Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 6, No. 1, (July 2013), pp. 181–189
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Prime bi-ideals in ternary semirings

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Received 15 September 2012; Accepted 27 October 2012

Abstract. This paper is based on the idea of bi-ideals in ternary semirings. The concept of prime, strongly prime and semiprime bi-ideals in ternary semirings stand defined. Besides, irreducible and strongly irreducible bi-ideals in ternary semirings have been thoroughly studied. It stands confirmed that" each bi-ideal of a regular ternary semiring A is strongly prime if exclusively each bi-ideal of A is idempotent and the set of bi-ideals of A is totally ordered by inclusion". The set of all strongly prime proper bi-ideals of a ternary semiring A forms a topology.

2010 AMS Classification: 06D72, 20N10

Keywords: Prime, Strongly prime, Semiprime, Irreducible and strongly irreducible bi-ideals in ternary semirings.

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1. Introduction

Algebraic structures play a very outstanding role in mathematics with classification in multifarious disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological areas etc. Ternary generalizations of algebraic structures are the very natural ways for further developement and indepth comprehension of their basic traits.

Cayley for the first time pioneered and launched first ternary algebraic operations in the way back in the 19th century. Cayley's ideas expounded and developed n-ary generalizations of matrices and their determinants [9, 13] and general theory of n-ary algebras [3, 10] and ternary rings [11] (For physical apllications in Nambu mechanics, supersymmetry, Yang-Baxter equations etc). Ternary structures and their generalizations creat some hopes because of their possibility of applications in physics. A few important physical applications are listed in [1, 2, 6, 7]. In pursuance of Lister's generalizations of ternary rings introduced in 1971, T. K. Dutta and S. Kar came up with the notion of ternary semirings.

T. K. Dutta and S. Kar initiated prime ideals and prime radical of ternary semirings in [4]. The same researchers launched semiprime ideals and irreducible ideals of ternary semirings in [5]. Furthermore S. Kar in [8] came up with the notion of quasi-ideals and bi-ideals in ternary semirings. Similarly, M. Shabir and M. Bano floated prime bi-ideals in ternary semigroups in [12]. In the opening section of this paper, we assemble requisite material on prime, strongly prime and semiprime bi-ideals in ternary semirings. Deploying the taxonomic order, we define irreducible and strongly irreducible bi-ideals in ternary semirings and specify some classes of ternary semirings by the characteristics of these ideals.

2. Preliminaries

Definition 2.1 ([4]). A non-empty set A together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a *ternary semiring*, if A is an additive commutative semigroup satisfying the following conditions:

(i) $(abc)\,de = a\,(bcd)\,e = ab\,(cde)$
(ii) $(a+b)\,cd = acd + bcd$
(iii) $a\,(b+c)\,d = abd + acd$
(iv) $ab\,(c+d) = abc + abd$

for all $a, b, c, d, e \in A$.

Definition 2.2 ([4]). Let A be a ternary semiring. If there exists an element $0 \in A$ such that $0 + x = x = x + 0$ and $0xy = x0y = xy0 = 0$ for all $x, y \in A$, then $'0'$ is called the *zero element* of the ternary semiring A. In this case we say that A is a *ternary semiring with zero*.

We shall always assume that $(A, +, \cdot)$ is a ternary semiring with zero.

Definition 2.3 ([4]). A ternary semiring A is said to be *commutative* if $abc = acb = cab = cba$ for all $a, b, c \in A$.

We note that if A is commutative, then also $abc = acb = cab = cba$ for all $a, b, c \in A$.

Definition 2.4 ([4]). Let B, C, D be three subsets of A. Then by BCD, we mean the set of all finite sums of the form $\sum b_i c_i d_i$ with $b_i \in B$, $c_i \in C$, $d_i \in D$.

Definition 2.5 ([4]). An additive subsemigroup T of a ternary semiring A is called a *ternary subsemiring* if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.6 ([4]). An additive subsemigroup I of A is called *left* (*right, lateral*) *ideal* of A if $s_1s_2i\,(is_1s_2, s_1is_2) \in I$ for all $s_1, s_2 \in A$ and $i \in I$. If I is a left, a right and a lateral ideal of A, then I is called an ideal of A.

Definition 2.7 ([5]). An element a of a ternary semiring A is said to be regular if there exists an element x in A such that $a = axa$. A ternary semiring A is called regular if every element of A is regular.

Definition 2.8 ([4]). An element a of a ternary semiring A is an idempotent element if $a^3 = a$. A ternary semiring A is called idempotent ternery semiring if every element of A is idempotent. An ideal I of A is called idempotent if $I^3 = I$.

Definition 2.9 ([8]). By a *bi-ideal* of a ternary semiring A, we mean a ternary subsemiring B of A such that $BABAB \subseteq B$.

Proposition 2.10. *The intersection of any family of bi-ideals of a ternary semiring A is a bi-ideal of A.*

Proof. Let $\{B_i : i \in I\}$ be any family of bi-ideals of a ternary semiring A. Let $x, y \in \bigcap_{i\in I} B_i$. Then $x, y \in B_i$ for all $i \in I$. As each B_i is a bi-ideal of A, we have $x + y \in B_i$ for all $i \in I$ implies $x + y \in \bigcap_{i\in I} B_i$. Now let $x, y, z \in \bigcap_{i\in I} B_i$. Then $x, y, z \in B_i$ for all $i \in I$. As each B_i is a bi-ideal of A, we have $xyz \in B_i$ for all $i \in I$ implies $xyz \in \bigcap_{i\in I} B_i$. Now as $B_iAB_iAB_i \subseteq B_i$ and $\bigcap_{i\in I} B_i \subseteq B_i$ for all $i \in I$, we have $\left(\bigcap_{i\in I} B_i\right) A \left(\bigcap_{i\in I} B_i\right) A \left(\bigcap_{i\in I} B_i\right) \subseteq B_iAB_iAB_i \subseteq B_i$ for all $i \in I$. This implies $\left(\bigcap_{i\in I} B_i\right) A \left(\bigcap_{i\in I} B_i\right) A \left(\bigcap_{i\in I} B_i\right) \subseteq \bigcap_{i\in I} B_i$. Thus $\bigcap_{i\in I} B_i$ is a bi-ideal of A. □

Proposition 2.11 ([8]). *Every left (right, lateral) ideal of a ternary semiring A is a bi-ideal of A.*

Proposition 2.12 ([8]). *If B is a bi-ideal of a regular ternary semiring A and X, Y, be any non-empty subsets of A, then BXY, XBY and XYB are bi-ideals of A.*

Corollary 2.13 ([8]). *If B_1, B_2, B_3 are any bi-ideals of a regular ternary semiring A, then $B_1B_2B_3$ is a bi-ideal of A.*

3. Prime, strongly prime and semiprime bi-ideal of ternary semiring

Definition 3.1. A bi-ideal B of a ternary semiring A is called a *prime* bi-ideal if $B_1B_2B_3 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ideals B_1, B_2, B_3 of A.

Definition 3.2. A bi-ideal B of a ternary semiring A is called a *strongly prime* bi-ideal of A if $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ideals B_1, B_2, B_3 of A.

Definition 3.3. A bi-ideal B of a ternary semiring A is called a *semiprime* bi-ideal of A if $B_1^3 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ideal B_1 of A.

Proposition 3.4. *Every strongly prime bi-ideal of a ternary semiring A is a prime bi-ideal of A.*

Proof. Let B be a strongly prime bi-ideal of a ternary semiring A. Let B_1, B_2, B_3 be three bi-ideals of A such that $B_1B_2B_3 \subseteq B$. This implies $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B$. Thus $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is a prime bi-ideal of A. □

Proposition 3.5. *Every prime bi-ideal of a ternary semiring A is a semiprime bi-ideal of A.*

Proof. Let B be a prime bi-ideal of a ternary semiring A. Now let B_1 be any bi-ideal of A such that $B_1^3 \subseteq B$. Then $B_1 \subseteq B$. Thus B is a semiprime bi-ideal of A. □

Remark 3.6. A prime bi-ideal of a ternary semiring is not necessarily a strongly prime bi-ideal and a semiprime bi-ideal of a ternary semiring is not necessarily a prime bi-ideal. This fact is clear from the following examples:

Example 3.7. Consider $A = \{0, a, b\}$. Define addition and multiplication on A as

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Then $(A, +, \cdot)$ is a ternary semiring with ternary operation of multiplication defined as, $abc = a(bc)$ and bi-ideals of A are $\{0\}, \{0, a\}, \{0, b\}$ and A. $\{0\}$ is a prime bi-ideal of A but not a strongly prime bi-ideal of A, as $\{0, a\}\{0, a\}\{0, b\} \cap \{0, a\}\{0, b\}\{0, a\} \cap \{0, b\}\{0, a\}\{0, a\} = \{0, a\} \cap \{0, a\} \cap \{0, b\} = \{0\} \subseteq \{0\}$. But neither $\{0, a\}$ nor $\{0, b\}$ is contained in $\{0\}$.

Example 3.8. Let $A = \{\Phi, \{a\}, \{b\}, \{a, b\}\}$. Define addition and ternary multiplication on A as $X + Y = X\Delta Y = (X \cup Y) - (X \cap Y)$ and $(X \cdot Y) \cdot Z = X \cap Y \cap Z$ for all $X, Y, Z \in A$. Then A is a ternary semiring. Bi-ideals of A are $\{\Phi\}, \{\Phi, \{a\}\}, \{\Phi, \{b\}\}$ and $\{\Phi, \{a\}, \{b\}, \{a, b\}\}$. Since $X^3 = X$ for all $X \in A$, so each bi-ideal of A is semiprime. In particular $\{\Phi\}$ is a semiprime bi-ideal of A but not a prime bi-ideal of A, because $\{\Phi, \{a\}\} \cdot \{\Phi, \{b\}\} \cdot \{\Phi, \{a\}, \{b\}, \{a, b\}\} = \{\Phi\} \subseteq \{\Phi\}$. But none of $\{\Phi, \{a\}\}, \{\Phi, \{b\}\}$ and $\{\Phi, \{a\}, \{b\}, \{a, b\}\}$ is contained in $\{\Phi\}$.

Proposition 3.9. *The intersection of any family of prime bi-ideals of a ternary semiring A is a semiprime bi-ideal of A.*

Proof. Let $\{B_i : i \in I\}$ be any family of prime bi-ideals of a ternary semiring A. We have to show that $\underset{i\in I}{\cap} B_i$ is a semiprime bi-ideal of A. By Proposition 2.10, $\underset{i\in I}{\cap} B_i$ is a bi-ideal of A. Now let B be any bi-ideal of A such that $B^3 \subseteq \underset{i\in I}{\cap} B_i$, implies $BBB = B^3 \subseteq B_i$ for all $i \in I$. Thus $B \subseteq B_i$ for all $i \in I$, because each B_i is a prime bi-ideal of A. This implies $B \subseteq \underset{i\in I}{\cap} B_i$. Hence $\underset{i\in I}{\cap} B_i$ is a semiprime bi-ideal of A. □

4. Irreducible and strongly irreducible bi-ideal of ternary semiring

Definition 4.1. A bi-ideal B of a ternary semiring A is called an *irreducible bi-ideal* of A if $B_1 \cap B_2 \cap B_3 = B$ implies either $B_1 = B$ or $B_2 = B$ or $B_3 = B$ for any bi-ideals B_1, B_2, B_3 of A.

Definition 4.2. A bi-ideal B of a ternary semiring A is called a *strongly irreducible bi-ideal* of A if $B_1 \cap B_2 \cap B_3 \subseteq B$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ideals B_1, B_2, B_3 of A.

Proposition 4.3. *Every strongly irreducible semiprime bi-ideal of a ternary semiring A is a strongly prime bi-ideal of A.*

Proof. Let B be a strongly irreducible semiprime bi-ideal of a ternary semiring A. Let B_1, B_2, B_3 be three bi-ideals of A such that

$$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B \qquad \textbf{(i)}$$

Then we have to show that either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. As $B_1 \cap B_2 \cap B_3 \subseteq B_1$, $B_1 \cap B_2 \cap B_3 \subseteq B_2$ and $B_1 \cap B_2 \cap B_3 \subseteq B_3$ implies $(B_1 \cap B_2 \cap B_3)^3 \subseteq B_1B_2B_3$, $(B_1 \cap B_2 \cap B_3)^3 \subseteq B_2B_3B_1$ and $(B_1 \cap B_2 \cap B_3)^3 \subseteq B_3B_1B_2$. Thus

$$(B_1 \cap B_2 \cap B_3)^3 \subseteq B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B \text{ (using } \mathbf{(i)}).$$

This implies $B_1 \cap B_2 \cap B_3 \subseteq B$, because B is a semiprime bi-ideal of A. Thus $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$, because B is a strongly irreducible bi-ideal of A. Hence B is a strongly prime bi-ideal of A. □

Proposition 4.4. *Let B be a bi-ideal of a ternary semiring A and $a \in A$ such that $a \notin B$. Then there exists an irreducible bi-ideal I of A such that $B \subseteq I$ and $a \notin I$.*

Proof. Let X be the collection of all bi-ideals of A which contain B but do not contain a, that is $X = \{Y_i : Y_i \text{ is a bi-ideal of } A,\ B \subseteq Y_i \text{ and } a \notin Y_i\}$. Then X is non-empty as $B \in X$. The collection X is a partially ordered set under inclusion. If $\{Y_i : i \in I\}$ is any totally ordered subset (chain) of X, then $\underset{i \in I}{\cup} Y_i = Y$ is also a bi-ideal of A containing B and $a \notin Y$. So Y is an upper bound of $\{Y_i : i \in I\}$. Thus every chain in X has an upper bound in X. Hence by Zorn's lemma, there exists a maximal element I (say) in X. This implies $B \subseteq I$ and $a \notin I$. Now we show that I is an irreducible bi-ideal of A. For this let C, D and E be three bi-ideals of A such that $I = C \cap D \cap E$. If C, D and E properly contain I, then $a \in C, a \in D$ and $a \in E$. Thus $a \in C \cap D \cap E = I$. Which is a contradiction to the fact that $a \notin I$. So either $I = C$ or $I = D$ or $I = E$. Hence I is an irreducible bi-ideal of A. □

Theorem 4.5. *For a ternary semiring A, the following assertions are equivalent:*

(1) *Every bi-ideal of A is idempotent.*
(2) $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 = B_1 \cap B_2 \cap B_3$ *for any bi-ideals B_1, B_2, B_3 of A.*
(3) *Each bi-ideal of A is semiprime.*
(4) *Each proper bi-ideal of A is the intersection of irreducible semiprime bi-ideals of A which contain it.*

Proof. (1) ⇒ (2) Let B_1, B_2, B_3 be any three bi-ideals of A. Then $B_1 \cap B_2 \cap B_3$ is also a bi-ideal of A, by Proposition 2.10. By hypothesis, we have $B_1 \cap B_2 \cap B_3 = (B_1 \cap B_2 \cap B_3)^3 = (B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3) \subseteq B_1B_2B_3$. Similarly $B_1 \cap B_2 \cap B_3 \subseteq B_2B_3B_1$ and $B_1 \cap B_2 \cap B_3 \subseteq B_3B_1B_2$. So $B_1 \cap B_2 \cap B_3 \subseteq B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2$. Now $B_1B_2B_3$, $B_2B_3B_1$and $B_3B_1B_2$, being the products of three bi-ideals of A, are bi-ideals of A, by Corollary 2.13. Also $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2$ is a bi-ideal of A, by Proposition 2.10. Then by hypothesis $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 = (B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2)^3$
$\subseteq (B_1B_2B_3)(B_3B_1B_2)(B_2B_3B_1) \subseteq (B_1AA)(AB_1A)(AAB_1)$
$= B_1(AAA)B_1(AAA)B_1 = B_1AB_1AB_1 \subseteq B_1$. Similarly $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B_2$ and $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B_3$. Thus $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B_1 \cap B_2 \cap B_3$. Hence $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 = B_1 \cap B_2 \cap B_3$.

(2) ⇒ (1) Let B be a bi-ideal of A. Then by hypothesis $B = B \cap B \cap B = BBB \cap BBB \cap BBB = BBB$.

(1) $\Rightarrow$ (3) Let B be a bi-ideal of A such that $B_1^3 \subseteq B$ for any bi-ideal B_1 of A. Then by hypothesis, we have $B_1 = B_1^3 \subseteq B$. Hence every bi-ideal of A is a semiprime bi-ideal of A.

(3) $\Rightarrow$ (4) Let each bi-ideal of A is semiprime. Now let B be a proper bi-ideal of A. If $\underset{\alpha}{\cap} I_\alpha$ is the intersection of all bi-ideals of A containing B, then $B \subseteq \underset{\alpha}{\cap} I_\alpha$. If this inclusion is proper, then there exists $a \in \underset{\alpha}{\cap} I_\alpha$ such that $a \notin B$. This implies $a \in I_\alpha$ for all α. As $a \notin B$, then by Proposition 4.4, there exists an irreducible bi-ideal I(say) of A such that $B \subseteq I$ and $a \notin I$. Which is a contradiction to the fact that $a \in I_\alpha$ for all α. So $B = \underset{\alpha}{\cap} I_\alpha$. By hypothesis, each bi-ideal of A is semiprime. Thus each proper bi-ideal of A is the intersection of irreducible semiprime bi-ideals of A which contain it.

(4) $\Rightarrow$ (1) Let each proper bi-ideal of A is the intersection of irreducible semiprime bi-ideals of A which contain it. Now if B is a bi-ideal of A, then B^3 is also a bi-ideal of A, by Corollary 2.13. If $B^3 = A$ (improper bi-ideal), then $A \subseteq B^3$. This implies $B \subseteq A \subseteq B^3$. Also $B^3 \subseteq B$. So $B^3 = B$ for each bi-ideal B of A. Now if B^3 is a proper bi-ideal of A, that is $B^3 \neq A$, then $B^3 = \cap\{B_\alpha :$ B_α is an irreducible semiprime bi-ideal of A such that $B^3 \subseteq B_\alpha$ for all $\alpha\}$. This implies $B \subseteq B_\alpha$ for all α, because each B_α is a semiprime bi-ideal of A. Thus $B \subseteq \cap B_\alpha = B^3$. Also $B^3 \subseteq B$, as B is closed under multiplication. Hence $B^3 = B$ for each bi-ideal B of A. □

Proposition 4.6. *If each bi-ideal of a ternary semiring A is idempotent then a bi-ideal B of A is strongly irreducible if and only if B is strongly prime.*

Proof. Let B be a strongly irreducible bi-ideal of A. Then we have to show that B is a strongly prime bi-ideal of A. For this let B_1, B_2, B_3 be any three bi-ideals of A such that $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B$. By Theorem 4.5, we have $B_1 \cap B_2 \cap B_3 = B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B$. But B is a strongly irreducible bi-ideal of A. Thus we have $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is a strongly prime bi-ideal of A.

Conversely suppose that B is a strongly prime bi-ideal of A. To show that B is strongly irreducible bi-ideal of A, let B_1, B_2, B_3 be any bi-ideals of A such that $B_1 \cap B_2 \cap B_3 \subseteq B$. By Theorem 4.5, we have $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 = B_1 \cap B_2 \cap B_3 \subseteq B$. But B is a strongly prime bi-ideal of A, we have $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is a strongly irreducible bi-ideal of A. □

Next we characterize those semirings in which each bi-ideal is strongly prime and also those semirings in which each bi-ideal is strongly irreducible.

Theorem 4.7. *Each bi-ideal of a ternary semiring A is strongly prime if and only if each bi-ideal of A is idempotent and the set of bi-ideals of A is totally ordered by inclusion.*

Proof. Suppose that each bi-ideal of A is strongly prime. This implies that each bi-ideal of A is semiprime. Thus by Theorem 4.5, each bi-ideal of A is idempotent. Now we show that the set of bi-ideals of A is totally ordered by inclusion. For this, let B_1, B_2 be two bi-ideals of A. Then by Theorem 4.5, we have $B_1 \cap B_2 = B_1 \cap B_2 \cap A = B_1B_2A \cap B_2AB_1 \cap AB_1B_2$, implies $B_1B_2A \cap B_2AB_1 \cap AB_1B_2 \subseteq B_1 \cap B_2$.

By hypothesis, B_1 and B_2 are strongly prime bi-ideals of A, so is $B_1 \cap B_2$. Then $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$ or $A \subseteq B_1 \cap B_2$. Thus $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Hence the set of bi-ideals of A is totally ordered by inclusion.

Conversely, assume that each bi-ideal of A is idempotent and the set of bi-ideals of A is totally ordered by inclusion. We have to show that each bi-ideal of A is strongly prime. For this, let B be an arbitrary bi-ideal of A and B_1, B_2, B_3 be any bi-ideals of A such that $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B$. By Theorem 4.5, we have $B_1 \cap B_2 \cap B_3 = B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B$ (i).

Since the set of bi-ideals of A is totally ordered by inclusion, so for B_1, B_2, B_3 we have the following six possibilities:

(ii) $B_1 \subseteq B_2 \subseteq B_3$ (iii) $B_1 \subseteq B_3 \subseteq B_2$ (iv) $B_2 \subseteq B_3 \subseteq B_1$
(v) $B_2 \subseteq B_1 \subseteq B_3$ (vi) $B_3 \subseteq B_1 \subseteq B_2$ (vii) $B_3 \subseteq B_2 \subseteq B_1$.

In these cases we have

(ii) $B_1 \cap B_2 \cap B_3 = B_1$ (iii) $B_1 \cap B_2 \cap B_3 = B_1$ (iv) $B_1 \cap B_2 \cap B_3 = B_2$
(v) $B_1 \cap B_2 \cap B_3 = B_2$ (vi) $B_1 \cap B_2 \cap B_3 = B_3$ (vii) $B_1 \cap B_2 \cap B_3 = B_3$.

Thus (i) gives, either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is strongly prime. □

Theorem 4.8. *If the set of bi-ideals of a ternary semiring A is totally ordered, then each bi-ideal of A is idempotent if and only if each bi-ideal of A is prime.*

Proof. Suppose each bi-ideal of A is idempotent and B is an arbitrary bi-ideal of A and B_1, B_2, B_3 be any bi-ideals of A such that $B_1B_2B_3 \subseteq B$. As the set of bi-ideals of A is totally ordered, then for B_1, B_2, B_3 we have the following six possibilities:

(i) $B_1 \subseteq B_2 \subseteq B_3$ (ii) $B_1 \subseteq B_3 \subseteq B_2$ (iii) $B_2 \subseteq B_3 \subseteq B_1$
(iv) $B_2 \subseteq B_1 \subseteq B_3$ (v) $B_3 \subseteq B_1 \subseteq B_2$ (vi) $B_3 \subseteq B_2 \subseteq B_1$.

For (i) and (ii), we have $B_1^3 = B_1B_1B_1 \subseteq B_1B_2B_3 \subseteq B$, implies $B_1 \subseteq B$, as B is idempotent. Similarly for other possibilities we have $B_2 \subseteq B$ or $B_3 \subseteq B$. Conversely, suppose that each bi-ideal of A is prime, so is semiprime, by Proposition 3.5. Thus by Theorem 4.5, each bi-ideal of A is idempotent. □

Proposition 4.9. *If the set of bi-ideals of a ternary semiring A is totally ordered, then the concepts of primeness and strongly primeness coincide.*

Proof. Let B be a prime bi-ideal of A. To show that B is a strongly prime bi-ideal of A, let B_1, B_2, B_3 be any bi-ideals of A such that $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B$. As the set of bi-ideals of semiring A is totally ordered, then for B_1, B_2, B_3 we have the following six possibilities:

(i) $B_1 \subseteq B_2 \subseteq B_3$ (ii) $B_1 \subseteq B_3 \subseteq B_2$ (iii) $B_2 \subseteq B_3 \subseteq B_1$
(iv) $B_2 \subseteq B_1 \subseteq B_3$ (v) $B_3 \subseteq B_1 \subseteq B_2$ (vi) $B_3 \subseteq B_2 \subseteq B_1$.

For (i) and (ii), we have $B_1^3 = B_1^3 \cap B_1^3 \cap B_1^3 \subseteq B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B$, implies $B_1 \subseteq B$, as B is a prime bi-ideal of A. Similarly for other possibilities we have $B_2 \subseteq B$ or $B_3 \subseteq B$. This shows that B is a strongly prime bi-ideal of A. Thus every prime bi-ideal of A is a strongly prime bi-ideal of A. Now let B be a strongly prime bi-ideal of A. To show that B is a prime bi-ideal of A, let B_1, B_2, B_3 be any bi-ideals of A such that $B_1B_2B_3 \subseteq B$. Implies $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B$. Implies either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$, as B is a strongly prime bi-ideal of A. This shows that B is a prime bi-ideal of A. Thus every strongly prime bi-ideal of A is a prime bi-ideal of A. □

Theorem 4.10. *For a ternary semiring A, the following assertions are equivalent:*
(1) *The set of bi-ideals of A is totally ordered by set inclusion.*
(2) *Each bi-ideal of A is strongly irreducible.*
(3) *Each bi-ideal of A is irreducible.*

Proof. (1) $\Rightarrow$ (2) Let the set of bi-ideals of A is totally ordered by set inclusion. To show that each bi-ideal of A is strongly irreducible, let B be an arbitrary bi-ideal of A and B_1, B_2, B_3 be any bi-ideals of A such that $B_1 \cap B_2 \cap B_3 \subseteq B$. Since the set of bi-ideals of A is totally ordered by set inclusion, then $B_1 \cap B_2 \cap B_3 = B_1$ or B_2 or B_3. Thus either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. So B is strongly irreducible. Hence each bi-ideal of A is strongly irreducible.

(2) $\Rightarrow$ (3) Let each bi-ideal of A is strongly irreducible. To show that each bi-ideal of A is irreducible, let B be an arbitrary bi-ideal of A and B_1, B_2, B_3 be any bi-ideals of A such that $B_1 \cap B_2 \cap B_3 = B$. This implies $B \subseteq B_1$, $B \subseteq B_2$ and $B \subseteq B_3$. On the other hand, by hypothesis we have $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence either $B_1 = B$ or $B_2 = B$ or $B_3 = B$. Thus B is an irreducible bi-ideal of A. Hence each bi-ideal of A is irreducible.

(3) $\Rightarrow$ (1) Let each bi-ideal of A is irreducible. To show that the set of bi-ideals of A is totally ordered by set inclusion, let B_1, B_2 be any two bi-ideals of A. Then by Proposition 2.10, $B_1 \cap B_2$ is also a bi-ideal of A and so is irreducible bi-ideal of A. Since $B_1 \cap B_2 \cap A = B_1 \cap B_2$, implies $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$ or $A = B_1 \cap B_2$, implies either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ or $B_1 = B_2 = A$. Hence the set of bi-ideals of A is totally ordered by set inclusion. □

Definition 4.11. Let A be a ternary semiring, β be the set of all bi-ideals of A and ρ be the set of all strongly prime proper bi-ideals of A. Define for each $B \in \beta$

$$\begin{aligned} \theta_B &= \{J \in \rho : B \nsubseteq J\} \\ \tau(\rho) &= \{\theta_B : B \in \beta\}. \end{aligned}$$

Theorem 4.12. *Let A be a ternary semiring with the property that each bi-ideal of A is idempotent. Then $\tau(\rho)$ forms a topology on the set ρ.*

Proof. Since $\{0\}$ is a bi-ideal of A and 0 belongs to every bi-ideal of A, therefore $\theta_{\{0\}} = \{J \in \rho : \{0\} \nsubseteq J\} = \Phi$ (the empty set) $\in \tau(\rho)$. Also A is a bi-ideal of A, then $\theta_A = \{J \in \rho : A \nsubseteq J\} = \rho \in \tau(\rho)$. Thus $\Phi, \rho \in \tau(\rho)$. Now we show that intersection of finite number of members of $\tau(\rho)$ belongs to $\tau(\rho)$. For this, let $\theta_{B_1}, \theta_{B_2} \in \tau(\rho)$. Then we have to show that $\theta_{B_1} \cap \theta_{B_2} \in \tau(\rho)$. For this we show that $\theta_{B_1} \cap \theta_{B_2} = \theta_{B_1 \cap B_2}$. Let $J \in \theta_{B_1} \cap \theta_{B_2}$, implies $J \in \theta_{B_1}$ and $J \in \theta_{B_2}$. So $J \in \rho$ and $B_1 \nsubseteq J$ and $B_2 \nsubseteq J$, by the definition of θ_{B_1} and θ_{B_2}. Now we suppose that $B_1 \cap B_2 \cap A = B_1 \cap B_2 \subseteq J$. Since each bi-ideal of A is idempotent, therefore by Theorem 4.5, we have $B_1B_2A \cap B_2AB_1 \cap AB_1B_2 = B_1 \cap B_2 \cap A \subseteq J$. But J is a strongly prime bi-ideal of A. This implies $B_1 \subseteq J$ or $B_2 \subseteq J$ $(A \nsubseteq J$ because J is a proper bi-ideal of $A)$. Which is a contradiction. So our supposition is wrong. Thus $B_1 \cap B_2 \nsubseteq J$. Also $B_1 \cap B_2$, being the intersection of two bi-ideals of A, is a bi-ideal of A, by Proposition 2.10. This implies $J \in \theta_{B_1 \cap B_2}$. So $\theta_{B_1} \cap \theta_{B_2} \subseteq \theta_{B_1 \cap B_2}$. Now let $J \in \theta_{B_1 \cap B_2}$. This implies $J \in \rho$ and $B_1 \cap B_2 \nsubseteq J$, by the definition of $\theta_{B_1 \cap B_2}$. Implies $B_1 \nsubseteq J$ and $B_2 \nsubseteq J$. So $J \in \theta_{B_1}$ and $J \in \theta_{B_2}$, implies $J \in \theta_{B_1} \cap \theta_{B_2}$. Thus $\theta_{B_1 \cap B_2} \subseteq \theta_{B_1} \cap \theta_{B_2}$. Hence $\theta_{B_1} \cap \theta_{B_2} = \theta_{B_1 \cap B_2} \in \tau(\rho)$.

Now we show that union of any number of members of $\tau(\rho)$ belong to $\tau(\rho)$. For this, let $\{\theta_{B_\alpha} : \alpha \in I\} \subseteq \tau(\rho)$. Then we have to show that $\underset{\alpha}{\cup}\theta_{B_\alpha} \in \tau(\rho)$. As

$$\begin{aligned} \underset{\alpha}{\cup}\theta_{B_\alpha} &= \{J \in \rho : B_\alpha \nsubseteq J \text{ for some } \alpha \in\} \\ &= \left\{ J \in \rho : \overset{\Lambda}{\underset{\alpha}{\cup}} B_\alpha \nsubseteq J \right\} \\ &= \theta_{\overset{\Lambda}{\underset{\alpha}{\cup}} B_\alpha} \in \tau(\rho) \end{aligned}$$

where $\overset{\Lambda}{\underset{\alpha}{\cup}} B_\alpha$ is the bi-ideal of A generated by $\underset{\alpha}{\cup} B_\alpha$, thus $\tau(\rho)$ is a topology on ρ. □

Acknowledgements. The Authors are very thankful to the learned referees/ editor in Chief Young Bae Jun for their suggestions to improve the present paper.

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