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Multivariate fuzzy-random normalized neural network approximation operators

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Abstract. In this article we study the rate of multivariate pointwise convergence in the q-mean to the Fuzzy-Random unit operator or its perturbation of very precise multivariate normalized Fuzzy-Random neural network operators of Cardaliaguet-Euvrard and "Squashing" types. These multivariate Fuzzy-Ranfom operators arise in a natural and common way among multivariate Fuzzy-Random neural network. These rates are given through multivariate Probabilistic-Jackson type inequalities involving the multivariate Fuzzy-Random modulus of continuity of the engaged multivariate Fuzzy-Random function or its Fuzzy partial derivatives. Also several interesting results in multivariate Fuzzy-Random Analysis are given of independent merit, which are used then in the proof of the main results of the paper.

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Keywords: Fuzzy-Random analysis, Fuzzy-Random neural networks and operators, Fuzzy-Random modulus of continuity, Fuzzy-Random functions, Jackson type and probabilistic inequalities.

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1. Introduction

Let $(X, \mathcal{B}, P)$ be a probability space. Consider the set of all fuzzy-random variables $\mathcal{L}_F(X, \mathcal{B}, P)$. Let $f : \mathbb{R}^d \to \mathcal{L}_F(X, \mathcal{B}, P)$, $d \in \mathbb{N}$, be a multivariate fuzzy-random function or fuzzy-stochastic process. Here for $\overrightarrow{t} \in \mathbb{R}^d$, $s \in X$ we denote $\left(f\left(\overrightarrow{t}\right)\right)(s) = f\left(\overrightarrow{t}, s\right)$ and actually we have $f : \mathbb{R}^d \times X \to \mathbb{R}_{\mathcal{F}}$, where $\mathbb{R}_{\mathcal{F}}$ is the set of fuzzy real numbers. Let $1 \leq q < +\infty$. Here we consider only multivariate fuzzy-random functions f which are (q-mean) uniformly continuous over $\mathbb{R}^d$. For each $n \in \mathbb{N}$, the multivariate fuzzy-random neural network we deal with has the following structure:

It is a three-layer feed forward network with one hidden layer. It has one input unit and one output unit. The hidden layer has $\left(2n^2+1\right)$ processing units. To each pair of connecting units (input to each processing unit) we assign the same weight $n^{1-\alpha}$, $0<\alpha<1$. The threshold values $\frac{\overrightarrow{k}}{n^{\alpha}}$ are one for each processing unit $\overrightarrow{k}$, $\overrightarrow{k}\in\mathbb{Z}^d$. The activation function b (or S) is the same for each processing unit. The Fuzzy-Random weights associated with the output unit are $f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{1}{\sum_{\overrightarrow{k}=-\overrightarrow{n}^2}^{\overrightarrow{n}^2} b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}$, one for each processing unit $\overrightarrow{k}$, $\odot$ denotes the scalar Fuzzy multiplication.

The above precisely described multivariate Fuzzy-Random neural networks induce some completely described multivariate Fuzzy-Random neural network operators of normalized Cardaliaguet-Euvrard and "Squashing "types.

We study here throughly the multivariate Fuzzy-Random pointwise convergence (in q-mean) of these operators to the unit operator and its perturbation. See Theorem 35, 38 and Comment 41. This is done with rates through multivariate Probabilistic-Jackson type inequalities involving Fuzzy-Random moduli of continuity of the engaged Fuzzy-Random function and its Fuzzy pertial derivatives.

On the way to establish these main results we produce some new independent and interersting results for multivariate Fuzzy-Random Analysis. The real ordinary theory of the above mentioned operators was presented earlier in [1], [2] and [9]. And the fuzzy case was treated in [3]. The fuzzy random case was studied first in [5]. Of course this article is strongly motivated from there and is a continuation.

The monumental revolutionizing work of L. Zadeh [13] is the foundation of this work, as well as another strong motivation. Fuzzyness in Computer Science and Engineering seems one of the main trends today. Also Fuzzyness has penetrated many areas of Mathematics and Statistics. These are other strong reasons for this work.

Our approach is quantitative and recent on the topic, started in [1], [2] and continued in [3], [7], [8]. It determines precisely the rates of convergence through natural very tight inequalities using the measurement of smoothness of the engaged multivariate Fuzzy-Random functions.

2. Background

We begin with

Definition 2.1. (see [12]) Let $\mu:\mathbb{R}\to[0,1]$ with the following properties:

(i) is normal, i.e., $\exists\ x_0\in\mathbb{R}:\mu(x_0)=1$.

(ii) $\mu(\lambda x+(1-\lambda)y)\geq\min\{\mu(x),\mu(y)\}$, $\forall\ x,y\in\mathbb{R}$, $\forall\ \lambda\in[0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e., $\forall\ x_0\in\mathbb{R}$ and $\forall\ \varepsilon>0$, $\exists$ neighborhood $V(x_0):\mu(x)\leq\mu(x_0)+\varepsilon$, $\forall\ x\in V(x_0)$.

(iv) the set $\overline{\text{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\text{supp}(\mu):=\{x\in\mathbb{R};\mu(x)>0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}}\in\mathbb{R}_{\mathcal{F}}$, for any $x_0\in\mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$ and $[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}$.

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall\, r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [12]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} < u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall\, r \in [0,1]$.

Define

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$$

by

$$D(u,v) := \sup_{r \in [0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\},$$

where $[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [12], with the properties

$$D(u \oplus w, v \oplus w) = D(u,v), \quad \forall\, u, v, w \in \mathbb{R}_{\mathcal{F}}, \tag{1}$$

$$\begin{aligned} D(k \odot u, k \odot v) &= |k|\, D(u,v), \quad \forall\, u, v \in \mathbb{R}_{\mathcal{F}}, \forall\, k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) &\leq D(u,w) + D(v,e), \quad \forall\, u, v, w, e \in \mathbb{R}_{\mathcal{F}}. \end{aligned}$$

Let $f, g : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be fuzzy real number valued functions. The distance between f, g is defined by

$$D^*(f,g) := \sup_{x \in \mathbb{R}} D(f(x), g(x)).$$

On $\mathbb{R}_{\mathcal{F}}$ we define a partial order by "$\leq$": $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \leq v$ iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, $\forall\, r \in [0,1]$.

We need

Lemma 2.2 ([4]). *For any $a, b \in \mathbb{R} : a \cdot b \geq 0$ and any $u \in \mathbb{R}_{\mathcal{F}}$ we have*

$$D(a \odot u, b \odot u) \leq |a - b| \cdot D(u, \widetilde{o}), \tag{2}$$

where $\widetilde{o} \in \mathbb{R}_{\mathcal{F}}$ is defined by $\widetilde{o} := \chi_{\{0\}}$.

Lemma 2.3 ([4]). (i) *If we denote $\widetilde{o} := \chi_{\{0\}}$, then $\widetilde{o} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e., $u \oplus \widetilde{o} = \widetilde{o} \oplus u = u$, $\forall\, u \in \mathbb{R}_{\mathcal{F}}$.*

(ii) *With respect to $\widetilde{o}$, none of $u \in \mathbb{R}_{\mathcal{F}}$, $u \neq \widetilde{o}$ has opposite in $\mathbb{R}_{\mathcal{F}}$.*

(iii) *Let $a, b \in \mathbb{R} : a \cdot b \geq 0$, and any $u \in \mathbb{R}_{\mathcal{F}}$, we have $(a+b) \odot u = a \odot u \oplus b \odot u$. For general $a, b \in \mathbb{R}$, the above property is false.*

(iv) *For any $\lambda \in \mathbb{R}$ and any $u, v \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$.*

(v) *For any $\lambda, \mu \in \mathbb{R}$ and $u \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda \odot (\mu \odot u) = (\lambda \cdot \mu) \odot u$.*

(vi) *If we denote* $\|u\|_{\mathcal{F}} := D(u, \widetilde{o})$, $\forall\ u \in \mathbb{R}_{\mathcal{F}}$, *then* $\|\cdot\|_{\mathcal{F}}$ *has the properties of a usual norm on* $\mathbb{R}_{\mathcal{F}}$, *i.e.,*

$$\|u\|_{\mathcal{F}} = 0 \text{ iff } u = \widetilde{o}, \quad \|\lambda \odot u\|_{\mathcal{F}} = |\lambda| \cdot \|u\|_{\mathcal{F}},$$

$$\|u \oplus v\|_{\mathcal{F}} \leq \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}}, \quad \|u\|_{\mathcal{F}} - \|v\|_{\mathcal{F}} \leq D(u, v). \tag{3}$$

Notice that $(\mathbb{R}_{\mathcal{F}}, \oplus, \odot)$ is not a linear space over $\mathbb{R}$; and consequently $(\mathbb{R}_{\mathcal{F}}, \|\cdot\|_{\mathcal{F}})$ is not a normed space.

As in Remark 4.4 ([4]) one can show easily that a sequence of operators of the form

$$L_n(f)(x) := \sum_{k=0}^{n*} f(x_{k_n}) \odot w_{n,k}(x), \quad n \in \mathbb{N}, \tag{4}$$

($\overset{*}{\sum}$ denotes the fuzzy summation) where $f : \mathbb{R}^d \to \mathbb{R}_{\mathcal{F}}$, $x_{k_n} \in \mathbb{R}^d$, $d \in \mathbb{N}$, $w_{n,k}(x)$ real valued weights, are linear over $\mathbb{R}^d$, i.e.,

$$L_n(\lambda \odot f \oplus \mu \odot g)(x) = \lambda \odot L_n(f)(x) \oplus \mu \odot L_n(g)(x), \tag{5}$$

$\forall \lambda, \mu \in \mathbb{R}$, any $x \in \mathbb{R}^d$; $f, g : \mathbb{R}^d \to \mathbb{R}_{\mathcal{F}}$. (Proof based on Lemma 2.3 (iv).)

We need

Definition 2.4. (see [12]) Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists a $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y + z$, then we call z the H-difference of x and y, denoted by $z := x - y$.

Definition 2.5. (see [12]) Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is differentiable at $x \in T$ if there exists a $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h}$$

exist and are equal to $f'(x)$. We call f' the derivative of f or H-derivative of f at x. If f is differentiable at any $x \in T$, we call f differentiable or H-differentiable and it has derivative over T the function f'.

We need also a particular case of the Fuzzy Henstock integral ($\delta(x) = \frac{\delta}{2}$) introduced in [12], Definition 2.1 there.

That is,

Definition 2.6. (see [11], p. 644) Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$. We say that f is Fuzzy-Riemann integrable to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u, v]; \xi\}$ of $[a, b]$ with the norms $\Delta(P) < \delta$, we have

$$D\left(\sum_{P}^{*} (v - u) \odot f(\xi), I\right) < \varepsilon. \tag{6}$$

We choose to write

$$I := (FR) \int_a^b f(x)\, dx. \tag{7}$$

We also call an f as above (FR)-integrable.

We mention the following fundamental theorem of Fuzzy Calculus:

Corollary 2.7 ([3]). *If $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ has a fuzzy continuous derivative f' on $[a,b]$, then $f'(x)$ is (FR)-integrable over $[a,b]$ and*

$$f(s) = f(t) \oplus (FR)\int_t^s f'(x)\,dx, \quad \text{for any } s \geq t,\ s,t \in [a,b]. \tag{8}$$

Note. In Corollary 2.7 when $s < t$ the formula is invalid! Since fuzzy real numbers correspond to closed intervals etc.

We need also

Lemma 2.8 ([3]). *If $f,g : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ are fuzzy continuous (with respect to metric D), then the function $F : [a,b] \to \mathbb{R}_+ \cup \{0\}$ defined by $F(x) := D(f(x), g(x))$ is continuous on $[a,b]$, and*

$$D\left((FR)\int_a^b f(u)\,du, (FR)\int_a^b g(u)\,du\right) \leq \int_a^b D(f(x), g(x))\,dx. \tag{9}$$

Lemma 2.9 ([3]). *Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ fuzzy continuous (with respect to metric D), then $D(f(x), \widetilde{o}) \leq M$, $\forall\, x \in [a,b]$, $M > 0$, that is f is fuzzy bounded.*

We mention

Lemma 2.10 ([5]). *Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ have an existing fuzzy derivative f' at $c \in [a,b]$. Then f is fuzzy continuous at c.*

Note. Higher order fuzzy derivatives and all fuzzy partial derivatives are defined the obvious and analogous way to the real derivatives, all based on Definitions 2.4, 2.5, here.

We need the fuzzy multivariate Taylor formula.

Theorem 2.11. ([7], p.54) *Let U be an open convex subset on $\mathbb{R}^n$, $n \in \mathbb{N}$ and $f : U \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy continuous function. Assume that all H-fuzzy partial derivatives of f up to order $m \in \mathbb{N}$ exist and are fuzzy continuous. Let $z := (z_1, ..., z_n)$, $x_0 := (x_{01}, ..., x_{0n}) \in U$ such that $z_i \geq x_{0i}$, $i = 1, ..., n$. Let $0 \leq t \leq 1$, we define $x_i := x_{0i} + t(z_i - z_{0i})$, $i = 1, 2, ..., n$ and $g_z(t) := f(x_0 + t(z - x_0))$. (Clearly $x_0 + t(z - x_0) \in U$). Then for $N = 1, ..., m$ we obtain*

$$g_z^{(N)}(t) = \left[\left(\sum_{i=1}^{n*} (z_i - x_{0i}) \odot \frac{\partial}{\partial x_i}\right)^N f\right](x_1, x_2, ..., x_n). \tag{10}$$

Furthermore it holds the following fuzzy multivariate Taylor formula

$$f(z) = f(x_0) \oplus \sum_{N=1}^{m-1*} \frac{g_z^{(N)}(0)}{N!} \oplus \mathcal{R}_m(0,1), \tag{11}$$

where

$$\mathcal{R}_m(0,1) := \frac{1}{(m-1)!} \odot (FR)\int_0^1 (1-s)^{m-1} \odot g_z^{(m)}(s)\,ds. \tag{12}$$

Comment 2.12. *(explaining formula (10)) When $N = n = 2$ we have ($z_i \geq x_{0i}$, $i = 1, 2$)*

$$g_z(t) = f(x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})), \quad 0 \leq t \leq 1.$$

We apply Theorems 2.18, 2.19, 2.21 of [7] repeatedly, etc. Thus we find

$$g_z'(t) = (z_1 - x_{01}) \odot \frac{\partial f}{\partial x_1}(x_1, x_2) \oplus (z_2 - x_{02}) \odot \frac{\partial f}{\partial x_2}(x_1, x_2).$$

Furthermore it holds

$$g_z''(t) = (z_1 - x_{01})^2 \odot \frac{\partial^2 f}{\partial x_1^2}(x_1, x_2) \oplus 2(z_1 - x_{01})(z_2 - x_{02}) \odot \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \oplus (z_2 - x_{02})^2 \odot \frac{\partial^2 f}{\partial x_2^2}(x_1, x_2). \tag{13}$$

When $n = 2$ and $N = 3$ we obtain

$$g_z'''(t) = (z_1 - x_{01})^3 \odot \frac{\partial^3 f}{\partial x_1^3}(x_1, x_2) \oplus 3(z_1 - x_{01})^2 (z_2 - x_{02}) \odot \frac{\partial^3 f(x_1, x_2)}{\partial x_1^2 \partial x_2}$$

$$\oplus 3(z_1 - x_{01})(z_2 - x_{02})^2 \odot \frac{\partial^3 f(x_1, x_2)}{\partial x_1 \partial x_2^2} \oplus (z_2 - x_{02})^3 \odot \frac{\partial^3 f}{\partial x_2^3}(x_1, x_2). \tag{14}$$

When $n = 3$ and $N = 2$ we get ($z_i \geq x_{0i}$, $i = 1, 2, 3$)

$$g_z''(t) = (z_1 - x_{01})^2 \odot \frac{\partial^2 f}{\partial x_1^2}(x_1, x_2, x_3) \oplus (z_2 - x_{02})^2 \odot \frac{\partial^2 f}{\partial x_2^2}(x_1, x_2, x_3)$$

$$\oplus (z_3 - x_{03})^2 \odot \frac{\partial^2 f}{\partial x_3^2}(x_1, x_2, x_3) \oplus 2(z_1 - x_{01})(z_2 - x_{02}) \tag{15}$$

$$\odot \frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_1 \partial x_2} \oplus 2(z_2 - x_{02})(z_3 - x_{03}) \odot \frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_2 \partial x_3}$$

$$\oplus 2(z_3 - x_{03})(z_1 - x_{01}) \odot \frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_3 \partial x_1},$$

etc.

3. Basic properties

We need

Definition 3.1. (see also [11], Definition 13.16, p. 654) Let $(X, \mathcal{B}, P)$ be a probability space. A fuzzy-random variable is a $\mathcal{B}$-measurable mapping $g : X \to \mathbb{R}_{\mathcal{F}}$ (i.e., for any open set $U \subseteq \mathbb{R}_{\mathcal{F}}$, in the topology of $\mathbb{R}_{\mathcal{F}}$ generated by the metric D, we have

$$g^{-1}(U) = \{s \in X; g(s) \in U\} \in \mathcal{B}). \tag{16}$$

The set of all fuzzy-random variables is denoted by $\mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Let $g_n, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $n \in \mathbb{N}$ and $0 < q < +\infty$. We say $g_n(s) \overset{\text{"}q\text{-mean"}}{\underset{n \to +\infty}{\to}} g(s)$ if

$$\lim_{n \to +\infty} \int_X D(g_n(s), g(s))^q P(ds) = 0. \tag{17}$$

Remark 3.2. (see [11], p. 654) If $f, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, let us denote $F : X \to \mathbb{R}_+ \cup \{0\}$ by $F(s) = D(f(s), g(s))$, $s \in X$. Here, F is $\mathcal{B}$-measurable, because $F = G \circ H$, where $G(u, v) = D(u, v)$ is continuous on $\mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, and $H : X \to \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, $H(s) = (f(s), g(s))$, $s \in X$, is $\mathcal{B}$-measurable. This shows that the above convergence in q-mean makes sense.

Definition 3.3. (see [11], p. 654, Definition 13.17) Let $(T, \mathcal{T})$ be a topological space. A mapping $f : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$ will be called fuzzy-random function (or fuzzy-stochastic process) on T. We denote $f(t)(s) = f(t, s)$, $t \in T$, $s \in X$.

Remark 3.4. (see [11], p. 655) Any usual fuzzy real function $f : T \to \mathbb{R}_{\mathcal{F}}$ can be identified with the degenerate fuzzy-random function $f(t, s) = f(t)$, $\forall\, t \in T$, $s \in X$.

Remark 3.5. (see [11], p. 655) Fuzzy-random functions that coincide with probability one for each $t \in T$ will be consider equivalent.

Remark 3.6. (see [11], p. 655) Let $f, g : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Then $f \oplus g$ and $k \odot f$ are defined pointwise, i.e.,

$$\begin{aligned} (f \oplus g)(t, s) &= f(t, s) \oplus g(t, s), \\ (k \odot f)(t, s) &= k \odot f(t, s), \quad t \in T, s \in X. \end{aligned}$$

Definition 3.7. (see also Definition 13.18, pp. 655-656, [11]) For a fuzzy-random function $f : \mathbb{R}^d \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$, we define the (first) fuzzy-random modulus of continuity

$$\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} =$$

$$\sup \left\{ \left(\int_X D^q(f(x, s), f(y, s)) P(ds) \right)^{\frac{1}{q}} : x, y \in \mathbb{R}^d,\ \ \|x - y\|_{l_1} \le \delta \right\},$$

$0 < \delta$, $1 \le q < \infty$.

Definition 3.8. Here $1 \le q < +\infty$. Let $f : \mathbb{R}^d \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$, be a fuzzy random function. We call f a (q-mean) uniformly continuous fuzzy random function over $\mathbb{R}^d$, iff $\forall\, \varepsilon > 0\ \exists\, \delta > 0$:whenever $\|x - y\|_{l_1} \le \delta$, $x, y \in \mathbb{R}^d$, implies that

$$\int_X (D(f(x, s), f(y, s)))^q P(ds) \le \varepsilon.$$

We denote it as $f \in C_{FR}^{U_q}(\mathbb{R}^d)$.

Proposition 3.9. *Let* $f \in C_{FR}^{U_q}(\mathbb{R}^d)$. *Then* $\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} < \infty$, *any* $\delta > 0$.

Proof. Let $\varepsilon_0 > 0$ be arbitrary but fixed. Then there exists $\delta_0 > 0 : \|x - y\|_{l_1} \le \delta_0$ implies

$$\int_X (D(f(x, s), f(y, s)))^q P(ds) \le \varepsilon_0 < \infty.$$

That is $\Omega_1^{(\mathcal{F})}(f, \delta_0)_{L^q} \le \varepsilon_0^{\frac{1}{q}} < \infty$. Let now $\delta > 0$ arbitrary, $x, y \in \mathbb{R}^d$ such that $\|x - y\|_{l_1} \le \delta$. Choose $n \in \mathbb{N} : n\delta_0 \ge \delta$ and set $x_i := x + \frac{i}{n}(y - x)$, $0 \le i \le n$. Then

$$D(f(x, s), f(y, s)) \le D(f(x, s), f(x_1, s)) + D(f(x_1, s), f(x_2, s))$$
$$+ ... + D(f(x_{n-1}, s), f(y, s)).$$

Consequently

$$\left(\int_X (D(f(x,s), f(y,s)))^q P(ds)\right)^{\frac{1}{q}} \le \left(\int_X (D(f(x,s), f(x_1,s)))^q P(ds)\right)^{\frac{1}{q}}$$

$$+ ... + \left(\int_X (D(f(x_{n-1},s), f(y,s)))^q P(ds)\right)^{\frac{1}{q}} \le n\Omega_1^{(\mathcal{F})}(f,\delta_0)_{L^q} \le n\varepsilon_0^{\frac{1}{q}} < \infty,$$

since $\|x_i - x_{i+1}\|_{l_1} = \frac{1}{n}\|x-y\|_{l_1} \le \frac{1}{n}\delta \le \delta_0$, $0 \le i \le n$.

Therefore $\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} \le n\varepsilon_0^{\frac{1}{q}} < \infty$. □

Proposition 3.10. *Let $f, g : \mathbb{R}^d \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$, be fuzzy random functions. It holds*

(i) $\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}$ *is nonnegative and nondecreasing in* $\delta > 0$.

(ii) $\lim_{\delta\downarrow 0}\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} = \Omega_1^{(\mathcal{F})}(f,0)_{L^q} = 0$, *iff* $f \in C_{FR}^{U_q}(\mathbb{R}^d)$.

(iii) $\Omega_1^{(\mathcal{F})}(f,\delta_1+\delta_2)_{L^q} \le \Omega_1^{(\mathcal{F})}(f,\delta_1)_{L^q} + \Omega_1^{(\mathcal{F})}(f,\delta_2)_{L^q}$, $\delta_1, \delta_2 > 0$.

(iv) $\Omega_1^{(\mathcal{F})}(f,n\delta)_{L^q} \le n\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}$, $\delta > 0$, $n \in \mathbb{N}$.

(v) $\Omega_1^{(\mathcal{F})}(f,\lambda\delta)_{L^q} \le \lceil\lambda\rceil\,\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} \le (\lambda+1)\,\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}$, $\lambda > 0$, $\delta > 0$, *where* $\lceil\cdot\rceil$ *is the ceilling of the number.*

(vi) $\Omega_1^{(\mathcal{F})}(f \oplus g,\delta)_{L^q} \le \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} + \Omega_1^{(\mathcal{F})}(g,\delta)_{L^q}$, $\delta > 0$. *Here $f \oplus g$ is a fuzzy random function.*

(vii) $\Omega_1^{(\mathcal{F})}(f,\cdot)_{L^q}$ *is continuous on* $\mathbb{R}_+$, *for* $f \in C_{FR}^{U_q}(\mathbb{R}^d)$.

Proof. (i) is obvious.

(ii) $\Omega_1(f,0)_{L^q} = 0$.

($\Rightarrow$) Let $\lim_{\delta\downarrow 0}\Omega_1(f,\delta)_{L^q} = 0$. Then $\forall\, \varepsilon > 0$, $\varepsilon^{\frac{1}{q}} > 0$ and $\exists\, \delta > 0$, $\Omega_1(f,\delta)_{L^q} \le \varepsilon^{\frac{1}{q}}$.
I.e. for any $x, y \in \mathbb{R}^d : \|x-y\|_{l_1} \le \delta$ we get

$$\int_X D^q(f(x,s), f(y,s))\, P(ds) \le \varepsilon.$$

That is $f \in C_{FR}^{U_q}(\mathbb{R}^d)$.

($\Leftarrow$) Let $f \in C_{FR}^{U_q}(\mathbb{R}^d)$. Then $\forall\, \varepsilon > 0\ \exists\, \delta > 0$: whenever $\|x-y\|_{l_1} \le \delta$, $x, y \in \mathbb{R}^d$, it implies

$$\int_X D^q(f(x,s), f(y,s))\, P(ds) \le \varepsilon.$$

I.e. $\forall\, \varepsilon > 0\ \exists\, \delta > 0 : \Omega_1(f,\delta)_{L^q} \le \varepsilon^{\frac{1}{q}}$. That is $\Omega_1(f,\delta)_{L^q} \to 0$ as $\delta \downarrow 0$.

(iii) Let $x_1, x_2 \in \mathbb{R}^d : \|x_1 - x_2\|_{l_1} \le \delta_1 + \delta_2$. Set $x = \frac{\delta_2}{\delta_1+\delta_2}x_1 + \frac{\delta_1}{\delta_1+\delta_2}x_2$, so that $x \in \overline{x_1 x_2}$. Hence $\|x - x_1\|_{l_1} \le \delta_1$ and $\|x_2 - x\|_{l_1} \le \delta_2$. We have

$$\left(\int_X D^q(f(x_1,s), f(x_2,s))\, P(ds)\right)^{\frac{1}{q}} \le$$

$$\left(\int_X D^q(f(x_1,s), f(x,s))\, P(ds)\right)^{\frac{1}{q}} + \left(\int_X D^q(f(x,s), f(x_2,s))\, P(ds)\right)^{\frac{1}{q}} \le$$

$$\Omega_1\left(f, \|x_1 - x\|_{l_1}\right)_{L^q} + \Omega_1\left(f, \|x_2 - x\|_{l_1}\right)_{L^q} \le$$

$$\Omega_1 (f, \delta_1)_{L^q} + \Omega_2 (f, \delta_2)_{L^q} .$$

Therefore (iii) is true.

(iv) and (v) are obvious.

(vi) Notice that

$$\left(\int_X D^q ((f \oplus g)(x,s), (f \oplus g)(y,s)) P(ds) \right)^{\frac{1}{q}} \leq$$

$$\left(\int_X D^q (f(x,s), f(y,s)) P(ds) \right)^{\frac{1}{q}} + \left(\int_X D^q (g(x,s), g(y,s)) P(ds) \right)^{\frac{1}{q}} .$$

That is (vi) is now clear.

(vii) By (iii) we get

$$\left| \Omega_1^{(\mathcal{F})} (f, \delta_1 + \delta_2)_{L^q} - \Omega_1^{(\mathcal{F})} (f, \delta_1)_{L^q} \right| \leq \Omega_1^{(\mathcal{F})} (f, \delta_2)_{L^q} .$$

Let now $f \in C_{FR}^{U_q} (\mathbb{R}^d)$, then by (ii) $\lim_{\delta_2 \downarrow 0} \Omega_1^{(\mathcal{F})} (f, \delta_2)_{L^q} = 0$. That is proving the continuity of $\Omega_1^{(\mathcal{F})} (f, \cdot)_{L^q}$ on $\mathbb{R}_+$. □

We give

Definition 3.11. ([6]) Let $f(t,s)$ be a stochastic process from $\mathbb{R}^d \times (X, \mathcal{B}, P)$ into $\mathbb{R}$, $d \in \mathbb{N}$, where $(X, \mathcal{B}, P)$ is a probability space. We define the q-mean multivariate first moduli of continuity of f by

$$\Omega_1 (f, \delta)_{L^q} :=$$

$$\sup \left\{ \left(\int_X |f(x,s) - f(y,s)|^q P(ds) \right)^{\frac{1}{q}} : x, y \in \mathbb{R}^d, \ \|x - y\|_{l_1} \leq \delta \right\}, \tag{18}$$

$\delta > 0$, $1 \leq q < \infty$.

For more see [6].

We also give

Proposition 3.12. *Assume that $\Omega_1^{(\mathcal{F})} (f, \delta)_{L^q}$ is finite, $\delta > 0$, $1 \leq q < \infty$. Then*

$$\Omega_1^{(\mathcal{F})} (f, \delta)_{L^q} \geq \sup_{r \in [0,1]} \max \left\{ \Omega_1 \left(f_-^{(r)}, \delta \right)_{L^q}, \Omega_1 \left(f_+^{(r)}, \delta \right)_{L^q} \right\}. \tag{19}$$

The reverse direction "≤" "is not possible.

Proof. We observe that

$$D(f(x,s), f(y,s)) =$$

$$\sup_{r \in [0,1]} \max \left\{ \left| f_-^{(r)} (x,s) - f_-^{(r)} (y,s) \right|, \left| f_+^{(r)} (x,s) - f_+^{(r)} (y,s) \right| \right\}$$

$$\geq \left| f_\pm^{(r)} (x,s) - f_\pm^{(r)} (y,s) \right|,$$

respectively in $+, -$.

Hence

$$\left(\int_X D^q\left(f\left(x,s\right),f\left(y,s\right)\right)P\left(ds\right)\right)^{\frac{1}{q}} \geq \left(\int_X \left|f_{\pm}^{(r)}\left(x,s\right)-f_{\pm}^{(r)}\left(y,s\right)\right|^q P\left(ds\right)\right)^{\frac{1}{q}},$$

respectively in $+,-$.

Therefore it holds

$$\sup_{\substack{x,y\in\mathbb{R}^d \\ \|x-y\|_{l_1}\leq\delta}} \left(\int_X D^q\left(f\left(x,s\right),f\left(y,s\right)\right)P\left(ds\right)\right)^{\frac{1}{q}} \geq$$

$$\sup_{r\in[0,1]}\max_{\{+,-\}}\left\{\sup_{\substack{x,y\in\mathbb{R}^d \\ \|x-y\|_{l_1}\leq\delta}}\left(\int_X \left|f_{\pm}^{(r)}\left(x,s\right)-f_{\pm}^{(r)}\left(y,s\right)\right|^q P\left(ds\right)\right)^{\frac{1}{q}}\right\},$$

proving the claim. □

Remark 3.13. For each $s\in X$ we define the usual first modulus of continuity of $f\left(\cdot,s\right)$ by

$$\omega_1^{(\mathcal{F})}\left(f\left(\cdot,s\right),\delta\right):=\sup_{\substack{x,y\in\mathbb{R}^d \\ \|x-y\|_{l_1}\leq\delta}} D\left(f\left(x,s\right),f\left(y,s\right)\right),\quad \delta>0. \tag{20}$$

Therefore

$$D^q\left(f\left(x,s\right),f\left(y,s\right)\right)\leq\left(\omega_1^{(\mathcal{F})}\left(f\left(\cdot,s\right),\delta\right)\right)^q,$$

$\forall\ s\in X$ and $x,y\in\mathbb{R}^d:\|x-y\|_{l_1}\leq\delta$, $\delta>0$.

Hence it holds

$$\left(\int_X D^q\left(f\left(x,s\right),f\left(y,s\right)\right)P\left(ds\right)\right)^{\frac{1}{q}}\leq\left(\int_X\left(\omega_1^{(\mathcal{F})}\left(f\left(\cdot,s\right),\delta\right)\right)^q P\left(ds\right)\right)^{\frac{1}{q}},$$

$\forall\ x,y\in\mathbb{R}^d:\|x-y\|_{l_1}\leq\delta$.

We have that

$$\Omega_1^{(\mathcal{F})}\left(f,\delta\right)_{L^q}\leq\left(\int_X\left(\omega_1^{(\mathcal{F})}\left(f\left(\cdot,s\right),\delta\right)\right)^q P\left(ds\right)\right)^{\frac{1}{q}}, \tag{21}$$

under the assumption that the right hand side of (21) is finite.

The reverse "$\geq$" of the last (21) is not true.

Also we have

Proposition 3.14 ([5]). (i) *Let $Y\left(t,\omega\right)$ be a real valued stochastic process such that Y is continuous in $t\in[a,b]$. Then Y is jointly measurable in $\left(t,\omega\right)$.*

(ii) *Further assume that the expectation $\left(E\left|Y\right|\right)\left(t\right)\in C\left([a,b]\right)$, or more generally $\int_a^b\left(E\left|Y\right|\right)\left(t\right)dt$ makes sense and is finite. Then*

$$E\left(\int_a^b Y\left(t,\omega\right)dt\right)=\int_a^b\left(EY\right)\left(t\right)dt. \tag{22}$$

According to [10], p. 94 we have the following

Definition 3.15. Let $(Y, \mathcal{T})$ be a topological space, with its σ-algebra of Borel sets $\mathcal{B} := \mathcal{B}(Y, \mathcal{T})$ generated by $\mathcal{T}$. If $(X, \mathcal{S})$ is a measurable space, a function $f : X \to Y$ is called measurable iff $f^{-1}(B) \in \mathcal{S}$ for all $B \in \mathcal{B}$.

By Theorem 4.1.6 of [10], p. 89 f as above is measurable iff

$$f^{-1}(C) \in \mathcal{S} \text{ for all } C \in \mathcal{T}.$$

We would need

Theorem 3.16. *(see [10], p. 95) Let $(X, \mathcal{S})$ be a measurable space and (Y, d) be a metric space. Let f_n be measurable functions from X into Y such that for all $x \in X$, $f_n(x) \to f(x)$ in Y. Then f is measurable. I.e., $\lim_{n\to\infty} f_n = f$ is measurable.*

We need also

Proposition 3.17. *Let f, g be fuzzy random variables from $\mathcal{S}$ into $\mathbb{R}_{\mathcal{F}}$. Then*
(i) *Let $c \in \mathbb{R}$, then $c \odot f$ is a fuzzy random variable.*
(ii) *$f \oplus g$ is a fuzzy random variable.*

Finally we need

Proposition 3.18 ([5])**.** *Let $f : [a, b] \to L_{\mathcal{F}}(X, \mathcal{B}, P)$, $a, b \in \mathbb{R}$, be a fuzzy-random function. We assume that $f(t, s)$ is fuzzy continuous in $t \in [a, b]$, $s \in X$. Then $(FR)\int_a^b f(t, s)\,dt$ exists and is a fuzzy-random variable.*

4. Main results

We need the following (see [9]) definitions.

Definition 4.1. A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L_1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a, b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero.

Definition 4.2. (see [9]) A function $b : \mathbb{R}^d \to \mathbb{R}$ $(d \geq 1)$ is said to be a d-dimensional bell-shaped function if it is integrable and its integral is not zero, and for all $i = 1, ..., d$,

$$t \to b(x_1, ..., t, ..., x_d)$$

is a centered bell-shaped function, where $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$ arbitrary.

Example 4.3. (from [9]) Let b be a centered bell-shaped function over $\mathbb{R}$, then $(x_1, ..., x_d) \to b(x_1)...b(x_d)$ is a d-dimensional bell-shaped function, e.g. b could be the characteristic function or the hat function on $[-1, 1]$.

Assumption 4.4. *Here $b(\overrightarrow{x})$ is of compact support $\mathcal{B}^* := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and it may have jump discontinuities there. Here we consider functions $f \in C_{FR}^{U_q}(\mathbb{R}^d)$.*

In this article we study among others in q-mean $(1 \leq q < \infty)$ the pointwise convergence with rates over $\mathbb{R}^d$, to the fuzzy-random unit operator or a perturbation *of it, of the following fuzzy-random multivariate neural network operators,* ($0 < \alpha < 1$, $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$, $s \in X$, $(X, \mathcal{B}, P)$ *a probability space,* $n \in \mathbb{N}$)

$$(M_n(f))(\overrightarrow{x}, s) =$$

$$\frac{\sum_{k_1=-n^2}^{n^2 *} \cdots \sum_{k_d=-n^2}^{n^2 *} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}, s\right) \odot b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}. \tag{23}$$

In short, we can write

$$(M_n(f))(\overrightarrow{x}, s) = \frac{\sum_{\overrightarrow{k}=-\overrightarrow{n^2}}^{\overrightarrow{n^2} *} f\left(\frac{\overrightarrow{k}}{n}, s\right) \odot b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{\sum_{\overrightarrow{k}=-\overrightarrow{n^2}}^{\overrightarrow{n^2}} b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}. \tag{24}$$

In this article we assume that

$$n \geq \max_{i \in \{1,...,d\}} \left\{ T_i + |x_i|, T_i^{-\frac{1}{\alpha}} \right\}, \tag{25}$$

see also [2], *p. 91.*

So, by (25) we can rewrite ($[\cdot]$ *is the integral part of a number, while* $\lceil \cdot \rceil$ *is the ceiling of a number)*

$$(M_n(f))(\overrightarrow{x}, s) =$$

$$\frac{\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1 + T_1 n^\alpha] *} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d + T_d n^\alpha] *} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}, s\right) \odot b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1 + T_1 n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d + T_d n^\alpha]} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}. \tag{26}$$

In short we can write

$$(M_n(f))(\overrightarrow{x}, s) = \frac{\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x} - \overrightarrow{T} n^\alpha \rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^\alpha] *} f\left(\frac{\overrightarrow{k}}{n}, s\right) \odot b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x} - \overrightarrow{T} n^\alpha \rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^\alpha]} b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}. \tag{27}$$

Denoting

$$V(\overrightarrow{x}) := \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x} - \overrightarrow{T} n^\alpha \rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^\alpha]} b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right), \tag{28}$$

we will write and use from now on that

$$(M_n(f))(\overrightarrow{x}, s) = \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*} f\left(\frac{\overrightarrow{k}}{n}, s\right) \odot \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}. \tag{29}$$

The above M_n are linear operators over $\mathbb{R}^d \times X$.

Related works were done in [7] *and* [8].

We present

Theorem 4.5. *Here all as above. Then*

$$\left(\int_X D^q\left((M_n(f))(\overrightarrow{x}, s), f(\overrightarrow{x}, s)\right) P(ds)\right)^{\frac{1}{q}} \leq \Omega_1^{(\mathcal{F})}\left(f, \frac{\sum_{i=1}^{d} T_i}{n^{1-\alpha}}\right)_{L^q}. \tag{30}$$

As $n \to \infty$, we get that

$$(M_n f)(\overrightarrow{x}, s) \overset{\text{"q-mean"}}{\to} f(\overrightarrow{x}, s)$$

with rates.

Proof. We observe that

$$D\left((M_n(f))(\overrightarrow{x}, s), f(\overrightarrow{x}, s)\right) =$$

$$D\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*} f\left(\frac{\overrightarrow{k}}{n}, s\right) \odot \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}, f(\overrightarrow{x}, s) \odot 1\right) =$$

$$D\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*} f\left(\frac{\overrightarrow{k}}{n}, s\right) \odot \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}, f(\overrightarrow{x}, s) \odot \frac{V(\overrightarrow{x})}{V(\overrightarrow{x})}\right) =$$

$$D\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*} f\left(\frac{\overrightarrow{k}}{n}, s\right) \odot \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}, \right.$$

$$\left. \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*} f(\overrightarrow{x}, s) \odot \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\right) \leq$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})} D\left(f\left(\frac{\overrightarrow{k}}{n}, s\right), f(\overrightarrow{x}, s)\right).$$

That is it holds

$$D\left((M_n(f))(\overrightarrow{x}, s), f(\overrightarrow{x}, s)\right) \leq \tag{31}$$

$$\sum_{\vec{k}=\lceil n\vec{x}-\vec{T}n^{\alpha}\rceil}^{[n\vec{x}+\vec{T}n^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(\vec{x}-\frac{\vec{k}}{n}\right)\right)}{V(\vec{x})} D\left(f\left(\frac{\vec{k}}{n}, s\right), f(\vec{x}, s)\right).$$

Hence

$$\left(\int_X D^q\left(\left(M_n(f)\right)(\vec{x}, s), f(\vec{x}, s)\right) P(ds)\right)^{\frac{1}{q}} \le$$

$$\sum_{\vec{k}=\lceil n\vec{x}-\vec{T}n^{\alpha}\rceil}^{[n\vec{x}+\vec{T}n^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(\vec{x}-\frac{\vec{k}}{n}\right)\right)}{V(\vec{x})} \left(\int_X D^q\left(f\left(\frac{\vec{k}}{n}, s\right), f(\vec{x}, s)\right) P(ds)\right)^{\frac{1}{q}} \le$$

$$\sum_{\vec{k}=\lceil n\vec{x}-\vec{T}n^{\alpha}\rceil}^{[n\vec{x}+\vec{T}n^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(\vec{x}-\frac{\vec{k}}{n}\right)\right)}{V(\vec{x})} \Omega_1^{(\mathcal{F})}\left(f, \left\|\vec{x}-\frac{\vec{k}}{n}\right\|_{l_1}\right)_{L^q} \le \tag{32}$$

$$\sum_{\vec{k}=\lceil n\vec{x}-\vec{T}n^{\alpha}\rceil}^{[n\vec{x}+\vec{T}n^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(\vec{x}-\frac{\vec{k}}{n}\right)\right)}{V(\vec{x})} \Omega_1^{(\mathcal{F})}\left(f, \frac{\sum_{i=1}^{d} T_i}{n^{1-\alpha}}\right)_{L^q} =$$

$$\Omega_1^{(\mathcal{F})}\left(f, \frac{\sum_{i=1}^{d} T_i}{n^{1-\alpha}}\right)_{L^q}. \tag{33}$$

Condition (25) implies that

$$\left|x_i - \frac{k_i}{n}\right| \le \frac{T_i}{n^{1-\alpha}}, \quad \text{all } i = 1, ..., d. \tag{34}$$

The proof of (30) is now finished. □

Remark 4.6. Consider the fuzzy-random perturbed unit operator

$$\left(T_n(f)\right)(\vec{x}, s) := f\left(\vec{x} - \frac{\vec{T}}{n^{1-\alpha}}, s\right),$$

$\forall\, (\vec{x}, s) \in \mathbb{R}^d \times X$; $\vec{T} := (T_1, ..., T_d)$, $n \in \mathbb{N}$, $1 \le q < \infty$.

We observe that

$$\int_X D^q\left(\left(T_n(f)(\vec{x}, s), f(\vec{x}, s)\right) P(ds)\right)^{\frac{1}{q}} =$$

$$\left(\int_X D^q\left(f\left(\vec{x} - \frac{\vec{T}}{n^{1-\alpha}}, s\right), f(\vec{x}, s)\right) P(ds)\right)^{\frac{1}{q}} \le \Omega_1^{(\mathcal{F})}\left(f, \frac{\sum_{i=1}^{d} T_i}{n^{1-\alpha}}\right)_{L^q}. \tag{35}$$

Given that $f \in C_{FR}^{U_q}\left(\mathbb{R}^d\right)$, we get that $\left(T_n\left(f\right)\right)\left(\overrightarrow{x}, s\right) \overset{\text{"q-mean"}}{\underset{n\to\infty}{\to}} f\left(\overrightarrow{x}, s\right), \forall \left(\overrightarrow{x}, s\right) \in \mathbb{R}^d \times X$.

Next we estimate in high order with rates the 1-mean difference

$$\int_X D\left(\left(M_n\left(f\right)\right)\left(\overrightarrow{x}, s\right), \left(T_n\left(f\right)\right)\left(\overrightarrow{x}, s\right)\right) P\left(ds\right), \quad n \in \mathbb{N}.$$

We make

Assumption 4.7. *Let $\overrightarrow{x} \in \mathbb{R}^d$, $d \in \mathbb{N}$, $s \in X$; where $(X, \mathcal{B}, P)$ is a probability space, n as in (25), b of compact support $\mathcal{B}^*$, $0 < \alpha < 1$, M_n as in (29).*

Let $f : \mathbb{R}^d \to L_{\mathcal{F}}\left(X, \mathcal{B}, P\right)$ be a fuzzy continuous in $\overrightarrow{x} \in \mathbb{R}^d$ random function.

We assume that all H-fuzzy partial derivatives of f up to order $N \in \mathbb{N}$ exist and are fuzzy continuous in $\overrightarrow{x} \in \mathbb{R}^d$ and all belong to $C_{FR}^{U_1}\left(\mathbb{R}^d\right)$.

Furthermore we assume that

$$\int_X D\left(f_\alpha\left(\overrightarrow{x}, s\right), \widetilde{o}\right) P\left(ds\right) < \infty, \tag{36}$$

for all $\alpha : |\alpha| = j$, $j = 1, ..., N$.

Call

$$\Omega_{1,N}^{(\mathcal{F})}\left(f_\alpha^{\max}, \delta\right)_{L^1} := \max_{|\alpha|=N} \Omega_1^{(\mathcal{F})}\left(f_\alpha, \delta\right)_{L^1}, \quad \delta > 0. \tag{37}$$

We give

Theorem 4.8. *All here as in Assumption 4.7. Then*

$$\int_X D\left(\left(M_n\left(f\right)\right)\left(\overrightarrow{x}, s\right), \left(T_n\left(f\right)\right)\left(\overrightarrow{x}, s\right)\right) P\left(ds\right) \leq$$

$$\sum_{j=1}^{N}\left[\sum_{|\alpha|=j}\left(\frac{1}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\right.$$

$$\left.\left\{\int_X D\left(f_\alpha\left(\overrightarrow{x}, s\right), \widetilde{o}\right) P\left(ds\right) + \Omega_1^{(\mathcal{F})}\left(f_\alpha, \frac{\sum_{i=1}^{d} T_i}{n^{1-\alpha}}\right)_{L^1}\right\}\right]$$

$$+\left[\sum_{|\alpha|=N}\left(\frac{1}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\right]\Omega_{1,N}^{(\mathcal{F})}\left(f_\alpha^{\max}, \frac{2\left(\sum_{i=1}^{d} T_i\right)}{n^{1-\alpha}}\right)_{L^1}. \tag{38}$$

By (38), as $n \to \infty$, we get

$$\int_X D\left(\left(M_n\left(f\right)\right)\left(\overrightarrow{x}, s\right), \left(T_n\left(f\right)\right)\left(\overrightarrow{x}, s\right)\right) P\left(ds\right) \to 0,$$

with rates in high order.

Proof. By (25) we get that

$$-\frac{T_i}{n^{1-\alpha}} \leq x_i - \frac{k_i}{n} \leq \frac{T_i}{n^{1-\alpha}}, \quad \text{all } i = 1, ..., d, \ x_i \in \mathbb{R}. \tag{39}$$

We consider the case

$$x_i - \frac{T_i}{n^{1-\alpha}} \leq \frac{k_i}{n}, \quad i = 1, ..., d. \tag{40}$$

Set

$$g_{\frac{\overrightarrow{k}}{n}}(t, s) := f\left(\overrightarrow{x} - \frac{\overrightarrow{T}}{n^{1-\alpha}} + t\left(\frac{\overrightarrow{k}}{n} - \overrightarrow{x} + \frac{\overrightarrow{T}}{n^{1-\alpha}}\right), s\right), \tag{41}$$

$0 \leq t \leq 1$, $\forall\ s \in X$.

We apply Theorem 2.11, and we have by the fuzzy multivariate Taylor formula that

$$f\left(\frac{\overrightarrow{k}}{n}, s\right) = f\left(\overrightarrow{x} - \frac{\overrightarrow{T}}{n^{1-\alpha}}, s\right) \oplus \sum_{j=1}^{N-1*} \frac{g^{(j)}_{\frac{\overrightarrow{k}}{n}}(0, s)}{j!} \oplus \mathcal{R}_N(0, 1, s), \tag{42}$$

where

$$\mathcal{R}_N(0, 1, s) := \frac{1}{(N-1)!} \odot (FR) \int_0^1 (1-\theta)^{N-1} \odot g^{(N)}_{\frac{\overrightarrow{k}}{n}}(\theta, s)\, d\theta, \tag{43}$$

$\forall\ s \in X$.

Here for $j = 1, ..., N$, we obtain

$$g^{(j)}_{\frac{\overrightarrow{k}}{n}}(\theta, s) = \left[\left(\sum_{i=1}^{d*}\left(\frac{k_i}{n} - x_i + \frac{T_i}{n^{1-\alpha}}\right) \odot \frac{\partial}{\partial x_i}\right)^j f\right](x_1, x_2, ..., x_d, s), \tag{44}$$

$0 \leq \theta \leq 1$, $\forall\ s \in X$.

More precisely we have for $j = 1, ..., N$, that

$$g^{(j)}_{\frac{\overrightarrow{k}}{n}}(0, s) = \sum^{*}_{\substack{\alpha := (\alpha_1, ..., \alpha_d),\ \alpha_i \in \mathbb{Z}^+ \\ i = 1, ..., d,\ |\alpha| := \sum_{i=1}^{d} \alpha_i = j}} \left(\frac{j!}{\prod_{i=1}^{d} \alpha_i!}\right) \left(\prod_{i=1}^{d}\left(\frac{k_i}{n} - x_i + \frac{T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)$$

$$\odot f_\alpha\left(\overrightarrow{x} - \frac{\overrightarrow{T}}{n^{1-\alpha}}, s\right), \tag{45}$$

$\forall\ s \in X$, and

$$g^{(N)}_{\frac{\overrightarrow{k}}{n}}(\theta, s) = \sum^{*}_{\substack{\alpha := (\alpha_1, ..., \alpha_d),\ \alpha_i \in \mathbb{Z}^+ \\ i = 1, ..., d,\ |\alpha| := \sum_{i=1}^{d} \alpha_i = N}} \left(\frac{N!}{\prod_{i=1}^{d} \alpha_i!}\right) \left(\prod_{i=1}^{d}\left(\frac{k_i}{n} - x_i + \frac{T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right) \tag{46}$$

$$\odot f_{\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}}+\theta\left(\frac{\overrightarrow{k}}{n}-\overrightarrow{x}+\frac{\overrightarrow{T}}{n^{1-\alpha}}\right),s\right),$$

$0\leq\theta\leq 1$, $\forall\ s\in X$.

Multiplying (42) by $\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}$ and applying $\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*}$ to both sides, we obtain

$$(47)\qquad (M_n(f))(\overrightarrow{x},s)=f\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}},s\right)\oplus\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*}\sum_{j=1}^{N-1*}\frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0,s)}{j!}$$

$$\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\oplus\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*}\mathcal{R}_N(0,1,s)\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})},$$

$\forall\ s\in X$.

Next we observe

$$D\left((M_n(f))(\overrightarrow{x},s),f\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}},s\right)\right)\overset{((47),(1))}{=}$$

$$D\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*}\sum_{j=1}^{N-1*}\frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0,s)}{j!}\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\oplus\right.$$

$$(48)\qquad \left.\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*}\mathcal{R}_N(0,1,s)\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})},\widetilde{o}\right)\overset{(1)}{=}$$

$$D\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*}\sum_{j=1}^{N-1*}\frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0,s)}{j!}\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\right.$$

$$\oplus\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*}\mathcal{R}_N(0,1,s)\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})},$$

$$\left.\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*}\frac{g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0,s)}{N!}\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\right)\overset{(1)}{\leq}$$

$$\sum_{j=1}^{N}\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)j!}D\left(g_{\frac{\overrightarrow{k}}{n}}^{(j)}\left(0,s\right),\widetilde{o}\right)+ \tag{49}$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}D\left(\mathcal{R}_N\left(0,1,s\right),\frac{g_{\frac{\overrightarrow{k}}{n}}^{(N)}\left(0,s\right)}{N!}\right).$$

So we obtain

$$D\left(\left(M_n\left(f\right)\right)\left(\overrightarrow{x},s\right),\left(T_n\left(f\right)\right)\left(\overrightarrow{x},s\right)\right)\leq$$

$$\sum_{j=1}^{N}\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)j!}D\left(g_{\frac{\overrightarrow{k}}{n}}^{(j)}\left(0,s\right),\widetilde{o}\right)+ \tag{50}$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}D\left(\mathcal{R}_N\left(0,1,s\right),\frac{g_{\frac{\overrightarrow{k}}{n}}^{(N)}\left(0,s\right)}{N!}\right),$$

$\forall\ s\in X$.

Notice that

$$\frac{g_{\frac{\overrightarrow{k}}{n}}^{(N)}\left(0,s\right)}{N!}=\frac{1}{(N-1)!}\odot(FR)\int_0^1(1-\theta)^{N-1}\odot g_{\frac{\overrightarrow{k}}{n}}^{(N)}\left(0,s\right)d\theta. \tag{51}$$

We estimate

$$D\left(\mathcal{R}_N\left(0,1,s\right),\frac{g_{\frac{\overrightarrow{k}}{n}}^{(N)}\left(0,s\right)}{N!}\right)\overset{(51)}{=}$$

$$D\left(\frac{1}{(N-1)!}\odot(FR)\int_0^1(1-\theta)^{N-1}\odot g_{\frac{\overrightarrow{k}}{n}}^{(N)}\left(\theta,s\right)d\theta,\right.$$

$$\left.\frac{1}{(N-1)!}\odot(FR)\int_0^1(1-\theta)^{N-1}\odot g_{\frac{\overrightarrow{k}}{n}}^{(N)}\left(0,s\right)d\theta\right)\overset{(9)}{\leq} \tag{52}$$

$$\frac{1}{(N-1)!}\int_0^1(1-\theta)^{N-1}D\left(g_{\frac{\overrightarrow{k}}{n}}^{(N)}\left(\theta,s\right),g_{\frac{\overrightarrow{k}}{n}}^{(N)}\left(0,s\right)\right)d\theta\overset{(46)}{\leq}$$

$$\frac{1}{(N-1)!}\int_0^1(1-\theta)^{N-1}\left[\sum_{|\alpha|=N}\left(\frac{N!}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{k_i}{n}-x_i+\frac{T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\right.$$

$$\left.D\left(f_\alpha\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}}+\theta\left(\frac{\overrightarrow{k}}{n}-\overrightarrow{x}+\frac{\overrightarrow{T}}{n^{1-\alpha}}\right),s\right),f_\alpha\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}},s\right)\right)\right]d\theta\leq$$

$$\frac{1}{(N-1)!}\left[\sum_{|\alpha|=N}\left(\frac{N!}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\int_0^1(1-\theta)^{N-1}\right.$$

$$D\left(f_\alpha\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}}+\theta\left(\frac{\overrightarrow{k}}{n}-\overrightarrow{x}+\frac{\overrightarrow{T}}{n^{1-\alpha}}\right),s\right),f_\alpha\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}},s\right)\right)d\theta\Bigg]. \tag{53}$$

Therefore it holds

$$\int_X D\left(\mathcal{R}_N(0,1,s),\frac{g^{(N)}_{\frac{\overrightarrow{k}}{n}}(0,s)}{N!}\right)P(ds)\overset{\text{(by (53) and Tonelli's ([10]) theorem)}}{\leq}$$

$$\frac{1}{(N-1)!}\left[\sum_{|\alpha|=N}\left(\frac{N!}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\int_0^1(1-\theta)^{N-1}\right. \tag{54}$$

$$\left(\int_X D\left(f_\alpha\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}}+\theta\left(\frac{\overrightarrow{k}}{n}-\overrightarrow{x}+\frac{\overrightarrow{T}}{n^{1-\alpha}}\right),s\right),\right.\right.$$

$$\left.\left.f_\alpha\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}},s\right)\right)P(ds)\right)d\theta\Bigg]\leq$$

$$\frac{1}{(N-1)!}\left[\sum_{|\alpha|=N}\left(\frac{N!}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\right.$$

$$\int_0^1(1-\theta)^{N-1}\Omega_1^{(\mathcal{F})}\left(f_\alpha,\theta\left(\sum_{i=1}^{d}\left(\frac{k_i}{n}-x_i+\frac{T_i}{n^{1-\alpha}}\right)\right)\right)_{L^1}d\theta\Bigg]\leq$$

$$\sum_{|\alpha|=N}\left(\frac{1}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\Omega_1^{(\mathcal{F})}\left(f_\alpha,\frac{2\left(\sum_{i=1}^{d}T_i\right)}{n^{1-\alpha}}\right)_{L^1}\leq \tag{55}$$

$$\left[\sum_{|\alpha|=N}\left(\frac{1}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\right]\Omega_{1,N}^{(\mathcal{F})}\left(f_\alpha^{\max},\frac{2\left(\sum_{i=1}^{d}T_i\right)}{n^{1-\alpha}}\right)_{L^1}.$$

We have proved that

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lfloor n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rfloor} \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}\left(\int_X D\left(\mathcal{R}_N\left(0,1,s\right),\frac{g^{(N)}_{\frac{\overrightarrow{k}}{n}}\left(0,s\right)}{N!}\right)P\left(ds\right)\right)$$

$$\leq \left[\sum_{|\alpha|=N}\left(\frac{1}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\right]\Omega_{1,N}^{(\mathcal{F})}\left(f_{\alpha}^{\max},\frac{2\left(\sum_{i=1}^{d}T_i\right)}{n^{1-\alpha}}\right)_{L^1}. \tag{56}$$

Next we extimate

$$\frac{1}{j!}D\left(g^{(j)}_{\frac{\overrightarrow{k}}{n}}\left(0,s\right),\widetilde{o}\right)=$$

(57)

$$\frac{1}{j!}D\left(\sum_{|\alpha|=j}^{*}\left(\frac{j!}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{k_i}{n}-x_i+\frac{T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\odot f_{\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}},s\right),\widetilde{o}\right)\leq$$

$$\sum_{|\alpha|=j}\left(\frac{1}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{k_i}{n}-x_i+\frac{T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)D\left(f_{\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}},s\right),\widetilde{o}\right)\leq$$

$$\sum_{|\alpha|=j}\left(\frac{1}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)$$

$$\left[D\left(f_{\alpha}\left(\overrightarrow{x},s\right),\widetilde{o}\right)+D\left(f_{\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}},s\right),f_{\alpha}\left(\overrightarrow{x},s\right)\right)\right]. \tag{58}$$

Consequently we have

$$\frac{1}{j!}\int_X D\left(g^{(j)}_{\frac{\overrightarrow{k}}{n}}\left(0,s\right),\widetilde{o}\right)P\left(ds\right)\leq\sum_{|\alpha|=j}\left(\frac{1}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)$$

$$\left\{\int_X D\left(f_{\alpha}\left(\overrightarrow{x},s\right),\widetilde{o}\right)P\left(ds\right)+\Omega_1^{(\mathcal{F})}\left(f_{\alpha},\frac{\sum_{i=1}^{d}T_i}{n^{1-\alpha}}\right)_{L^1}\right\}. \tag{59}$$

Furthermore it holds

$$\sum_{j=1}^{N} \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right) j!} \int_X D\left(g_{\frac{\overrightarrow{k}}{n}}^{(j)}\left(0,s\right),\widetilde{o}\right) P\left(ds\right) \leq$$

$$\sum_{j=1}^{N} \left[\sum_{|\alpha|=j} \left(\frac{1}{\prod_{i=1}^{d} \alpha_i!} \right) \left(\prod_{i=1}^{d} \left(\frac{2T_i}{n^{1-\alpha}} \right)^{\alpha_i} \right) \right.$$

$$\left. \left\{ \int_X D\left(f_\alpha\left(\overrightarrow{x},s\right),\widetilde{o}\right) P\left(ds\right) + \Omega_1^{(\mathcal{F})}\left(f_\alpha, \frac{\sum_{i=1}^{d} T_i}{n^{1-\alpha}} \right)_{L^1} \right\} \right]. \tag{60}$$

By (50) we get

$$\int_X D\left(\left(M_n\left(f\right)\right)\left(\overrightarrow{x},s\right),\left(T_n\left(f\right)\right)\left(\overrightarrow{x},s\right)\right) P\left(ds\right) \leq$$

$$\sum_{j=1}^{N} \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right) j!} \int_X D\left(g_{\frac{\overrightarrow{k}}{n}}^{(j)}\left(0,s\right),\widetilde{o}\right) P\left(ds\right) + \tag{61}$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)} \left(\int_X D\left(\mathcal{R}_N\left(0,1,s\right), \frac{g_{\frac{\overrightarrow{k}}{n}}^{(N)}\left(0,s\right)}{N!} \right) P\left(ds\right) \right)$$

$$\overset{((56),\,(60))}{\leq} \sum_{j=1}^{N} \left[\sum_{|\alpha|=j} \left(\frac{1}{\prod_{i=1}^{d} \alpha_i!} \right) \left(\prod_{i=1}^{d} \left(\frac{2T_i}{n^{1-\alpha}} \right)^{\alpha_i} \right) \right.$$

$$\left. \left\{ \int_X D\left(f_\alpha\left(\overrightarrow{x},s\right),\widetilde{o}\right) P\left(ds\right) + \Omega_1^{(\mathcal{F})}\left(f_\alpha, \frac{\sum_{i=1}^{d} T_i}{n^{1-\alpha}} \right)_{L^1} \right\} \right] + \tag{62}$$

$$\left[\sum_{|\alpha|=j} \left(\frac{1}{\prod_{i=1}^{d} \alpha_i!} \right) \left(\prod_{i=1}^{d} \left(\frac{2T_i}{n^{1-\alpha}} \right)^{\alpha_i} \right) \right] \Omega_{1,N}^{(\mathcal{F})}\left(f_\alpha^{\max}, \frac{2\left(\sum_{i=1}^{d} T_i\right)}{n^{1-\alpha}} \right)_{L^1},$$

proving (38). □

Remark 4.9. Inequality (38) reveals that the operators M_n behave in good approximation like the simple operators T_n. So T_n is a good simplification of M_n.

We give the following definition

Definition 4.10. Let the nonnegative function $S : \mathbb{R}^d \to \mathbb{R}$, $d \geq 1$, S has compact support $\mathcal{B}^* := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and is nondecreasing there for each coordinate. S can be continuous only on either $\prod_{i=1}^{d} (-\infty, T_i]$ or $\mathcal{B}^*$ and can have jump discontinuities. We call S the multivariate "squashing function" (see also [9]).

Comment 4.11. *If the operators M_n, see (23), replace b by S, we derive the normalized "squashing "operators K_n. Then Theorems 4.5, 4.8 remain valid for K_n, just replace M_n by K_n there.*

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