, we have $G \sqcap H \in \mathcal{T}$.

Let $\{G_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{T}$ and $x \widetilde{\in} \bigsqcup_{\lambda \in \Lambda} G_\lambda$. Then there exists $\lambda \in \Lambda$ such that $x \widetilde{\in} G_\lambda$ and by the definition of $\mathcal{T}$ there exists $U_x \in \mathcal{N}(x)$ such that $x \widetilde{\in} U_x \sqsubseteq G_\lambda \sqsubseteq \bigsqcup_{\lambda \in \Lambda} G_\lambda$. Hence, we have $\bigsqcup_{\lambda \in \Lambda} G_\lambda \in \mathcal{T}$.

Now, let us show that $\mathcal{N}(x) = \mathcal{N}_{\mathcal{T}}(x)$. Let $U^*(e) := \{y \in X \mid U \in \mathcal{N}(y)\}$, for each $U \in \mathcal{N}(x)$. We can show that U^* satisfies the following properties, $U^* \in \mathcal{T}$ and $x\widetilde{\in}U^* \sqsubseteq U$. Since $U \in \mathcal{N}(x)$, we have $x\widetilde{\in}U^*$ (from definition of U^*). By (4), since $U \in \mathcal{N}(y)$ for all $y\widetilde{\in}U^*$, there exists $V \in \mathcal{N}(y)$ such that $U \in \mathcal{N}(z)$, for each $z\widetilde{\in}V$. So we have $z\widetilde{\in}U^*$. Therefore $V \sqsubseteq U^*$, and we obtain $U^* \in \mathcal{T}$, from the definition of $\mathcal{T}$. On the other hand, $U \in \mathcal{N}(y)$ for each $y\widetilde{\in}U^*$ and hence $y\widetilde{\in}U$, that is $U^* \sqsubseteq U$. Consequently, since $U^* \in \mathcal{T}$ and $x\widetilde{\in}U^* \sqsubseteq U$, we have $U \in \mathcal{N}_{\mathcal{T}}(x)$. Hence, $\mathcal{N}(x) \subseteq \mathcal{N}_{\mathcal{T}}(x)$. Let $U \in \mathcal{N}_{\mathcal{T}}(x)$. From the definition of neighborhood structure there exists $G \in \mathcal{T}$ such that $x\widetilde{\in}G \sqsubseteq U$. Then by the definition of $\mathcal{T}$, there exists $V_x \in \mathcal{N}(x)$ such that $x\widetilde{\in}V_x \sqsubseteq G$. Therefore $V_x \sqsubseteq G \sqsubseteq U$ is satisfied and by (2), we have $U \in \mathcal{N}(x)$. Hence, we obtain that $\mathcal{N}_{\mathcal{T}}(x) \subseteq \mathcal{N}(x)$. Finally, we obtain the desired equality. □

Definition 3.5 ([9])**.** Let $(X, \mathcal{T})$ be a soft topological space and $F \in S(X, E)$. The soft closure of F is the soft set $\overline{F} = \sqcap\{G \in S(X, E) \mid G$ is a soft closed set and $F \sqsubseteq G\}$.

Proposition 3.6 ([9])**.** *Let $(X, \mathcal{T})$ be a soft topological space and $F \in S(X, E)$. F is soft closed set if and only if $F = \overline{F}$.*

Definition 3.7. Let $(X, \mathcal{T})$ be a soft topological space, $F \in S(X, E)$ and $x \in X$. Then x is called a soft adherence point of F if each soft neighborhood of x meets (intersects) F, i.e., $F \sqcap N \neq \Phi$, for each $N \in \mathcal{N}_{\mathcal{T}}(x)$.

Definition 3.8 ([9])**.** Let $(X, \mathcal{T})$ be a soft topological space and $x, y \in X$ such that $x \neq y$. $(X, \mathcal{T})$ is called soft Hausdorff space or soft T_2-space if there exist soft open sets F and G such that $x\widetilde{\in}F$, $y\widetilde{\in}G$ and $F \sqcap G = \Phi$.

4. Soft uniformity structure

In this section, we define soft uniformity, soft uniform base and also we give the relations between soft uniformity and soft topology.

Definition 4.1. We call $\mathcal{D} \subseteq S(X \times X, E)$ is a soft uniformity structure if it satisfies the following:

(1) $\Delta \sqsubseteq U$, for each $U \in \mathcal{D}$.
(2) $U_1 \sqcap U_2 \in \mathcal{D}$, for each $U_1, U_2 \in \mathcal{D}$.
(3) For each $U \in \mathcal{D}$ there exists $V \in \mathcal{D}$ such that $V \circ V \sqsubseteq U$.
(4) For each $U \in \mathcal{D}$ there exists $V \in \mathcal{D}$ such that $V^{-1} \sqsubseteq U$.
(5) If $U \in \mathcal{D}$ and $U \sqsubseteq V$, then $V \in \mathcal{D}$.

The pair $(X, \mathcal{D})$ is called a soft uniform space.

Definition 4.2. Let $(X, \mathcal{D})$ be a soft uniform space. The soft uniform structure $\mathcal{D}$ on X is called a soft Hausdorff uniform structure if it provides $\sqcap\{U \mid U \in \mathcal{D}\} = \Delta$.

Definition 4.3. Let $(X, \mathcal{D})$ be a soft uniform space and $\mathcal{B}$ be a subfamily of $\mathcal{D}$. If for each $D \in \mathcal{D}$ there exists a $B \in \mathcal{B}$ such that $B \sqsubseteq D$, then $\mathcal{B}$ is called a base for the soft uniform structure $\mathcal{D}$.

Remark 4.4. Let X be a nonempty set, E be the parameter set and $\mathcal{B}$ be a subfamily of $S(X \times X, E)$. $\mathcal{B}$ is a base for a soft uniform structure on X if it satisfies the following conditions:

(1) $\Delta \sqsubseteq \mathcal{B}$, for each $B \in \mathcal{B}$.
(2) For each $B_1, B_2 \in \mathcal{B}$ there exists $B_3 \in \mathcal{B}$ such that $B_3 \sqsubseteq B_1 \sqcap B_2$.
(3) For each $B \in \mathcal{B}$ there exists $C \in \mathcal{B}$ such that $C \circ C \sqsubseteq B$.
(4) For each $B \in \mathcal{B}$ there exists $C \in \mathcal{B}$ such that $C^{-1} \sqsubseteq B$.

Definition 4.5. Let $(X, \mathcal{D})$ be a soft uniform space and $\mathcal{S}$ be a subfamily of $\mathcal{D}$. If for all finite intersections of elements of $\mathcal{S}$ form a base for $\mathcal{D}$, then $\mathcal{S}$ is called a subbase for the soft uniform structure $\mathcal{D}$.

Example 4.6. (1) Given any nonempty set X, the collection $\mathcal{D}$ of all soft subsets of $X \times X$ which contain soft diagonal Δ is a uniformity on X, called the discrete soft uniformity. It has for a base the collection consisting of the single soft set Δ.

(2) Given any set X, the collection $\mathcal{D}$ consisting of the single soft set $\widetilde{E}_{X\times X} : E \to P(X \times X), \widetilde{E}_{X\times X}(e) = X \times X$ for each $e \in E$, is a uniformity on X, called the trivial soft uniformity.

(3) Let $E = \mathbb{R}$ and let $D : E \to P(\mathbb{R} \times \mathbb{R})$ be defined as follows:

$$D(e) = \Delta(e) \cup \{(x, y) \mid x > e, y > e\}, \text{ for all } e \in E.$$

Then the soft sets D form a subbase for a soft uniformity on $\mathbb{R}$.

Remark 4.7. (1) If $D \in \mathcal{D}$, then $D^{-1} \in \mathcal{D}$, for any uniformity $\mathcal{D}$ on X.

(2) The requirements (3) and (4) in the definition of a soft uniformity are together equivalent to the single requirement:

$$\text{If } D \in \mathcal{D}, \text{ then } U \circ U^{-1} \sqsubseteq D \text{ for some } U \in \mathcal{D}.$$

First suppose (3) and (4) hold. Then given $D \in \mathcal{D}$, find $U \in \mathcal{D}$ such that $U \circ U \sqsubseteq D$ and $V \in \mathcal{D}$ such that $V^{-1} \sqsubseteq U$. Let $W = U \sqcap V$. Then $W \circ W^{-1} \sqsubseteq D$. Thus the condition above holds.

On the other hand, if the condition above holds, then given $D \in \mathcal{D}$, find $U \in \mathcal{D}$ such that $U \circ U^{-1} \sqsubseteq D$. Then $U^{-1} \sqsubseteq D$ easily, and if $V = U \sqcap U^{-1}$, then $V \in \mathcal{D}$ and $V \circ V \sqsubseteq D$. Thus (3) and (4) hold.

(3) The symmetric soft sets D in $\mathcal{D}$ (i.e., those for which $D = D^{-1}$) form a base for $\mathcal{D}$.

Definition 4.8. For $x \in X, e \in E$ and $D \in \mathcal{D}$, we define a soft set $D[x] : E \to P(X)$ as follows:

$$D[x](e) = \{y \in X \mid (x, y) \in D(e)\}.$$

This is extended to the soft set H on X as follows:

$$D[H] = \bigsqcup_{x \widetilde{\in} H} D[x],$$

that is, for each $e \in E$,

$$D[H](e) = \bigcup_{x \in H(e)} D[x](e) = \{y \in X \mid (x, y) \in D(e) \text{ for some } x \in H(e)\}.$$

Theorem 4.9. (1) *For each $x \in X$, the collection $\mathcal{U}_x = \{D[x] \mid D \in \mathcal{D}\}$ forms a soft neighborhood at x, making X a soft topological space.*

(2) *The soft topology is Hausdorff iff $\mathcal{D}$ is soft Hausdorff.*

Proof. (1) It is enough to show that the family of $\mathcal{U}_x$ satisfies the conditions of Theorem 3.4.

(1): Since for $D[x] \in \mathcal{U}_x$, we have $D \in \mathcal{D}$ and $\Delta \sqsubseteq D$, then $\Delta(e) \subseteq D(e)$, that is, $(x,x) \in D(e)$, for each $e \in E$. So, $x \in D[x](e)$, for each $e \in E$, i.e., $x \widetilde{\in} D[x]$.

(2): If $D[x] \in \mathcal{U}_x$ and $D[x] \sqsubseteq V$, then $V \in \mathcal{U}_x$. In fact, if we define $W(e) = D(e) \cup \{(x,y) \mid y \in V(e)\}$, for each $e \in E$, it is easy to see that $W \in \mathcal{D}$ and $V = W[x]$.

(3): Let $D_1[x], D_2[x] \in \mathcal{U}_x$, then by the definition we have $D_1, D_2 \in \mathcal{D}$ and $D_1 \sqcap D_2 \in \mathcal{D}$.

$$\begin{aligned} y \in (D_1[x] \sqcap D_2[x])(e) &\iff y \in D_1[x](e) \cap D_2[x](e) \\ &\iff (x,y) \in D_1(e) \text{ and } (x,y) \in D_2(e) \\ &\iff (x,y) \in D_1(e) \cap D_2(e) \\ &\iff (x,y) \in (D_1 \sqcap D_2)(e) \\ &\iff y \in (D_1 \sqcap D_2)[x](e), \text{ for each } e \in E. \end{aligned}$$

So, $D_1[x] \sqcap D_2[x] = (D_1 \sqcap D_2)[x]$, i.e., $D_1[x] \sqcap D_2[x] \in \mathcal{U}_x$.

(4): Let $D[x] \in \mathcal{U}_x$. So by the definition, we have $D \in \mathcal{D}$. Then there exists $V \in \mathcal{D}$ such that $V \circ V \sqsubseteq D$. According to this $V[x] \in \mathcal{U}_x$ satisfies the desired property. Because, if $y \widetilde{\in} V[x]$, i.e., $y \in V[x](e)$, then $(x,y) \in V(e)$, for each $e \in E$. Then, $(y,z) \in V(e)$, for all $z \in V[y](e)$ and hence $(x,z) \in (V \circ V)(e) \subseteq D(e)$. It is clear that $z \in D[x](e)$, i.e., $z \widetilde{\in} D[x]$. As a result, from the inequality $V[y] \sqsubseteq U[x]$ and (2), we have $D[x] \in \mathcal{U}_y$.

(2) Let $(X, \mathcal{D})$ be a soft Hausdorff uniform space, that is $\sqcap\{D \mid D \in \mathcal{D}\} = \Delta$ and $x, y \in X$ such that $x \neq y$. So, there exists $e \in E$ such that $(x,y) \notin \Delta(e)$. By the hypothesis, there exists $D \in \mathcal{D}$ such that $(x,y) \notin D(e)$. Since the symmetric elements of $\mathcal{D}$ constitutes a base there exists $V \in \mathcal{D}$ with $V = V^{-1}$ such that $V \circ V \sqsubseteq D$. From here it is clear that $V[x] \sqcap V[y] = \Phi$. Otherwise, $z \widetilde{\in} (V[x] \sqcap V[y])$, i.e., $z \in (V[x] \sqcap V[y])(e)$, for each $e \in E$. So, $(x,z) \in V(e)$ and $(z,y) \in V^{-1}(e) = V(e)$. Then, $(x,y) \in (V \circ V)(e) \subseteq D(e)$, for all $e \in E$. This is a contradiction.

Conversely, let the soft topology be Hausdorff. If $(x,y) \widetilde{\notin} \Delta$, then there exists $e \in E$ with $(x,y) \notin \Delta(e)$. Then for $x \neq y$ there exists $D[x] \in \mathcal{U}_x, V[y] \in \mathcal{U}_y$ such that $D[x] \sqcap V[y] = \Phi$. From here it is clear that $D, V \in \mathcal{D}$ and hence $D \sqcap V \in \mathcal{D}$. Since $(x,y) \widetilde{\notin} D \sqcap V$, we have $(x,y) \widetilde{\notin} \sqcap\{D \mid D \in \mathcal{D}\}$. Hence, $\sqcap\{D \mid D \in \mathcal{D}\} = \Delta$. □

Definition 4.10. The topology thus associated with a diagonal uniformity $\mathcal{D}$ will be called the uniform topology $\mathcal{T}_{\mathcal{D}}$ generated by $\mathcal{D}$. Whenever the soft topology on a soft topological space X can be obtained in this way from a soft uniformity, X is called a uniformizable soft topological space.

Example 4.11. (1) The discrete soft uniformity on a set X generates the discrete soft topology.

(2) The trivial soft uniformity on a set X generates the trivial soft topology.

(3) Let us take into account the soft uniformity base for $\mathbb{R}$ for which consists of the soft sets

$$D(e) = \Delta(e) \cup \{(x,y) \mid x > e, y > e\}, \text{ for all } e \in E$$

which is defined in Example 4.6 (3). For any $x \in \mathbb{R}, D[x] = x_E$ whenever $e \geq x$ and consequently this soft uniformity generates the discrete soft topology.

Theorem 4.12. *The soft open and symmetric elements of $\mathcal{D}$ forms a base for $\mathcal{D}$.*

Proof. A soft open symmetric soft set can be obtained by intersecting an open soft set with its complement, so it suffices to show that the open soft sets form a base, for which purpose it is enough to verify that if $D \in \mathcal{D}$, then $D^c \in \mathcal{D}$. Pick a symmetric U such that $U \circ U \circ U \sqsubseteq D$. So, it is enough to show $U \sqsubseteq D^c$. But if $(x,y)\widetilde{\in}U$, then $U[x] \times U[y] \sqsubseteq D$, for if $(w,z)\widetilde{\in}U[x] \times U[y]$, then $(x,w)\widetilde{\in}U, (y,z)\widetilde{\in}U$ and hence, since $(x,y)\widetilde{\in}U, (w,z)\widetilde{\in}U \circ U \circ U \sqsubseteq D$ has a neighborhood in D, so $U \sqsubseteq D^c$. □

Theorem 4.13. *Let $(X, \mathcal{D})$ be a soft uniform space and $F \in S(X,E)$. Then we have $\overline{F} = \sqcap_{D\in\mathcal{D}} D[F]$.*

Proof. If we consider $x\widetilde{\in}D[y]$ if and only if $y\widetilde{\in}D^{-1}[x]$ and $D^{-1} \in \mathcal{D}$, for all $D \in \mathcal{D}$, we have

$$\begin{aligned} x\widetilde{\in}\overline{F} &\iff D[x] \sqcap F \neq \Phi, \text{ for each } D \in \mathcal{D}. \\ &\iff \exists\, y\widetilde{\in}F : y\widetilde{\in}D[x] \text{ for each } D \in \mathcal{D}. \\ &\iff \exists\, y\widetilde{\in}F : x\widetilde{\in}D^{-1}[y] \text{ for each } D \in \mathcal{D}. \\ &\iff x\widetilde{\in}D^{-1}[F], \text{ for each } D \in \mathcal{D}. \\ &\iff x\widetilde{\in}D[F], \text{ for each } D \in \mathcal{D}. \end{aligned}$$

□

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