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Semi-infinite programming to solve linear programming with triangular fuzzy coefficients

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Abstract. Fuzzy set theory has been applied to many fields, such as operation research, control theory and management science. This paper presents a new method for solving fuzzy linear programming problem with triangular coefficients in constraints and objective function. It is shown that such problem can be reduced to a fuzzy linear semi-infinite programming problem. Than we present two methods for solving linear semi-infinite programming problem with fuzzy coefficients in objective function. Finally, a numerical example is included to illustrate the solution procedure.

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1. Introduction

Linear Programming (LP) is a one of the most important model in Operational Research (OR). Linear programming problems should be considered as a special kind of decision model, the decision space is defined by the constraints, the goal is defined by the objective function and the type of decision is decision making under certainty. The conventional model of linear programming can be stated as:

$$\begin{array}{ll} \max & c^T x \\ s.t. & Ax \leq b, \\ & x \geq 0, \end{array} \tag{1.1}$$

where c and x are n-dimensional column vectors, A is an $m \times n$ $(m \leq n)$ matrix, b is $(m \leq n)$ dimensional column vector, and 0 is the n-dimensional zero vector. Note that in this model, all coefficients of A, b, and c are crisp numbers, and each constraint

must be satisfied strictly. Linear programming is one of the most widely used decision making tools for solving real world problems. One of the main assumptions used in this technique is that the input data have complete accuracy.
However, more often than not, real world situations are characterized by imprecision (fuzziness) rather than exactness. Therefore, a number of researchers have shown interest in the area of fuzzy linear programming. The idea of fuzzy set was first proposed by Zadeh, as a mean of handling uncertainty that is due to rather than to randomness. After that Bellman and Zadeh [3] proposed that a fuzzy decision might be defined as the fuzzy set, defined by the intersection of fuzzy objective and constraint goals. This paper studies a linear programming problem with fuzzy coefficients in constraints and objective function. To describe this problem, we consider the following linear program in the conventional form:

$$\begin{array}{ll} max & \sum_{j=1}^{n} \tilde{c}_j x_j \\ s.t. & \sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \tilde{b}_i, \quad i = 1, \cdots, m, \\ & x_j \geq 0, \quad j = 1, \cdots, n, \end{array} \tag{1.2}$$

where $\tilde{a}_{ij}$, $\tilde{b_i}$ and $\tilde{c_j}$, $i = 1, ..., m, j = 1, ..., n$, are fuzzy coefficients in terms of fuzzy sets. Ramik and Rimanek [17] also dealt with problem [6] with fuzzy parameters in the constraints. Later, Delgado, Verdegay and Vila [6] studied a general model for fuzzy linear programming problems which involve fuzziness both in the coefficients and in the accomplishment of the constraints. In this paper, we focus on the linear programming problem [6] with fuzzy coefficients in the elements of A, b and c.
We will show that such problems can be reduced to a Linear Semi-Infinite Programming ($LSIP$) problem with fuzzy cost coefficients. Hence LSIP problem with fuzzy cost coefficients that we show them in an abbreviated form of ($FLSIP$), have a valuable role to solve linear programming problems with fuzzy coefficients in constraints. Thus the optimality conditions of solutions to $FLSIP$ are investigated and then an algorithm is proposed to solve the original problem. Also a numerical example is given to illustrate its performance.

2. Preliminaries

The ordering indices are so diverse that it is necessary to organize them into several lines to investigate them more efficiently. In [19], ordering indices are classified into three categories. In the first class, each index is associated with a mapping F from the set of fuzzy quantities to the real line $\mathbf{R}$ in order to transform the involved fuzzy quantities into real numbers (see also in [12, 13, 14, 15, 16]). Fuzzy quantities are then compared according to the corresponding real numbers. In the second class, reference sets are set up and all the fuzzy quantities to be ranked are compared with the reference sets. In the last class, a fuzzy relation is constructed to make pairwise comparisons between the fuzzy quantities involved. These pairwise comparisons serve as a basis to obtain the final ranking orders. For the details, readers are referred to [18, 19].

Let **R** be the real line, then a fuzzy set $\tilde{N}$ in **R** is defined to be a set of ordered pairs $\tilde{N} = \{(x, \mu_{\tilde{N}}(x)) \mid x \in \mathbf{R}\}$, where $\mu_{\tilde{N}}(x)$ is called the membership function for the fuzzy set. The membership function maps each element of **R** to a membership value between 0 and 1.

A fuzzy set $\tilde{N}$ on **R** is convex, if for any $x, y \in \mathbf{R}$ and any $\lambda \in [0,1]$, we have $\mu_{\tilde{N}}(\lambda x + (1-\lambda)y) \geq \min\ \{\mu_{\tilde{N}}(x), \mu_{\tilde{N}}(y)\}$.
In this paper, the support of a fuzzy set $\tilde{N}$ is denoted by supp $\tilde{N}$. A fuzzy number $\tilde{N}$ is defined as an upper semi continuous, bounded (bounded support), convex and normal fuzzy quantity. For a fuzzy number $\tilde{N}$, the interval $N_1 = \{x \mid \mu_{\tilde{N}}(x) = 1\}$ is called the kernel or the modal values interval of $\tilde{N}$. The non- decreasing function on the left of the modal values interval and the non-increasing function on the right of the modal values interval will be referred to as the left and right spread, respectively. A special case is the trapezoidal fuzzy number whose left and right spread are straight lines. The trapezoidal fuzzy number with support (a, b) and modal values interval $[c, d]$ is denoted by (a, c, d, b). When $c = d$, a trapezoidal fuzzy number reduces to a triangular fuzzy number. We denote the triangular fuzzy number with support (a, b) and modal value c by (a, c, b).

Definition 2.1. A fuzzy number $\tilde{N}$ shall be called a (fuzzy) zero, symbolized by $\tilde{o}$, if its membership function is as follows:

$$\mu_{\tilde{N}}(x) = \min\ \{\max\ [0, 1 + x/\alpha], \max\ [0, 1 - x/\ \beta]\}, \text{ for all } x \in \mathbf{R} \text{ and } \alpha, \beta > 0.$$

Its figure is shown below.

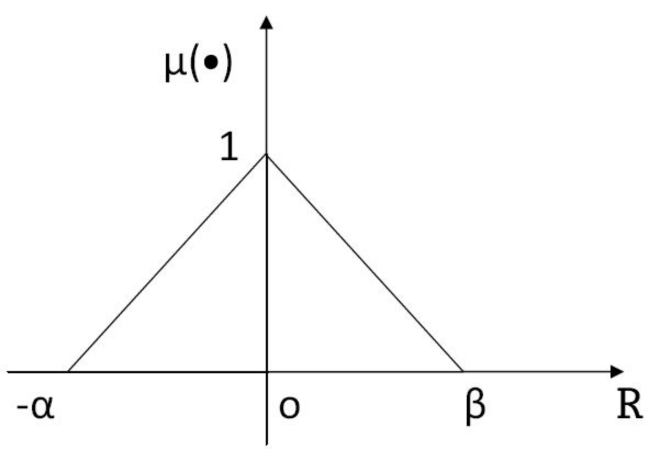

FIGURE 1. Zero triangular fuzzy number

Given a ranking method, the notations $\tilde{N} \succ \tilde{M}$, $\tilde{N} \sim \tilde{M}$ and $\tilde{N} \succsim \tilde{M}$ mean that $\tilde{N}$ has a higher ranking than $\tilde{M}$, the same ranking as $\tilde{M}$ and at least the same ranking as $\tilde{M}$, respectively; $\tilde{N} \prec \tilde{M}$ and $\tilde{N} \precsim \tilde{M}$ are equivalent to $\tilde{M} \succ \tilde{N}$ and $\tilde{M} \succsim \tilde{N}$, respectively.

3. The model

Assume $\tilde{N}_1, \tilde{N}_2, ..., \tilde{N}_n$ are the fuzzy quantities to be ranked and $n_{i\alpha}^{-}=\inf \tilde{N}_{i\alpha}$, $n_{i\alpha}^{+} = \sup \tilde{N}_{i\alpha}$ and $\alpha \in [0,1]$. $\tilde{N}_{i\alpha}$ denotes the α-cut of $\tilde{N}_i$, i.e. $\tilde{N}_{i\alpha}=\{x \mid x \in \mathbf{R}$ and $\mu_{\tilde{N}_i}(x) \geq \alpha\}$.

Figure 2 shows the membership function of the convex fuzzy number $\tilde{N}$.

Let $\mathrm{F}(\tilde{N})$ be the set of all fuzzy numbers. Based upon the Extension Principle [20], we have the following results the proofs are straight for word based on the extention principle and hence we omit them here.

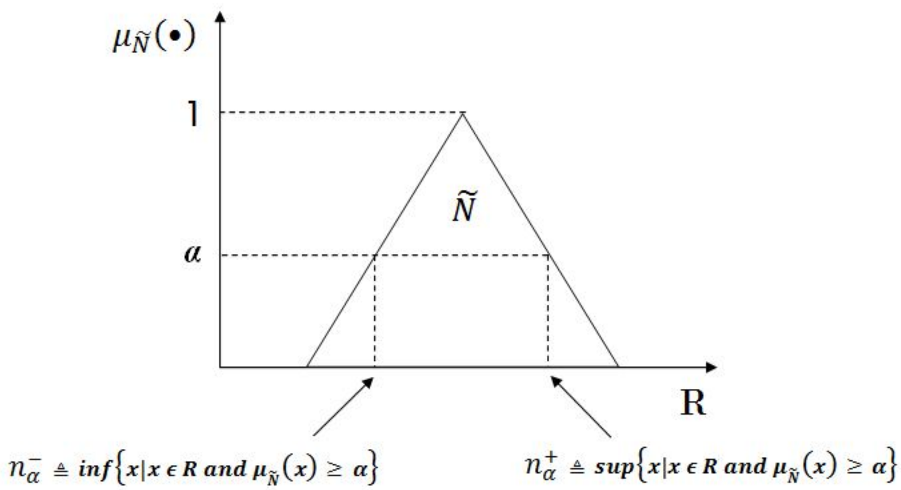

Figure 2. The membership function of a fuzzy number $\tilde{N}$

Theorem 3.1. *If $\tilde{N}_1,\ \tilde{N}_2, ...,\ \tilde{N}_n \in F(\tilde{N})$, then $\tilde{M} \triangleq \tilde{N}_1 + \tilde{N}_2 + ... + \tilde{N}_n \in F(\tilde{N})$ and*

$$\begin{aligned} m_{i\alpha}^{-} &= n_{1\alpha}^{-} + n_{2\alpha}^{-} + ... + n_{n\alpha}^{-}, \\ & \qquad\qquad \forall\ \alpha\ \epsilon\ [0,1]. \\ m_{i\alpha}^{+} &= n_{1\alpha}^{+} + n_{2\alpha}^{+} + ... + n_{n\alpha}^{+}, \end{aligned}$$

Theorem 3.2. *If $\tilde{N} \in F(\tilde{N})$ and k is a positive real number, then $\tilde{M} \triangleq k.\tilde{N} \in F(\tilde{N})$ and*

$$\begin{aligned} m_{\alpha}^{-} &= k.n_{\alpha}^{-}, \\ & \qquad \forall\ \alpha\ \epsilon\ [0,1]. \\ m_{\alpha}^{+} &= k.n_{\alpha}^{+}, \end{aligned}$$

Theorem 3.3. *If $\tilde{N} \in F(\tilde{N})$ and k is a negative real number, then $\tilde{M} \triangleq k.\tilde{N} \in F(\tilde{N})$ and*

$$\begin{aligned} m_{\alpha}^{-} &= k.n_{\alpha}^{+}, \\ & \qquad \forall\ \alpha\ \epsilon\ [0,1]. \\ m_{\alpha}^{+} &= k.n_{\alpha}^{-}, \end{aligned}$$

Theorem 3.4. *If $\tilde{N} \in F(\tilde{N})$ and $k = 0$, then $k.\tilde{N} \triangleq 0$.*

After introducing the concept of fuzzy numbers with their properties, we have to discuss the issue of ranking fuzzy numbers. There are many ranking methods available for the comparison relation between two fuzzy numbers [5, 11, 12, 13]. Here we adopt the commonly used concept of α-preference [1, 4], and provide the following ranking method.

Definition 3.5. For $\tilde{N}_1, \tilde{N}_2 \in F(\tilde{N})$ and $\alpha \,\epsilon\, [0,1]$, $\tilde{N}_1 \geq_\alpha \tilde{N}_2$ if and only if

$$\begin{array}{l} L_{\tilde{N}_1}(t) \geq L_{\tilde{N}_2}(t), \\ \qquad\qquad\qquad \forall\, t \,\epsilon\, [\alpha, 1]. \\ R_{\tilde{N}_1}(t) \geq R_{\tilde{N}_2}(t), \end{array}$$

Figure 3 illustrates such a relation of $\tilde{N}_1 \geq_\alpha \tilde{N}_2$ for some $\alpha \,\epsilon\, [0,1]$. According to

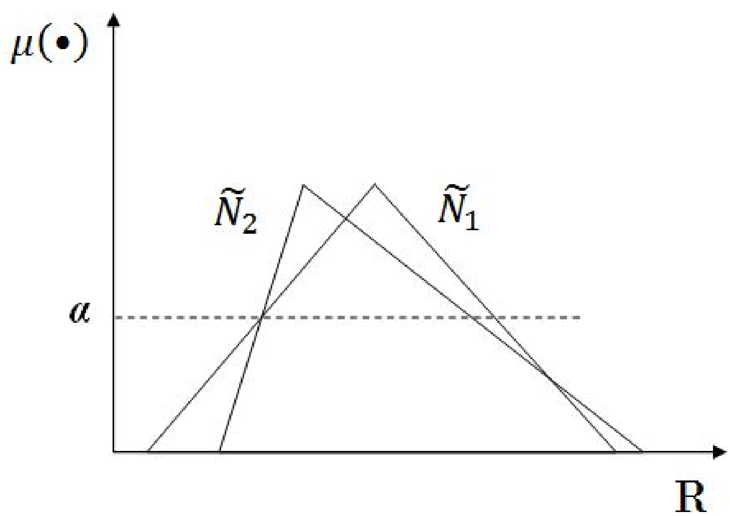

FIGURE 3. A relation of $\tilde{N}_1 \geq_\alpha \tilde{N}_2$ for some $\alpha \in [0,1]$

the fuzzy ranking method provided above, given $\alpha \in [0,1]$, a Fuzzy Linear Programming (FLP) problem considered can be described as follows:

$$(3.1) \qquad (FLP): \begin{cases} min & \sum_{j=1}^{n} \tilde{c}_j x_j \\ s.t. & \sum_{j=1}^{n} \tilde{a}_{ij} x_j \geq_\alpha \tilde{b}_i, \quad i = 1, \cdots, q, \\ & x_j \geq 0, \quad j = 1, \cdots, n, \end{cases}$$

where $\tilde{c_j}, \tilde{a_{ij}}, \tilde{b_i} \in F(\tilde{N})$, for $i = 1, 2, ..., q,\ j = 1, 2, ..., n$. Here, $\sum_{j=1}^{n} \tilde{a}_{ij}x_j \geq_\alpha \tilde{b}_i, i = 1, 2, ..., q$ means that

$$\begin{array}{l} \sum_{j=1}^{n} L_{\tilde{a}_{ij}(t)}.x_j \geq L_{\tilde{b}_i(t)}, \\ \sum_{j=1}^{n} R_{\tilde{a}_{ij}(t)}.x_j \geq R_{\tilde{b}_i(t)}, \end{array} \quad \forall\ t\ \epsilon\ [\alpha, 1]. \tag{3.2}$$

Now substituting expression (3.2) into problem (3.1) yields the following problem:

$$\begin{array}{ll} min & \sum_{j=1}^{n} \tilde{c_j}x_j \\ s.t. & \sum_{j=1}^{n} L_{\tilde{a}_{ij}(t)}.x_j \geq L_{\tilde{b}_i(t)}, \quad \forall\ t\ \epsilon\ [\alpha, 1],\ i = 1, \cdots, q. \\ & \sum_{j=1}^{n} R_{\tilde{a}_{ij}(t)}.x_j \geq R_{\tilde{b}_i(t)}, \\ & x_j \geq 0, \quad j = 1, \cdots, n, \end{array} \tag{3.3}$$

Let $f_{ij}(t) \triangleq L_{\tilde{a}_{ij}}(t)$, for $i = 1, ..., q, j = 1, ..., n, f_{ij}(t) \triangleq R_{\tilde{a}_{i-q,j}}(t)$, $b_i(t) \triangleq R_{\tilde{b}_{i-q}}(t)$, for $i = q + 1, ..., 2q, j = 1, ..., n$, $m \triangleq 2q$, and $\mathbf{T} \triangleq [\alpha, 1]$. Then we have the following equivalent problem:

(3.4)

$$(FLSIP) : \begin{cases} min & \sum_{j=1}^{n} \tilde{c}_j x_j \\ & \\ s.t. & \begin{pmatrix} f_{11}(t) & \dots & f_{1n}(t) \\ \vdots & \ddots & \vdots \\ f_{m1}(t) & \dots & f_{mn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \geq \begin{pmatrix} b_1(t) \\ \vdots \\ b_m(t) \end{pmatrix}, \ \forall t \in T. \end{cases}$$

where $\mathbf{T}$ is a compact metric space, $f_{ij}(t)$ and $b_i(t)$, $i = 1, ..., m, j = 1, ..., n$, are real-valued continuous functions on $\mathbf{T}$. Notice that problem (3.4) is a Fuzzy Linear Semi-Infinite ($FLSIP$) Programming problem with n variables and infinitely many constraints. Its feasible region and the optimal objective value are denoted by Fuzzy Programming (FP) and $v(FLSIP)$, respectively, in this paper. Here for solving the $FLSIP$ problem, we reduce the problemto a Linear Semi-Infinite Programming ($LSIP$) problem. Hence to investigate the optimality conditions of solutions to $LSIP$, some basic analysis for $LSIP$ are presented in the next section and the details can be found in [8].

Now we explain how can reduce an $FLSIP$ problem to an $LSIP$ problem using a linear ranking function as well as given in [14]. Hence consider the linear semi-infinite programming problem with fuzzy objective value, $\tilde{c_j} = ({c_j}^m, {c_j}^\alpha, {c_j}^\beta)$, for all $j = 1, ..., n$, where ${c_j}^m, {c_j}^\alpha$ and ${c_j}^\beta$ are respectively, the core, the left spread and the right spread of the triangular fuzzy number $\tilde{c_j}$. Up to rest the paper here we use a linear ranking function which as first introduced by Yager (see in [12] and also [15]). For example, if we let $F(\mathbf{R})$ denote the set of all triangular fuzzy numbers, a linear ranking function $\mathbf{R}: F(\mathbf{R}) \to \mathbf{R}$ on $\tilde{c_j} = ({c_j}^m, {c_j}^\alpha, {c_j}^\beta) \in F(\mathbf{R})$ is defined as $\mathbf{R}(\tilde{c_j}) = 1/2{c_j}^m + 1/4({c_j}^\beta - {c_j}^\alpha)$.

Now using a linear ranking function problem (3.4) can be convert to:

(3.5)

$$(LSIP): \begin{cases} min & \sum_{j=1}^{n} c_j x_j \\ s.t. & \begin{pmatrix} f_{11}(t) & \dots & f_{1n}(t) \\ \vdots & \ddots & \vdots \\ f_{m1}(t) & \dots & f_{mn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \geq \begin{pmatrix} b_1(t) \\ \vdots \\ b_m(t) \end{pmatrix}, \ \forall t \in T. \end{cases}$$

The current problem is now an $LSIP$ problem that can be solved by the process which is given in Section 6. Two next sections are concern to the fundamental theorems on the optimal solutions.

4. Basic analysis

Let $\mathbf{T}$ be a compact metric space, $C(\mathbf{T})$ be the space of all real-valued continuous functions on $\mathbf{T}$, $M(\mathbf{T})$ be the space of bounded regular Borel measures on $\mathbf{T}$, $C^+(\mathbf{T}) \triangleq \{h \in C(\mathbf{T}) \mid h(t) \geq 0, \forall\ t \in \mathbf{T}\}$, and $M^+(\mathbf{T}) \triangleq \{\mu \in M(\mathbf{T}) \mid \mu(B) \geq 0, \forall\ B \in B(\mathbf{T})$, where $B(\mathbf{T})$ is the set of all Borel set in $\mathbf{T}$. Consider the dual problem of $(LSIP)$ [10]:

$$(4.1) \qquad (DLSIP): \begin{cases} max & \sum_{i=1}^{m} \int_{\mathbf{T}} b_i(t) d\mu_i, \\ s.t. & \sum_{i=1}^{m} \int_{\mathbf{T}} f_{ij}(t) d\mu_i \leq c_j, \quad j = 1, \cdots, n, \\ & \mu_i \ \in \ M^+(\mathbf{T}), \ i = 1, \cdots, m. \end{cases}$$

Let FD be the feasible region of $DLSIP$ and $v(DLSIP)$ the optimal objective value of $DLSIP$. From a result of [19], it follows that if the optimal value of $DLSIP$ has finite value and there is a $\mu^0 = (\mu_1^0, \mu_2^0, ..., \mu_m^0) \in (M^+(\mathbf{T}))^{\mathbf{m}}$ such that $\sum_{i=1}^{m} \int_{\mathbf{T}} f_{ij}(t) d\mu_i^0 \leq c_j,\ j = 1, ..., n$, then the strong duality holds for $DLSIP$. This is stated in Theorem 4.1.

Theorem 4.1. *Assume that* $(FD) \neq \varnothing$ *and* $-\infty < v(DLSIP) < \infty$. *If there exists* $\mu^0 = (\mu_1^0, \mu_2^0, ..., \mu_m^0) \in (M^+(\mathbf{T})))^{\mathbf{m}}$ *such that* $\sum_{i=1}^{m} \int_{\mathbf{T}} f_{ij}(t) d\mu_i^0 \leq c_j, j = 1, ..., n$, *then* $(FP) \neq \varnothing$ *and* $v(LSIP) = v(DLSIP)$.

Applying Theorem 4.1, we have the following result.

Theorem 4.2. *Assume that* $v(LSIP) = v(DLSIP)$, *then* $x^* \in (FP)$ *solves* $LSIP$ *and* $\mu^* \in (FD)$ *solves* $DLSIP$ *if and only if*

$$\sum_{j=1}^{n} f_{ij}(t)\ x_j^* - b_i(t) = 0, \forall\ t\ \in \text{supp}(\mu_i^*),\ i = 1, \cdots, m.$$

next we discuss the existence theorem for $LSIP$.

Theorem 4.3. *If FP is bounded, then $(LSIP)$ has an optimal solution which is an extreme point of FP.*

Proof. It is obvious that the feasible set FP is bounded and closed, and hence, is a compact set in $\mathbf{R}^n$. Since the objective function of $LSIP$ is a continuous linear function on the compact set $FP \subset \mathbf{R}^n$, it will attain its minimum at an extreme point of FP. □

From Theorem 4.3, we see that the extreme points of the feasible set FP play an important role for optimal solutions of $LSIP$. We will discuss the relationship between the optimal solutions and extreme points of the feasible region of $LSIP$ in the next section.

5. Extreme points

To study the extreme points of the feasible region of $LSIP$, we recall some useful definitions for general linear programming. Let E and F be real linear spaces, and $A : E \to F$ a linear operator. Consider the following linear program:

$$(LP) : \begin{cases} min & \langle c^*, x\rangle, \\ s.t. & Ax = b, \\ & x \in P, \end{cases} \tag{5.1}$$

where c^* is a linear functional in $E, b \in F$ and P is a positive convex cone in E. For $x^0 \in P$ we define, $B(x^0) = \{x \in E \mid x^0 \pm \lambda x \in P \text{ for some real } \lambda > 0\}$.
Reference [2] showed that x^0 is an extreme point of the feasible region for LP if and only if $B(x^0) \cap N(A) = \{0\}$ where 0 denotes the zero vector and $N(A) = \{x \in E \mid Ax = 0\}$, the null space of A.
In order to investigate the conditions under which a feasible solution becomes an extreme point, the inequality constraint of $LSIP$ are transformed to equality constraint. Let $g = (g_1, ..., g_m) \in (c^+(\mathbf{T}))^m$ be the vector of "slack variables" of $LSIP$, and consider a new semi-infinite programming problem with equality constraint $(LSIP)_e$:

$$(LSIP) : \begin{cases} min & \sum\limits_{j=1}^{n} c_j x_j, \\ s.t. & \sum\limits_{j=1}^{n} f_{ij}(t)x_j - g_i(t) = b_i(t), \forall\, t \in \mathbf{T} \quad i = 1, \cdots, m, \\ & x_j \geq 0, \quad j = 1, \cdots, n, \\ & g_i(t) \in c^+(\mathbf{T}),\ i = 1, \cdots, m. \end{cases} \tag{5.2}$$

Let $(FP)_e$ be the feasible region of $(LSIP)_e$ and $(x, g) \in (FP)_e$. Suppose that exactly p components of the variable x are greater than zero, and, without loss of generality, we assume that the first p components of x are positive, *i.e.*, $x = (x_1, ..., x_p, 0, ..., 0)^{\mathbf{T}}$. Let $t_1^{(i)},\ t_2^{(i)}, ...,\ t_{l_i}^{(i)} \in \mathbf{T}$, such that $g_i(t_1^{(i)}) = g_i(t_2^{(i)}) = ... = g_i(t_{l_i}^{(i)}) = 0$.

Define the $l_i \times p$ matrices

$$K_i \triangleq \begin{pmatrix} f_{i1}(t_1^{(i)}) & \ldots & f_{ip}(t_1^{(i)}) \\ \vdots & \ddots & \vdots \\ f_{i1}(t_{l_i}^{(i)}) & \ldots & f_{ip}(t_{l_i}^{(i)}) \end{pmatrix}, \quad i = 1, \cdots, m,$$

(5.3)

$$K \triangleq \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{pmatrix}.$$

Then we have the following theorem [8].

Theorem 5.1. *Let K and $(x, g) \in (FP)_e$ be defined as above. If rank $(K) = p$, then (x, g) is an extreme point of $(FP)_e$.*

Next we check the conditions for an extreme point (x, g) to be an optimal solution for $(LSIP)_e$. Let (x, g) be an extreme point of $(FP)_e$ and

$$\{s_{k^{(1)}}\}_{k=1,\ldots,r_1} \subset \{t_{k^{(1)}}\}_{k=1,\ldots,l_1}, \ldots, \{s_{k^{(m)}}\}_{k=1,\ldots,r_m} \subset \{t_{k^{(m)}}\}_{k=1,\ldots,l_m}.$$

Suppose that exactly p components of x are greater than zero, and without loss of generality, the first p components of x are positive. Define

$$\bar{k}_i \triangleq \begin{pmatrix} f_{i1}(s_1^{(i)}) & \ldots & f_{ip}(s_1^{(i)}) \\ \vdots & \ddots & \vdots \\ f_{i1}(s_{r_i}^{(i)}) & \ldots & f_{ip}(s_{r_i}^{(i)}) \end{pmatrix}, \quad i = 1, \cdots, m,$$

(5.4) *and*

$$\bar{k} \triangleq \begin{pmatrix} \bar{k}_1 \\ \bar{k}_2 \\ \vdots \\ \bar{k}_m \end{pmatrix}.$$

Let $\underline{x} = (x_1, ..., x_p)^{\mathbf{T}}$ and $\underline{c} = (c_1, ..., c_p)^{\mathbf{T}}$. We have the following theorem [8].

Theorem 5.2. *Suppose that (x, g) and $\bar{k}$ are defined as above. If we find $\bar{k}$ such that*

i) $\bar{k}$ *is invertible,*

ii) $$(\bar{k}^T)^{-1}\underline{c} \triangleq \begin{pmatrix} u_1^1 \\ \vdots \\ u_{r_1}^1 \\ \vdots \\ u_1^m \\ \vdots \\ u_{r_m}^m \end{pmatrix} \geq 0,$$

iii) $\sum_{i=1}^{m} \sum_{k=1}^{r_i} f_{ij}(s_k^{(i)}) u_k^i - c_j \leq 0$, *for* $j = p+1, ..., n$, *then* (x, g) *is an optimal solution of* $(LSIP)_e$.

6. Solution procedure

There are many semi-infinite programming algorithms [7, 9] available for solving linear semi infinite programming problems. The difficulty lies in how to effectively deal with the infinite number of constraint. Based on a recent review [9], the "cutting plane approach" is an effective one for such application. Following the basic concept of the cutting plane approach, we can easily design an iterative algorithm which adds m constraint at a time until an optimal solution is identified. To be more specific, at the k^{th} iteration, given $\mathbf{T}_k = \{t^1, t^2, ..., t^k\}$, where $t^k = (t_1^k, t_2^k, ..., t_m^k) \in \mathbf{T}^m$, and $k \geq 1$, we consider the following linear programming problem (LP^k):

$$(LP^k): \begin{cases} min \quad \sum_{j=1}^{n} c_j x_j \\ s.t. \quad \left(\begin{array}{ccc} f_{11}(t_1^1) & \dots & f_{1n}(t_1^1) \\ \vdots & \ddots & \vdots \\ f_{m1}(t_m^1) & \dots & f_{mn}(t_m^1) \\ \hline \vdots & \ddots & \vdots \\ \hline f_{11}(t_1^k) & \dots & f_{1n}(t_1^k) \\ \vdots & \ddots & \vdots \\ f_{m1}(t_m^k) & \dots & f_{mn}(t_m^k) \end{array} \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \geq \left(\begin{array}{c} b_1(t_1^1) \\ \vdots \\ b_m(t_m^1) \\ \hline \vdots \\ \hline b_1(t_1^k) \\ \vdots \\ b_m(t_m^k) \end{array} \right) \\ \quad x_j \geq 0, \quad j = 1, \cdots, n. \end{cases}$$

Let F^k be the feasible region of problem (6). $x^k = (x_1^k, ..., x_n^k)$ is an optimal solution of problem (6). We define the "constraint violation functions" as follows:

$$v_i^{k+1}(t) \triangleq \sum_{j=1}^{n} f_{ij}(t) x_j^k - b_i(t), \quad \forall\, t \in \mathbf{T} \quad i = 1, \cdots, m.$$

Since $f_{ij}(t)$ and $b_i(t)$ are continuous over $\mathbf{T}$ and $\mathbf{T}$ is compact, the function $v_i^{k+1}(t)$ achieves its minimum over $\mathbf{T}$, for $i = 1, ..., m$. Let $t_i^{k+1}(t)$ be such a minimizer and consider the value of $v_i^{k+1}(t_i^{k+1})$, for $i = 1, ..., m$. If the value is greater than or equal to zero, for $i = 1, ..., m$, then x^k becomes a feasible solution of $LSIP$, and hence, x^k is optimal for $LSIP$ because the feasible region F^k of problem (6) is no smaller than the feasible region FP of $LSIP$. Otherwise, x^k is not optimal and $t^{k+1} = (t_1^{k+1}, ..., t_m^{k+1}) \notin \mathbf{T}_k$. We then augment $\mathbf{T}_k$ to a larger set $\mathbf{T}_{k+1} = \{t^1, ..., t^k, t^{k+1}\}$. By repeating this process, x^k will converge to the optimal solution of $LSIP$. This background provides a foundation for us to outline a cutting plane algorithm for solving $LSIP$.

Algorithm 6.1 *CPLSIP* Algorithm

Step 1 [Initialization] Set $k = 1$; Choose any $t_i^1 \in \mathbf{T}$; Set $\mathbf{T}_1 = \{t^1\}$.

Step 2 Solve $(LP)^k$ and obtain an optimal solution x^k.

Step 3 Find a minimizer $t_i^{k+1} \in$ of $v_i^{k+1}(t)$. over $\mathbf{T}$, for $i = 1, ..., m$.

Step 4 If $v_i^{k+1}(t_i^{k+1}) \geq 0$, for $i = 1, ..., m$, then stop with x^k being an optimal solution of $LSIP$.

Otherwise, set $\mathbf{T}_{k+1} = \mathbf{T}_k \cup \{t^{k+1}\}$ and $k \leftarrow k + 1$; go to Step 1.

When $LSIP$ has at least one feasible solution, *i.e.*, $FP \neq \emptyset$, it is easy to see that the $CPLSIP$ Algorithm either terminates in a finite number of iterations with an optimal solution or generates a sequence of points $\{x^k \mid k = 1, 2, ...\}$. Our objective for the remaining part of this section is to show that if the $CPLSIP$ Algorithm does not terminate in finite iterations, then $\{x^k\}$ has a subsequence which converges to an optimal solution of $LSIP$. We now show a convergence proof for the $CPLSIP$ Algorithm which we omit the proof here.

Theorem 6.1. *Let $\{x^k\}$ be a sequence generated by the $CPLSIP$ Algorithm. If there exists an $M > 0$ such that $\| x^k \| \leq M, \forall k$, then there is a subsequence of x^k which converges to an optimal solution of $LSIP$.*

7. Numerical example

Now we are a place to illustrate the proposed approach and solution procedures. Let us consider the following fuzzy linear programming problem:

Example 7.1. Consider the following fuzzy linear programming problem:

$$(FLP): \begin{cases} max & z = (13, 13/2, 7/2)x_1 + (16, 4, 8)x_2, \\ s.t. & \langle 3, 2, 1\rangle x_1 + \langle 6, 4, 1\rangle x_2 \leq_\alpha \langle 13, 5, 2\rangle, \\ & \langle 4, 1, 2\rangle x_1 + \langle 6, 5, 4\rangle x_2 \leq_\alpha \langle 7, 4, 2\rangle, \\ & x_1,\ x_2 \geq 0, \\ & \alpha \in [0, 1], \end{cases}$$

hence the mentioned fuzzy linear programming will reduced to the following from:

$$(LSIP): \begin{cases} min & -7.25x_1 - 9x_2 \\ s.t. & \begin{pmatrix} -2t_1 - 1 & -4t_1 - 2 \\ -t_2 - 3 & -5t_2 - 1 \\ -t_3 - 4 & t_3 - 7 \\ 2t_4 - 6 & 4t_4 - 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} -5t_1 - 8 \\ -4t_2 - 3 \\ 2t_3 - 15 \\ 2t_4 - 9 \end{pmatrix} \quad \forall t_i \in [\alpha, 1], \\ & x_1,\ x_2 \geq 0. \end{cases}$$

Given any $\alpha \in [0,1]$, say $\alpha = 0.6$ in this example and an arbitrary point, say $t^1 = (t_1^1, t_2^1, t_3^1, t_4^1) = (0.7, 0.8, 0.7, 0.8)$, we have a regular linear program,

$$(LP^1): \begin{cases} min & -7.25x_1 - 9x_2 \\ s.t. & \begin{pmatrix} -2t_1^1 - 1 & -4t_1^1 - 2 \\ -t_2^1 - 3 & -5t_2^1 - 1 \\ -t_3^1 - 4 & t_3^1 - 7 \\ 2t_4^1 - 6 & 4t_4^1 - 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} -5t_1^1 - 8 \\ -4t_2^1 - 3 \\ 2t_3^1 - 15 \\ 2t_4^1 - 9 \end{pmatrix} \quad \forall\ t_i \in [\alpha, 1], \\ & x_1,\ x_2 \geq 0. \end{cases}$$

Solving (LP^1) results in an optimal solution $x^1 = (x_1^1, x_2^1) = (1.63158, 0)$. Define

$$\begin{aligned} v_1^2(t_1) &= (-2t_1 - 1)x_1^1 + (-4t_1 - 2)x_2^1 - (-5t_1 - 8) = 1.73684t_1 + 6.36842, \\ v_2^2(t_2) &= (-t_2 - 3)x_1^1 + (-5t_2 - 1)x_2^1 - (-4t_2 - 3) = 2.36842t_2 - 7.89472, \\ v_3^2(t_3) &= (t_3 - 4)x_1^1 + (t_3 - 7)x_2^1 - (-2t_3 - 15) = 0.36842t_3 + 8.47368, \\ v_4^2(t_4) &= (2t_4 - 6)x_1^1 + (4t_4 - 10)x_2^1 - (t_4 - 9) = 1.26316t_4 - 0.78948. \end{aligned}$$

The minimizers of $v_1^2(t_1), v_2^2(t_2), v_3^2(t_3), v_4^2(t_4)$ over $[\alpha, 1] = [0.6, 1]$ are $(0.6, 0.6, 1, 0.6)$, respectively.

Hence we choose $t_2 = (t_1^2, t_2^2, t_3^2, t_4^2) = (0.6, 0.6, 1, 0.6)$.

Since $v_1^2(t_1^2) \geq 0, v_2^2(t_2^2) \leq 0, v_3^2(t_3^2) \geq 0, v_4^2(t_4^2) \leq 0$, the $CPLSIP$ Algorithm iterates with a new linear program,

$$(LP^2): \begin{cases} min & -7.25x_1 - 9x_2 \\ s.t. & \begin{pmatrix} -2t_1^1 - 1 & -4t_1^1 - 2 \\ -t_2^1 - 3 & -5t_2^1 - 1 \\ -t_3^1 - 4 & t_3^1 - 7 \\ 2t_4^1 - 6 & 4t_4^1 - 10 \\ \cdots & \cdots \\ -2t_1^2 - 1 & -4t_1^2 - 2 \\ -t_2^2 - 3 & -5t_2^2 - 1 \\ -t_3^2 - 4 & t_3^2 - 7 \\ 2t_4^2 - 6 & 4t_4^2 - 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} -5t_1^1 - 8 \\ -4t_2^1 - 3 \\ 2t_3^1 - 15 \\ 2t_4^1 - 9 \\ \cdots \\ -5t_1^2 - 8 \\ -4t_2^2 - 3 \\ 2t_3^2 - 15 \\ 2t_4^2 - 9 \end{pmatrix} \\ & x_1,\ x_2 \geq 0. \end{cases}$$

Solving (LP^2) result in an optimal solution $x^2 = (x_1^2, x_2^2) = (0.136364, 0.940191)$. Define

$$\begin{aligned} v_1^3(t_1) &= (-2t_1 - 1)x_1^2 + (-4t_1 - 2)x_2^2 - (-5t_1 - 8) = 0.966508t_1 + 5.983254, \\ v_2^3(t_2) &= (-t_2 - 3)x_1^2 + (-5t_2 - 1)x_2^2 - (-4t_2 - 3) = -0.837319t_2 + 1.65077, \\ v_3^3(t_3) &= (t_3 - 4)x_1^2 + (t_3 - 7)x_2^2 - (-2t_3 - 15) = 0.923445t_3 + 7.873027, \\ v_4^3(t_4) &= (2t_4 - 6)x_1^2 + (4t_4 - 10)x_2^2 - (t_4 - 9) = 2.03492t_4 - 1.220094. \end{aligned}$$

The minimizers of $v_1^3(t_1), v_2^3(t_2), v_3^3(t_3), v_4^3(t_4)$ over $[0.6, 1]$ are $0.6, 1, 1, 0.6$ respectively. Hence, we choose $t = (t_1^3, t_2^3, t_3^3, t_4^3) = (0.6, 1, 1, 0.6)$.

Now, since $v_1^3(t_1^3) \geq 0, v_2^3(t_2^3 \geq 0, v_3^3(t_3^3 \geq 0, v_4^3(t_4^3 \geq 0$, the algorithm stops and results an optimal solution $x^* = x^2 = \begin{pmatrix} 0.136364 \\ \\ 0.940191 \end{pmatrix}$. The fuzzy linear program with $\alpha = 0.6$.

8. Conclusion

In this paper, a linear programming problem with fuzzy coefficient in A, b and c is studied. By using the concept of α-preference we have shown that such problems can be reduced to a linear semi-infinite programming problem. We have also studied the relation between the optimal solutions and extreme points of the linear semi-infinite program are established. A cutting plane algorithm is proposed for solving a linear programming problem with fuzzy coefficients in terms of linear semi-infinite programming.

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