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Isomorphism on interval-valued fuzzy graphs

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Abstract. In this paper we discuss some properties of the self complementary and self weak complementary interval-valued fuzzy graphs, and get a sufficient condition for a interval-valued fuzzy graph to be the self weak complementary interval-valued fuzzy graph. Also we investigate relations between operations union, join, and complement on interval-valued fuzzy graphs.

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1. Introduction

In 1965, Zadeh [18] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. Since then, the theory of fuzzy sets has became a vigorous area of research in different disciplines including logic, topology, algebra, analysis, information theory, artificial intelligence, operations research, neural networks and planning etc [6], [8], [9], [10], [12]. The fuzzy graph theory as a generalization of Euler,s graph theory was first introduced by Rosenfeld [15] in 1975. The fuzzy relations between fuzzy sets were first considered by Rosenfeld and he developed the structure of fuzzy graphs obtaining analogs of several graph theoretical concepts. Later, Bhattacharya [5] gave some remarks on fuzzy graphs, and some operation of fuzzy graphs were introduced by Mordeson and Peng [11]. The complement of fuzzy graphs was studied by Sunitha and Vijayakumar [16]. In 1975, Zadeh [17] introduced the notion of interval valued fuzzy subset of a set as an extension of fuzzy set in which the values of the membership degrees are interval of numbers instead of the numbers. Hongmei and Lianhua gave the definition of interval-valued fuzzy graph in [7]. In 2011, Muhammad Akram, Wieslaw A. Dudek [1] gave some operations on interval-valued fuzzy graph. M. Akram and B. Davvaz [3] investigated strong intuitionistic fuzzy graphs. The definition of bipolar fuzzy graphs was proposed by Muhammad Aram [2]. Then M. Akram and M.G. Karunambigai [4] define length, distance, eccentricity, radius and diameter of a bipolar fuzzy graph and introduce the concept of self centered bipolar fuzzy graphs. A. Nagoorgani and J. Malarvizhi [13, 14] investigated isomorphism properties on fuzzy graphs. Also they defined the self complementary fuzzy graphs. Bhutani in [6] introduced the concept of weak isomorphism and isomorphism between fuzzy graphs. In this study we define the self complement and self weak complement interval-valued fuzzy graphs and some properties of its are discussed. We study some properties of isomorphism and complement on interval-valued fuzzy graphs.

2. Preliminaries

A fuzzy graph with S, a non empty finite set as the underlying set is a pair $G:(\sigma,\mu)$ where $\sigma : S \to [0,1]$ is a fuzzy subset of $S, \mu : S\times S \to [0,1]$ is a symmetric fuzzy relation on the fuzzy subset σ such that $\mu(x,y) \leq \min(\sigma(x),\sigma(y)),\ \ \forall\, x,y \in S$. A fuzzy relation μ is symmetric if $\mu(x,y) = \mu(x,y)$ for all $x,y \in S$. The underlying crisp graph of the fuzzy graph $G:(\sigma,\mu)$ is denoted as $G^*:(\sigma^*,\mu^*)$ where $\sigma^* = \{x \in S : \sigma(x) > 0\}$ and $\mu^* = \{(x,y) \in S\times S : \mu(x,y) > 0\}$. If $\mu(x,y) > 0$, then x and y are called neighbors. For simplicity, an edge (x,y) will be denoted by xy.

The interval-valued fuzzy set A in V is defined by

$$A = \{(x,[\mu_{A^-}(x),\mu_{A^+}(x)]) : x \in V\},$$

where $\mu_{A^-}(x)$ and $\mu_{A^+}(x)$ are fuzzy subsets of V such that $\mu_{A^-}(x) \leq \mu_{A^+}(x)$ for all $x \in V$.

If $G^*:(V,E)$ is a graph, then by an interval-valued fuzzy relation B on a set E we mean an interval-valued set such that for all $xy \in E$,

$$\begin{aligned} \mu_{B^-}(x,y) &\leq \min(\mu_{A^-}(x),\mu_{A^-}(y)),\\ \mu_{B^+}(x,y) &\leq \min(\mu_{A^+}(x),\mu_{A^+}(y)). \end{aligned}$$

Definition 2.1. By an interval-valued fuzzy graph of a graph $G^*:(V,E)$ we mean a pair $G=(A,B)$, where $A=[\mu_{A^-},\mu_{A^+}]$ is an interval-valued fuzzy set on V and $B=[\mu_{B^-},\mu_{B^+}]$ is an interval-valued fuzzy relation on E.

Definition 2.2. Let $G:(\sigma,\mu)$ be a fuzzy graph on underlying set S. The complement of G is defined as $\overline{G}:(\sigma,\overline{\mu})$ where

$$\overline{\mu}(x,y) = \min(\sigma(x),\sigma(y)) - \mu(x,y) \quad \forall\, x,y \in S.$$

When $G:(\sigma,\mu)$ is a fuzzy graph, $\overline{G}:(\sigma,\overline{\mu})$ is also a fuzzy graph.

Definition 2.3. The complement of an interval-valued fuzzy graph $G:(A,B)$ of a graph $G^*:(V,E)$ is an interval-valued fuzzy graph $\overline{G}:(\overline{A},\overline{B})$ of $\overline{G^*}:(V,V\times V)$, where $\overline{A}=A=[\mu_{A^-},\mu_{A^+}]$ and $\overline{B}=[\overline{\mu_{B^-}},\overline{\mu_{B^+}}]$ is defined by

$$\begin{aligned} \overline{\mu_{B^-}}(x,y) &= \min(\mu_{A^-}(x),\mu_{A^-}(y)) - \mu_{B^-}(xy) \quad \forall\, x,y \in V\\ \overline{\mu_{B^+}}(x,y) &= \min(\mu_{A^+}(x),\mu_{A^+}(y)) - \mu_{B^+}(xy) \quad \forall\, x,y \in V \end{aligned}$$

Definition 2.4. Let $G:(\sigma,\mu)$ and $G':(\sigma',\mu')$ be two fuzzy graphs with underlying sets S and S', respectively. A homomorphism of fuzzy graphs $h:G\to G'$ is a map $h:S\to S'$ which satisfies $\sigma(x)\le\sigma'(h(x)),\quad \forall\, x\in S$ and

$$\mu(xy)\le\mu'(h(x)h(y))\quad \forall\, x,y\in S.$$

A weak isomorphism $h:G\to G'$ is a bijective homomorphism that satisfies

$$\sigma(x)=\sigma'(h(x)),\quad \forall\, x\in S.$$

A co-weak isomorphism $h:G\to G'$ is a bijective homomorphism that satisfies

$$\mu(xy)=\mu'(h(x)h(y)),\quad \forall\, x,y\in S.$$

An isomorphism $h:G\to G'$ is a bijective homomorphism that satisfies

$$\begin{cases}\sigma(x)=\sigma'(h(x)) & \forall\, x\in S\\ \mu(xy)=\mu'(h(x)h(y)) & \forall\, x,y\in S\end{cases}$$

Throughout this paper $G_1:(A_1,B_1)$ and $G_2:(A_2,B_2)$ are taken to be the interval-valued fuzzy graphs of $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$, respectively.

Definition 2.5. Let $G_1:(A_1,B_1)$ and $G_2:(A_2,B_2)$ be two interval-valued fuzzy graphs. A homomorphism $f:G_1\to G_2$ is a mapping $f:V_1\to V_2$ such that

(a) $\mu_{A_1^-}(x)\le\mu_{A_2^-}(f(x)),\mu_{A_1^+}(x)\le\mu_{A_2^+}(f(x)),\quad \forall\, x\in V_1,$

(b) $\mu_{B_1^-}(xy)\le\mu_{B_2^-}(f(x)f(y)),\mu_{B_1^+}(xy)\le\mu_{B_2^+}(f(x)f(y)),\quad \forall\, x,y\in V_1,$

A bijective homomorphism with the property

(c) $\mu_{A_1^-}(x)=\mu_{A_2^-}(f(x)),\mu_{A_1^+}(x)=\mu_{A_2^+}(f(x)),\quad \forall\, x\in V_1$

is called a weak isomorphism.

A bijective homomorphism $f:G_1\to G_2$ such that

(d) $\mu_{B_1^-}(xy)=\mu_{B_2^-}(f(x)f(y)),\mu_{B_1^+}(xy)=\mu_{B_2^+}(f(x)f(y)),\quad \forall\, x,y\in V_1$

is called a co-weak isomorphism.

A bijective mapping $f:G_1\to G_2$ satisfying (c) and (d) is called an isomorphism.

Definition 2.6. The union $G_1\cup G_2=(A_1\cup A_2,B_1\cup B_2)$ of two interval-valued fuzzy graphs G_1 and G_2 is defined as follows:

$$(A)\begin{cases}(\mu_{A_1^-}\cup\mu_{A_2^-})(x)=\mu_{A_1^-} & if\ x\in V_1, x\notin V_2\\ (\mu_{A_1^-}\cup\mu_{A_2^-})(x)=\mu_{A_2^-} & if\ x\in V_2, x\notin V_1\\ (\mu_{A_1^-}\cup\mu_{A_2^-})(x)=\max(\mu_{A_1^-}(x),\mu_{A_2^-}(x)) & if\ x\in V_1\cap V_2,\end{cases}$$

$$(B)\begin{cases}(\mu_{A_1^+}\cup\mu_{A_2^+})(x)=\mu_{A_1^+} & if\ x\in V_1, x\notin V_2\\ (\mu_{A_1^+}\cup\mu_{A_2^+})(x)=\mu_{A_2^+} & if\ x\in V_2, x\notin V_1\\ (\mu_{A_1^+}\cup\mu_{A_2^+})(x)=\max(\mu_{A_1^+}(x),\mu_{A_2^+}(x)) & if\ x\in V_1\cap V_2,\end{cases}$$

$$(C)\begin{cases}(\mu_{B_1^-}\cup\mu_{B_2^-})(xy)=\mu_{B_1^-}(xy) & if\ xy\in E_1, xy\notin E_2\\ (\mu_{B_1^-}\cup\mu_{B_2^-})(xy)=\mu_{B_2^-}(xy) & if\ xy\in E_2, xy\notin E_1\\ (\mu_{B_1^-}\cup\mu_{B_2^-})(xy)=\max(\mu_{B_1^-}(xy),\mu_{B_2^-}(xy)) & if\ xy\in E_1\cap E_2,\end{cases}$$

$$(D)\begin{cases}(\mu_{B_1^+}\cup\mu_{B_2^+})(xy)=\mu_{B_1^+}(xy) & if\ xy\in E_1, xy\notin E_2\\ (\mu_{B_1^+}\cup\mu_{B_2^+})(xy)=\mu_{B_2^+}(xy) & if\ xy\in E_2, xy\notin E_1\\ (\mu_{B_1^+}\cup\mu_{B_2^+})(xy)=\max(\mu_{B_1^+}(xy),\mu_{B_2^+}(xy)) & if\ xy\in E_1\cap E_2,\end{cases}$$

Proposition 2.7 ([1]). *The union of two interval-valued fuzzy graphs is an interval-valued fuzzy graph.*

Definition 2.8. The join $G_1+G_2 = (A_1+A_2, B_1+B_2)$ of two interval-valued fuzzy graphs G_1 and G_2 is defined as follows:

$$(A)\begin{cases} (\mu_{A_1^-}+\mu_{A_2^-})(x) = (\mu_{A_1^-}\cup\mu_{A_2^-})(x) \\ (\mu_{A_1^+}+\mu_{A_2^+})(x) = (\mu_{A_1^+}\cup\mu_{A_2^+})(x) \end{cases} if\ x\in V_1\cup V_2,$$

$$(B)\begin{cases} (\mu_{B_1^-}+\mu_{B_2^-})(xy) = (\mu_{B_1^-}\cup\mu_{B_2^-})(xy) \\ (\mu_{B_1^+}+\mu_{B_2^+})(xy) = (\mu_{B_1^+}\cup\mu_{B_2^+})(xy) \end{cases} if\ xy\in E_1\cup E_2,$$

$$(C)\begin{cases} (\mu_{B_1^-}+\mu_{B_2^-})(xy) = \min(\mu_{A_1^-}(x),\mu_{A_2^-}(y)) \\ (\mu_{B_1^+}+\mu_{B_2^+})(xy) = \min(\mu_{A_1^+}(x),\mu_{A_2^+}(y)) \end{cases} if\ xy\in E',$$

where E' is the set of all edges joining the nodes of V_1 and V_2.

Proposition 2.9 ([1]). *The join of two interval-valued fuzzy graphs is an interval-valued fuzzy graph.*

3. Self complement and self weak complement interval-valued fuzzy graphs

In this section we will discuss some properties of the self complement and self weak complement on interval-valued fuzzy graphs.

Definition 3.1. An interval-valued fuzzy graph $G = (A, B)$ of a graph $G^* = (V, E)$ is said to be self weak complementary interval-valued fuzzy graph if G is weak isomorphic with it's complement $\overline{G}$, i.e there exists a bijective homomorphism $f : G \to \overline{G}$ such that for all $x, y \in V$

$$\begin{cases} \mu_{A^-}(x) = \overline{\mu_{A^-}}(f(x)) \\ \mu_{A^+}(x) = \overline{\mu_{A^+}}(f(x)) \end{cases}$$

and

$$\begin{cases} \mu_{B^-}(x,y) \le \overline{\mu_{B^-}}(f(x)f(y)) \\ \mu_{B^+}(x,y) \le \overline{\mu_{B^+}}(f(x)f(y)). \end{cases}$$

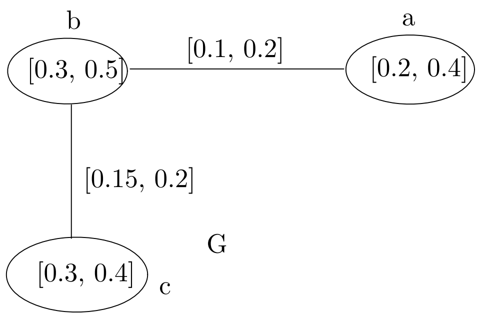
Definition 3.2. An interval-valued fuzzy graph G is said to be self complementary if $G \cong \overline{G}$.

Example 3.3. Consider a graph $G^* = (V, E)$ such that $V = \{a, b, c\}$, $E = \{ab, bc\}$. Then an interval-valued fuzzy graph $G = (A, B)$, where

$$A = \Big\langle\Big(\frac{a}{0.2},\frac{b}{0.3},\frac{c}{0.3}\Big),\Big(\frac{a}{0.4},\frac{b}{0.5},\frac{c}{0.4}\Big)\Big\rangle,$$

$$B = \Big\langle\Big(\frac{ab}{0.1},\frac{bc}{0.15}\Big),\Big(\frac{ab}{0.2},\frac{bc}{0.2}\Big)\Big\rangle,$$

is self weak complementary. In fact, identity bijective mapping is a weak isomorphism from G to $\overline{G}$.

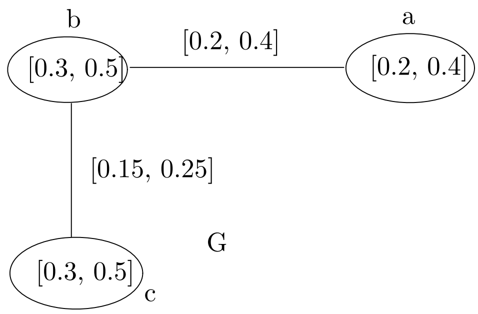


Example 3.4. Consider a graph $G^* = (V, E)$ such that $V = \{a, b, c\}$, $E = \{ab, bc\}$. Then an interval-valued fuzzy graph $G = (A, B)$, where

$$A = \Big\langle \Big(\frac{a}{0.2}, \frac{b}{0.3}, \frac{c}{0.3}\Big), \Big(\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.5}\Big) \Big\rangle,$$

$$B = \Big\langle \Big(\frac{ab}{0.2}, \frac{bc}{0.15}\Big), \Big(\frac{ab}{0.4}, \frac{bc}{0.25}\Big) \Big\rangle,$$

is self-complementary. In fact, bijective mapping $f : G \to \overline{G}$ such that $a \to a$, $b \to c$, $c \to b$ is an isomorphism.



Theorem 3.5. *Let $G = (A, B)$ be a self complementary interval-valued fuzzy graph of a graph $G^* : (V, E)$. Then*

$$\begin{cases} \sum_{x \neq y} \mu_{B^-}(xy) = \frac{1}{2} \sum_{x \neq y} \min(\mu_{A^-}(x), \mu_{A^-}(y)) \\ \sum_{x \neq y} \mu_{B^+}(xy) = \frac{1}{2} \sum_{x \neq y} \min(\mu_{A^+}(x), \mu_{A^+}(y)). \end{cases}$$

Proof. Let $G = (A, B)$ be a self complementary interval-valued fuzzy graph of a graph $G^*(V, E)$. Then there exists an isomorphism $g : G \to \overline{G}$ such that for all $x, y \in V$ we have

$$\begin{cases} \mu_{A^-}(x) = \overline{\mu_{A^-}}(g(x)) \\ \mu_{A^+}(x) = \overline{\mu_{A^+}}(g(x)) \end{cases} \quad \forall x \in V$$

and

$$\begin{cases} \mu_{B^-}(xy) = \overline{\mu_{B^-}}(g(x)g(y)) \\ \mu_{B^+}(xy) = \overline{\mu_{B^+}}(g(x)g(y)) \end{cases} \quad \forall x, y \in V.$$

Now by definition of $\overline{G}$, for all $x, y \in V$ we have

$$\begin{cases} \overline{\mu}_{B^-}(g(x)g(y)) = \min(\mu_{A^-}(g(x)), \mu_{A^-}(g(y))) - \mu_{B^-}(g(x)g(y)) \\ \overline{\mu}_{B^+}(g(x)g(y)) = \min(\mu_{A^+}(g(x)), \mu_{A^+}(g(y))) - \mu_{B^+}(g(x)g(y)) \end{cases}$$

i.e.,

$$\begin{cases} \mu_{B^-}(xy) = \min(\mu_{A^-}(g(x)), \mu_{A^-}(g(x))) - \mu_{B^-}(g(x)g(y)) \\ \mu_{B^+}(xy) = \min(\mu_{A^+}(g(x)), \mu_{A^+}(g(x))) - \mu_{B^+}(g(x)g(y)) \end{cases}.$$

Hence

$$\begin{cases} \sum_{x \neq y} \mu_{B^-}(xy) + \sum_{x \neq y} \mu_{B^-}(g(x)g(y)) = \sum_{x \neq y} \min(\mu_{A^-}(x), \mu_{A^-}(y)) \\ \sum_{x \neq y} \mu_{B^+}(xy) + \sum_{x \neq y} \mu_{B^+}(g(x)g(y)) = \sum_{x \neq y} \min(\mu_{A^+}(x), \mu_{A^+}(y)) \end{cases}$$

i.e.,

$$\begin{cases} \sum_{x \neq y} \mu_{B^-}(xy) = \frac{1}{2} \sum_{x \neq y} \min(\mu_{A^-}(x), \mu_{A^-}(y)) \\ \sum_{x \neq y} \mu_{B^+}(xy) = \frac{1}{2} \sum_{x \neq y} \min(\mu_{A^+}(x), \mu_{A^+}(y)) \end{cases}.$$

□

Theorem 3.6. *Let $G = (A, B)$ be a self weak complementary interval-valued fuzzy graph of a graph $G^* : (V, E)$. Then*

$$\begin{cases} \sum_{x \neq y} \mu_{B^-}(xy) \leq \frac{1}{2} \sum_{x \neq y} \min(\mu_{A^-}(x), \mu_{A^-}(y)) \\ \sum_{x \neq y} \mu_{B^+}(xy) \leq \frac{1}{2} \sum_{x \neq y} \min(\mu_{A^+}(x), \mu_{A^+}(y)). \end{cases}$$

Proof. Let $G = (A, B)$ be a self weak complementary interval-valued fuzzy graph of a graph $G^* : (V, E)$. Then there exists a weak isomorphism $h : G \to \overline{G}$ such that for all $x, y \in V$ we have

$$\begin{cases} \mu_{A^-}(x) = \overline{\mu_{A^-}}(h(x)) = \mu_{A^-}(h(x)) \\ \mu_{A^+}(x) = \overline{\mu_{A^+}}(h(x)) = \mu_{A^+}(h(x)) \\ \mu_{B^-}(xy) \leq \overline{\mu_{B^-}}(h(x)h(y)) \\ \mu_{B^+}(xy) \leq \overline{\mu_{B^+}}(h(x)h(y)) \end{cases}$$

Using the definition of complement in the above inequality, for all $x, y \in V$ we have

$$\begin{cases} \mu_{B^-}(xy) \leq \overline{\mu_{B^-}}(h(x)h(y)) = \min(\mu_{A^-}(h(x)), \mu_{A^-}(h(y))) - \mu_{B^-}(h(x)h(y)) \\ \mu_{B^+}(xy) \leq \overline{\mu_{B^+}}(h(x)h(y)) = \min(\mu_{A^+}(h(x)), \mu_{A^+}(h(y))) - \mu_{B^+}(h(x)h(y)) \end{cases}$$

$$\Rightarrow \begin{cases} \mu_{B^-}(xy) + \mu_{B^-}(h(x)h(y)) \leq \min(\mu_{A^-}(h(x)), \mu_{A^-}(h(y))) \\ \mu_{B^+}(xy) + \mu_{B^+}(h(x)h(y)) \leq \min(\mu_{A^+}(h(x)), \mu_{A^+}(h(y))) \end{cases}$$

$$\Rightarrow \begin{cases} \sum_{x \neq y} \mu_{B^-}(xy) + \sum_{x \neq y} \mu_{B^-}(h(x)h(y)) \leq \sum_{x \neq y} \min(\mu_{A^-}(h(x)), \mu_{A^-}(h(x))) \\ \sum_{x \neq y} \mu_{B^+}(xy) + \sum_{x \neq y} \mu_{B^+}(h(x)h(y)) \leq \sum_{x \neq y} \min(\mu_{A^+}(h(x)), \mu_{A^+}(h(x))) \end{cases}$$

$$\Rightarrow \begin{cases} 2\sum\limits_{x\neq y}\mu_{B^-}(xy) \leq \sum\limits_{x\neq y}\min(\mu_{A^-}(x),\mu_{A^-}(y)) \\ 2\sum\limits_{x\neq y}\mu_{B^+}(xy) \leq \sum\limits_{x\neq y}\min(\mu_{A^+}(x),\mu_{A^+}(y)) \end{cases}.$$

Hence

$$\begin{cases} \sum\limits_{x\neq y}\mu_{B^-}(xy) \leq \frac{1}{2}\sum\limits_{x\neq y}\min(\mu_{A^-}(x),\mu_{A^-}(y)) \\ \sum\limits_{x\neq y}\mu_{B^+}(xy) \leq \frac{1}{2}\sum\limits_{x\neq y}\min(\mu_{A^+}(x),\mu_{A^+}(y)) \end{cases}$$

□

Theorem 3.7. *Let $G = (A, B)$ be a interval-valued fuzzy graph of a graph $G^* : (V, E)$. If*

$$\begin{cases} \mu_{B^-}(xy) \leq \frac{1}{2}\min(\mu_{A^-}(x),\mu_{A^-}(y)) \\ \mu_{B^+}(xy) \leq \frac{1}{2}\min(\mu_{A^+}(x),\mu_{A^+}(y)) \end{cases}$$

for all $x, y \in V$, then $G = (A, B)$ is a self weak complementary interval-valued fuzzy graph.

Proof. Consider the identity map $I : V \to V$, $\begin{cases} \mu_{A^-}(x) = \mu_{A^-}(I(x)) \\ \mu_{A^+}(x) = \mu_{A^+}(I(x)) \end{cases} \quad \forall\ x \in V.$

By definition of $\overline{\mu_B}$ we have

$$\begin{cases} \overline{\mu_{B^-}}(xy) = \min(\mu_{A^-}(x),\mu_{A^-}(y)) - \mu_{B^-}(xy) \\ \overline{\mu_{B^+}}(xy) = \min(\mu_{A^+}(x),\mu_{A^+}(y)) - \mu_{B^+}(xy) \end{cases} \quad \forall\ x, y \in V.$$

Hence

$$\begin{cases} \overline{\mu_{B^-}}(xy) \geq \min(\mu_{A^-}(x),\mu_{A^-}(y)) - \frac{1}{2}\min(\mu_{A^-}(x),\mu_{A^-}(y)) \\ = \frac{1}{2}\min(\mu_{A^-}(x),\mu_{A^-}(y)) \geq \mu_{B^-}(xy) \\ \overline{\mu_{B^+}}(xy) \geq \min(\mu_{A^+}(x),\mu_{A^+}(y)) - \frac{1}{2}\min(\mu_{A^+}(x),\mu_{A^+}(y)) \\ = \frac{1}{2}\min(\mu_{A^+}(x),\mu_{A^+}(y)) \geq \mu_{B^+}(xy) \end{cases} \quad \forall\ x, y \in V$$

i.e.,

$$\begin{cases} \mu_{B^-}(xy) \leq \overline{\mu_{B^-}}(I(x)I(y)) \\ \mu_{B^+}(xy) \leq \overline{\mu_{B^+}}(I(x)I(y)) \end{cases} \quad \forall\ x, y \in V.$$

□

4. Complement and Isomorphism of an Interval-valued fuzzy graph

In this section we discuss some of the properties of isomorphism and complement on interval-valued fuzzy graphs.

Theorem 4.1. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two interval-valued fuzzy graph such that $V_1 \cap V_2 = \phi$. Then $\overline{G_1 + G_2} \cong \overline{G_1} \cup \overline{G_2}$*

Proof. We shall prove that the identity map is the required isomorphism. Let I : $V_1 \cup V_2 \to V_1 \cup V_2$ be the identity map. We prove that for all $x, y \in V$

$$\begin{cases} (\overline{\mu_{A_1^-} + \mu_{A_2^-}})(x) = (\overline{\mu_{A_1^-}} \cup \overline{\mu_{A_2^-}})(x) \\ (\overline{\mu_{A_1^+} + \mu_{A_2^+}})(x) = (\overline{\mu_{A_1^+}} \cup \overline{\mu_{A_2^+}})(x) \end{cases}$$

and

$$\begin{cases} (\overline{\mu_{B_1^-} + \mu_{B_2^-}})(xy) = (\overline{\mu_{B_1^-}} \cup \overline{\mu_{B_2^-}})(xy) \\ (\overline{\mu_{B_1^+} + \mu_{B_2^+}})(xy) = (\overline{\mu_{B_1^+}} \cup \overline{\mu_{B_2^+}})(xy). \end{cases}$$

For all $x, y \in V$ we have

$$\begin{aligned} \overline{(\mu_{A_1^-} + \mu_{A_2^-})}(x) = (\mu_{A_1^-} \cup \mu_{A_2^-})(x) &= \begin{cases} \mu_{A_1^-}(x) & if\ x \in V_1 \\ \mu_{A_2^-}(x) & if\ x \in V_2 \end{cases} \\ &= \begin{cases} \overline{\mu_{A_1^-}}(x) & if\ x \in V_1 \\ \overline{\mu_{A_2^-}}(x) & if\ x \in V_2 \end{cases} \\ &= (\overline{\mu_{A_1^-}} \cup \overline{\mu_{A_2^-}})(x), \\ \overline{(\mu_{A_1^+} + \mu_{A_2^+})}(x) = (\mu_{A_1^+} + \mu_{A_2^+})(x) &= \begin{cases} \mu_{A_1^+}(x) & if\ x \in V_1 \\ \mu_{A_2^+}(x) & if\ x \in V_2 \end{cases} \\ &= \begin{cases} \overline{\mu_{A_1^+}}(x) & if\ x \in V_1 \\ \overline{\mu_{A_2^+}}(x) & if\ x \in V_2 \end{cases} \\ &= (\overline{\mu_{A_1^+}} \cup \overline{\mu_{A_2^+}})(x), \end{aligned}$$

Also

$$(\overline{\mu_{B_1^-} + \mu_{B_2^-}})(xy) = \min((\mu_{A_1^-} + \mu_{A_2^-})(x), (\mu_{A_1^-} + \mu_{A_2^-})(y)) - (\mu_{B_1^-} + \mu_{B_2^-})(xy)$$

$$= \begin{cases} \min((\mu_{A_1^-} \cup \mu_{A_2^-})(x), (\mu_{A_1^-} \cup \mu_{A_2^-})(y)) - (\mu_{B_1^-} \cup \mu_{B_2^-})(xy) \text{ if } xy \in E_1 \cup E_2 \\ \min((\mu_{A_1^-} \cup \mu_{A_2^-})(x), (\mu_{A_1^-} \cup \mu_{A_2^-})(y)) - \min(\mu_{A_1^-}(x), \mu_{A_2^-}(y)) \text{ if } xy \in E' \end{cases}$$

$$= \begin{cases} \min(\mu_{A_1^-}(x), \mu_{A_2^-}(y)) - \mu_{B_1^-}(xy) & \text{if } \ xy \in E_1 \\ \min(\mu_{A_2^-}(x), \mu_{A_2^-}(y)) - \mu_{B_2^-}(xy) & \text{if } \ xy \in E_2 \\ \min(\mu_{A_1^-}(x), \mu_{A_1^-}(y)) - \min(\mu_{A_1^-}(x), \mu_{A_2^-}(y)) & \text{if } \ xy \in E' \end{cases}$$

$$= \begin{cases} \overline{\mu_{B_1^-}}(xy) & \text{if } \ xy \in E_1 \\ \overline{\mu_{B_2^-}}(xy) & \text{if } \ xy \in E_2 \\ 0 & \text{if } \ xy \in E' \end{cases}$$

$$= (\overline{\mu_{B_1^-}} \cup \overline{\mu_{B_2^-}})(xy).$$

$$(\overline{\mu_{B_1^+} + \mu_{B_2^+}})(xy) = \min((\mu_{A_1^+} + \mu_{A_2^+})(x), (\mu_{A_1^+} + \mu_{A_2^+})(y)) - (\mu_{B_1^+} + \mu_{B_2^+})(xy)$$

$$= \begin{cases} \min((\mu_{A_1^+} \cup \mu_{A_2^+})(x), (\mu_{A_1^+} \cup \mu_{A_2^+})(y)) - (\mu_{B_1^+} \cup \mu_{B_2^+})(xy) \text{ if } \ xy \in E_1 \cup E_2 \\ \min((\mu_{A_1^+} \cup \mu_{A_2^+})(x), (\mu_{A_1^+} \cup \mu_{A_2^+})(y)) - \min(\mu_{A_1^+}(x), \mu_{A_2^+}(y)) \text{ if } \ xy \in E' \end{cases}$$

$$= \begin{cases} \min(\mu_{A_1^+}(x), \mu_{A_1^+}(y)) - \mu_{B_1^+}(xy) & \text{if } \ xy \in E_1 \\ \min(\mu_{A_2^+}(x), \mu_{A_2^+}(y)) - \mu_{B_2^+}(xy) & \text{if } \ xy \in E_2 \\ \min(\mu_{A_1^+}(x), \mu_{A_2^+}(y)) - \min(\mu_{A_1^+}(x), \mu_{A_2^+}(y)) & \text{if } \ xy \in E' \end{cases}$$

$$= \begin{cases} \overline{\mu_{B_1^+}}(xy) & if \quad xy \in E_1 \\ \overline{\mu_{B_2^+}}(xy) & if \quad xy \in E_2 \\ 0 & if \quad xy \in E' \end{cases}$$

$$= (\overline{\mu_{B_1^+}} \cup \overline{\mu_{B_2^+}})(xy).$$

□

Theorem 4.2. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two interval-valued fuzzy graph such that $V_1 \cap V_2 = \phi$. Then $\overline{G_1 \cup G_2} \cong \overline{G_1} \cup \overline{G_2}$.*

Proof. We shall prove that the identity map is the required isomorphism. For all $x, y \in V$ we have

$$\begin{aligned}
(\overline{\mu_{A_1^-} \cup \mu_{A_2^-}})(x) = (\mu_{A_1^-} \cup \mu_{A_2^-})(x) &= \begin{cases} \mu_{A_1^-}(x) & if \quad x \in V_1 \\ \mu_{A_2^-}(x) & if \quad x \in V_2 \end{cases} \\
&= (\overline{\mu_{A_1^-}} \cup \overline{\mu_{A_2^-}})(x) = (\overline{\mu_{A_1^-}} + \overline{\mu_{A_2^-}})(x), \\
(\overline{\mu_{A_1^+} \cup \mu_{A_2^+}})(x) = (\mu_{A_1^+} \cup \mu_{A_2^+})(x) &= \begin{cases} \mu_{A_1^+}(x) & if \quad x \in V_1 \\ \mu_{A_2^+}(x) & if \quad x \in V_2 \end{cases} \\
&= \begin{cases} \overline{\mu}_{A_1^+}(x) & if \quad x \in V_1 \\ \overline{\mu}_{A_2^+}(x) & if \quad x \in V_2 \end{cases} \\
&= (\overline{\mu_{A_1^+}} \cup \overline{\mu_{A_2^+}})(x),
\end{aligned}$$

$$\overline{(\mu_{B_1^-} \cup \mu_{B_2^-})}(xy) = \min((\mu_{A_1^-} \cup \mu_{A_2^-})(x), (\mu_{A_1^-} \cup \mu_{A_2^-})(y)) - (\mu_{B_1^-} \cup \mu_{B_2^-})(xy)$$

$$= \begin{cases} \min(\mu_{A_1^-}(x), \mu_{A_1^-}(y)) - \mu_{B_1^-}(xy) & if \quad xy \in E_1 \\ \min(\mu_{A_2^-}(x), \mu_{A_2^-}(y)) - \mu_{B_2^-}(xy) & if \quad xy \in E_2 \\ \min(\mu_{A_1^-}(x), \mu_{A_2^-}(y)) - 0 & if \quad x \in V_1, y \in V_2 \end{cases}$$

$$= \begin{cases} \overline{\mu_{B_1^-}}(xy) & if \quad xy \in E_1 \\ \overline{\mu_{B_2^-}}(xy) & if \quad xy \in E_2 \\ \min(\mu_{A_1^-}(x), \mu_{A_2^-}(y)) & if x \in V_1, y \in V_2 \end{cases}$$

$$= \begin{cases} (\overline{\mu_{B_1^-}} \cup \overline{\mu_{B_2^-}})(xy) & if \quad xy \in E_1 \\ (\overline{\mu_{B_1^-}} \cup \overline{\mu_{B_2^-}})(xy) & if \quad xy \in E_2 \\ \min(\mu_{A_1^-}(x), \mu_{A_2^-}(y)) & if xy \in E' \end{cases}$$

$$= (\overline{\mu_{B_1^-}} + \overline{\mu_{B_2^-}})(xy),$$

$$\overline{(\mu_{B_1^+} \cup \mu_{B_2^+})}(xy) = \min((\mu_{A_1^+} \cup \mu_{A_2^+})(x), (\mu_{A_1^+} \cup \mu_{A_2^+})(y)) - (\mu_{B_1^+} \cup \mu_{B_2^+})(xy)$$

$$= \begin{cases} \min(\mu_{A_1^+}(x), \mu_{A_1^+}(y)) - \mu_{B_1^+}(xy) & if \quad xy \in E_1 \\ \min(\mu_{A_2^+}(x), \mu_{A_2^+}(y)) - \mu_{B_2^+}(xy) & if \quad xy \in E_2 \\ \min(\mu_{A_1^+}(x)\mu_{A_2^+}(y)) - 0 & if \quad x \in V_1, y \in V_2 \end{cases}$$

$$= \begin{cases} \overline{\mu_{B_1^+}}(xy) & if \quad xy \in E_1 \\ \overline{\mu_{B_2^+}}(xy) & if \quad xy \in E_2 \\ \min(\mu_{A_1^+}(x), \mu_{A_2^+}(y)) & if x \in V_1, y \in V_2 \end{cases}$$

$$= \begin{cases} (\overline{\mu_{B_1^+}} \cup \overline{\mu_{B_2^+}})(xy) & if \quad xy \in E_1 \\ (\overline{\mu_{B_1^+}} \cup \overline{\mu_{B_2^+}})(xy) & if \quad xy \in E_2 \\ \min(\mu_{A_1^+}(x), \mu_{A_2^+}(y)) & if xy \in E' \end{cases}$$

$$= (\overline{\mu_{B_1^+}} + \overline{\mu_{B_2^+}})(xy).$$ □

Theorem 4.3. *If there is a weak isomorphism between two interval-valued fuzzy graphs G_1 and G_2, then there is a weak isomorphism between $\overline{G_1}$ and $\overline{G_2}$.*

Proof. If h is a weak isomorphism between G_1 and G_2, then $h : V_1 \to V_2$ is a bijective map that satisfies

$$(1) : \begin{cases} \mu_{A_1^-}(x) = \mu_{A_2^-}(h(x)) \\ \mu_{A_1^+}(x) = \mu_{A_2^+}(h(x)) \end{cases} \forall\ x \in V_1$$

and

$$(2) : \begin{cases} \mu_{B_1^-}(xy) \leq \mu_{B_2^-}(h(x)h(y)) \\ \mu_{B_1^+}(xy) \leq \mu_{B_2^+}(h(x)h(y)) \end{cases} \forall\ x, y \in V_1.$$

As $h^{-1} : V_2 \to V_1$ is also bijective, for every $x_2 \in V_2$, there is an $x_1 \in V_1$ such that $h^{-1}(x_2) = x_1$. Using this in (1), we have

$$(3) : \begin{cases} \mu_{A_2^-}(x_2) = \mu_{A_1^-}(h^{-1}(x_2)) \\ \mu_{A_2^+}(x_2) = \mu_{A_1^+}(h^{-1}(x_2)) \end{cases} \forall\ x_2 \in V_2.$$

Again by using (1) and (2)

$$\overline{\mu_{B_1^-}}(x_1y_1) = \min(\mu_{A_1^-}(x_1), \mu_{A_1^-}(y_1)) - \mu_{B_1^-}(x_1y_1) \quad x_1, y_1 \in V_1,$$

$$\overline{\mu_{B_1^-}}(h^{-1}(x_2)h^{-1}(y_2)) \geq \min(\mu_{A_2^-}(h(x_1), \mu_{A_2^-}(h(y_1)) - \mu_{B_2^-}(h(x_1h(y_1))\ x_1, y_1 \in V_1,$$

$$= \min(\mu_{A_2^-}(x_2), \mu_{A_2^-}(y_2)) - \mu_{B_2^-}(x_2y_2)$$

$$= \overline{\mu_{B_2^-}}(x_2y_2), \quad \forall\ x_2, y_2 \in V_2,$$

i.e.,

$$\overline{\mu_{B_2^-}}(x_2y_2) \leq \overline{\mu_{B_1^-}}(h^{-1}(x_2)h^{-1}(y_2)), \quad \forall\ x_2, y_2 \in V_2.$$

Also

$$\overline{\mu_{B_1^+}}(x_1y_1) = \min(\mu_{A_1^+}(x_1), \mu_{A_1^+}(y_1)) - \mu_{B_1^+}(x_1y_1) \quad \forall\ x_1, y_1 \in V_1$$

$$\begin{aligned} \Rightarrow \overline{\mu_{B_1^+}}(h^{-1}(x_2)h^{-1}(y_2)) &\geq \min(\mu_{A_2^+}(h(x_1)), \mu_{A_2^+}(h(y_1))) - \mu_{B_2^+}(h(x_1)h(y_1)) \\ &= \min(\mu_{A_2^+}(x_2), \mu_{A_2^+}(y_2)) - \mu_{B_2^+}(x_2y_2) \\ &= \overline{\mu_{B_2^+}}(x_2y_2) \quad \forall\ x_2, y_2 \in V_2, \end{aligned}$$

i.e., $\overline{\mu_{B_2^+}}(x_2y_2) \leq \overline{\mu_{B_1^+}}(h^{-1}(x_2)h^{-1}(y_2)), \quad \forall\, x_2, y_2 \in V_2$.
Therefore $h^{-1} : V_2 \to V_1$ is a weak isomorphism between $\overline{G_1}$ and $\overline{G_2}$. □

Notation 4.4. We denote by $Aut(G)$, the automorphism group of a interval-valued fuzzy graph G.

Theorem 4.5. *Let $G = (A, B)$ be a interval-valued fuzzy graph of a graph $G^* : (V, E)$. Then the automorphism group of G and $\overline{G}$ are identical.*

Proof. We prove that for any injective map $h : V \to V$, $h \in Aut(G)$ if and only if $h \in Aut(\overline{G})$. We have

$$\overline{\mu_{A^-}}(h(x)) = \mu_{A^-}(h(x)) = \mu_{A^-}(x) = \overline{\mu_{A^-}}(x) \quad \forall\, x \in V,$$
$$\overline{\mu_{A^+}}(h(x)) = \mu_{A^+}(h(x)) = \mu_{A^+}(x) = \overline{\mu_{A^+}}(x) \quad \forall\, x \in V,$$

Also, for all $x, y \in V$,

$$\overline{\mu_{B^-}}(h(x)h(y)) = \overline{\mu_{B^-}}(xy)$$
$$\Leftrightarrow \min(\mu_{A^-}(h(x)), \mu_{A^-}(h(y))) - \mu_{B^-}(h(x)h(y)) = \min(\mu_{A^-}(x), \mu_{A^-}(y)) - \mu_{B^-}(xy)$$
$$\Leftrightarrow \mu_{B^-}(h(x)h(y)) = \mu_{B^-}(xy),$$

and

$$\overline{\mu_{B^+}}(h(x)h(y)) = \overline{\mu_{B^+}}(xy)$$
$$\Leftrightarrow \min(\mu_{A^+}(h(x)), \mu_{A^+}(h(y))) - \mu_{B^+}(h(x)h(y)) = \min(\mu_{A^+}(x), \mu_{A^+}(y)) - \mu_{B^+}(xy)$$
$$\Leftrightarrow \mu_{B^+}(h(x)h(y)) = \mu_{B^+}(xy).$$

This complete the proof. □

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