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On generalized redefined fuzzy prime ideals of ordered semigroups

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Abstract. In this paper, with a new idea, we redefine generalized fuzzy ideal and investigate some of its properties. We generalize fuzzy prime ideals and introduce the notions of $(\in, \in \vee q)-$fuzzy prime ideal of ordered semigroups. We characterize $(\in, \in \vee q)-$fuzzy prime ideal by its image and level ideals. We establish necessary and sufficient condition for fuzzy set to be a $(\in, \in \vee q)-$fuzzy prime ideal.

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1. Introduction

The important concept of a fuzzy set put forth by Zadeh in 1965 [10] has opened up keen insights and applications in a wide range of scientific fields. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, groupoids, topology, etc. A theory of fuzzy sets on ordered semigroups has been recently developed. Fuzzy sets in ordered semigroup were first studied by Kehayopulu and Tsingelis in [4], then they defined fuzzy analogies for several notions, which have proven useful in the theory of ordered semigroups. Moreover, Gupta and Kantroo [3] proved that each ordered groupoid having the greatest element(poe-groupoid) in terms of fuzzy sets. They also characterized the bi-ideals of ordered semigroups in terms of fuzzy bi-ideals and regular ordered semigroups in terms of fuzzy right, left ideals fuzzy quasi-ideals [5].

Malik and Morderson [7] defined prime fuzzy ideals in a ring R, and proved that a fuzzy ideal f of a ring is prime if and only if $Imf = \{1, \beta\}$, where $\beta \in [0, 1)$, and the ideal $f_1 = \{x \in R : f(x) = 1\}$ is prime. Dheena and Mohanraj [1] generalized fuzzy prime ideals in a semiring R to $(\lambda, \mu)-$fuzzy prime ideals. Motivated by the study of fuzzy prime ideals in near-subtraction semigroup by Dheena and Mohanraj [2], rings, semigroups and ordered semigroups, and also motivated by Kehayopulu, Tsingelis and Xie's works in ordered semigroups in terms of fuzzy subsets, we attempt in the paper to study the generalized fuzzy prime ideals in an ordered semigroup in detail.

In this paper, with a new idea, we redefine generalized fuzzy ideal and investigate some of its properties. We introduce the notions of $(\in, \in \vee q)-$fuzzy prime ideal of ordered semigroups. We characterize $(\in, \in \vee q)-$fuzzy prime ideal by its image and level ideals. A characterization of prime fuzzy ideals of an ordered semigroup is given. It is shown that a non-constant fuzzy ideal μ of an ordered semigroup S is prime if and only if $|Im\ \mu \cap [0, 0.5]| \leq 2$ and $\mu_{0.5}$ is a prime ideal. We establish necessary and sufficient condition for fuzzy set to be a $(\in, \in \vee q)-$fuzzy prime ideal.

2. Preliminaries

Definition 2.1. By an ordered semigroup(po-semigroup), we mean a structure $(S, ., \leq)$ in which the following conditions are satisfied:
(OS1) $(S, \cdot)$is an semigroup.
(OS2) $(S, \leq)$is a poset.
(OS3) $a \leq b \Rightarrow xa \leq xb$ and $ax \leq bx$ for all $a, b, x \in S$.

For $A \subseteq S$, we denote $(A] := \{t \in S | t \leq h \ for\ some\ h \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $\{(a]\}$. For non-empty subsets A, B of S, we denote

$$AB := \{ab | a \in A, b \in B\}$$

Definition 2.2. Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset A of S is called an ideal of S if

(1) $(\forall a \in S)(\forall b \in A)(a \leq b \Rightarrow a \in A)$.
(2) $AS \subseteq A$ and $SA \subseteq A$.

Definition 2.3. Let $(S, ., \leq)$ be an ordered semigroup. An ideal P in S is said to be prime ideal if $A \cdot B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$, for all ideals A, B of S.

By a fuzzy set μ of S, we mean a mapping $\mu : S \longrightarrow [0, 1]$.

Definition 2.4. A fuzzy set μ of S of the form

$$\mu(y) = \begin{cases} t \in (0, 1] & \text{if } \ y = x, \\ 0 & \text{if } \ y \neq x. \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

To say that $x_t \in \mu$(resp.$x_t q \mu$) means that $\mu(x) \geq t$(resp.$\mu(x) + t > 1$) and in this case, x_t is said to belong to(resp. be quasi-coincident with) a fuzzy set μ. To say that $x_t \in \vee q\mu$(resp.$x_t \in \wedge q\mu$) means that $x_t \in \mu$ or $x_t q\mu$(resp.$x_t \in \mu$ and $x_t q\mu$). We denote $x_t \bar{\in} \mu$ if $\mu(x) < t$ and $x_t \bar{q} \mu$ if $\mu(x) + t \leq 1$.

Definition 2.5. Let μ be a fuzzy set of an ordered semigroup S. Then μ is said to be a fuzzy ideal of S if

1. $x \leq y \Rightarrow \mu(x) \geq \mu(y)$.
2. $\mu(xy) \geq \mu(x) \vee \mu(y)$, for all $x, y \in S$.

Definition 2.6. Let μ be a fuzzy set of an ordered semigroup S. Then the level set μ_t is defined as follows:

$$\mu_t = \{x|\mu(x) \geq t\}$$

Definition 2.7. Let A be a non-empty subset of an ordered semigroup S. Then the characteristic function $\chi_A(x)$ is a fuzzy set of S defined as follows:

$$\chi_A(x) = \begin{cases} 1 & \text{if } \ x \in A \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.8 ([5])**.** For $a \in S$, we define

$$A_a = \{(y,z) \in S \times S | a \leq yz\}$$

Definition 2.9. Let λ and μ be any fuzzy sets of an ordered semigroup S. The fuzzy product of λ and μ denoted by $\lambda \circ \mu$ is defined as follows:

$$(\lambda \circ \mu)(a) = \begin{cases} \bigvee\limits_{(y,z)\in A_a} \{\lambda(y) \wedge \mu(z)\} & if \ A_a \neq \varnothing \\ 0 & if \ A_a = \varnothing. \end{cases}$$

3. $(\in, \in \vee q)$-fuzzy ideals

Hereafter S denotes an ordered semigroup unless otherwise specified.

Definition 3.1. Let μ be a fuzzy set of S. Then μ is said to be a $(\in, \in \vee q)$-fuzzy ideal if

$F1$. $x \leq y, y_t \in \mu$ implies $x_t \in \vee q \mu$

$F2$. $x_t \in \mu, y \in S$ implies $(xy)_t$ and $(yx)_t \in \vee q\mu$, for all $x, y \in S$ and for all $t \in (0, 1]$.

Theorem 3.2 ([6])**.** *Let μ be a fuzzy set of S. Then μ is a $(\in, \in \vee q)$-fuzzy ideal of S if and only if*

$F3$. $x \leq y$ implies $\mu(x) \geq \mu(y) \wedge 0.5$

$F4$. $\mu(xy) \geq [\mu(x) \vee \mu(y)] \wedge 0.5$, for all $x, y \in S$.

Theorem 3.3 ([6])**.** *Let μ be a fuzzy set of S. Then μ is a $(\in, \in \vee q)$-fuzzy ideal of S if and only if μ_t is an ideal in S for $t \in [0, 0.5]$ whenever non-empty.*

Definition 3.4 ([8])**.** Let μ and λ be any two fuzzy sets of S. Then $\mu \subseteq \vee q\lambda$, if $x_t \in \mu$ implies $x_t \in \vee q\lambda$ for all $x \in S$ and $t \in (0, 1]$.

Lemma 3.5 ([8])**.** *For any two fuzzy sets μ and λ of S, $\mu \subseteq \vee q\lambda$ if and only if $\lambda(x) \geq min\ \{\mu(x), 0.5\}$, for all $x \in S$.*

Definition 3.6 ([9])**.** Let μ be a fuzzy set of S. We define a fuzzy set $(\mu]$ as follows:

$$(\mu](x) = \bigvee_{y \geq x} \mu(y) \text{ for all } x \in S.$$

A fuzzy set μ of $S$is called strongly convex if $\mu = (\mu]$.

Definition 3.7. A fuzzy set μ of S is weakly convex if $(\mu] \subseteq \vee q\mu$.

Theorem 3.8. *Let μ be a fuzzy set of S. Then μ is a weakly convex fuzzy set of S if and only if $x \leq y$ implies $\mu(x) \geq \mu(y) \wedge 0.5$ for all $x, y \in S$.*

Proof. Let μ be a weakly convex fuzzy set of S. Let $x \leq y, x, y \in S$. Then, by Lemma 3.5, $\mu(x) \geq (\mu](x) \wedge 0.5 = (\bigvee_{w \geq x} \mu(w)) \wedge 0.5 \geq \mu(y) \wedge 0.5$.

Conversely, $x \leq y$ implies $\mu(x) \geq \mu(y) \wedge 0.5$, for all $x, y \in S$. Therefore $\mu(x) \geq \bigvee_{w \geq x} \mu(w) \wedge 0.5 = (\mu](x) \wedge 0.5$ for all $x \in S$. Thus, by Lemma 3.5, $(\mu] \subseteq \vee q\mu$. □

Remark 3.9. (1) Every strongly convex fuzzy set of S is a weakly convex fuzzy set of S.

(2) Every weakly convex fuzzy set of S need not be a strongly convex fuzzy set of S as shown by the following example.

Example 3.10. Let $(S, \cdot, \leq)$ be an ordered semigroup where $S = \{0, a, b, c\}$. The order relation "$\leq$" is given by $\leq := \{(0, a), (0, b), (0, c), (a, c), (b, c)\}$ and the multiplication is given as follows:

·	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

$$\mu(x) = \begin{cases} 0.8 & \text{if } x = c \\ 0.5 & \text{if } x \neq c \end{cases}$$

Clearly, μ is a weakly convex fuzzy set of S but μ is not a strongly convex fuzzy set, since $0.5 = \mu(a) < (\mu](a) = 0.8$.

Theorem 3.11. *Let μ be a fuzzy set of S. Then μ is a $(\in, \in \vee q)$-fuzzy ideal of S if and only if*

(1)$(\mu] \subseteq \vee q\mu$.

(2)$\chi_S \circ \mu \subseteq \vee q\mu$ and $\mu \circ \chi_S \subseteq \vee q\mu$.

Proof. Let μ be a $(\in, \in \vee q)$-fuzzy ideal of S. Then by Theorem 3.2 and 3.8, $(\mu] \subseteq \vee q\mu$. Now $(y, z) \in A_x$ implies $x \leq yz$. Then by Theorem 3.2, $\mu(x) \geq \mu(yz) \wedge 0.5 \geq \mu(z) \wedge 0.5$. Therefore

$$\begin{aligned} \mu(x) &\geq (\bigvee_{(y,z) \in A_x} \mu(z)) \wedge 0.5 \\ &= (\bigvee_{(y,z) \in A_x} 1 \wedge \mu(z)) \wedge 0.5 \\ &= (\bigvee_{(y,z) \in A_x} \chi_S(y) \wedge \mu(z)) \wedge 0.5 \\ &= (\chi_S \circ \mu)(x) \wedge 0.5. \end{aligned}$$

Thus by Lemma 3.5, $\chi_S \circ \mu \subseteq \vee q\mu$. Similarly, $\mu \circ \chi_S \subseteq \vee q\mu$.

Conversely, by Lemma 3.5,

$$\begin{aligned}\mu(x) &\geq (\chi_S \circ \mu)(x) \wedge 0.5\\ &= (\bigvee_{(y,z)\in A_x} \chi_S(y) \wedge \mu(z)) \wedge 0.5\\ &= (\bigvee_{(y,z)\in A_x} \mu(z)) \wedge 0.5\\ &= \bigvee_{(y,z)\in A_x} \mu(z) \wedge 0.5.\end{aligned}$$

Thus $\mu(xy) \geq \mu(y) \wedge 0.5$. Similarly, $\mu \circ \chi_S \subseteq \vee q\mu$ implies $\mu(xy) \geq \mu(x) \wedge 0.5$. Thus $\mu(xy) \geq (\mu(x) \wedge 0.5) \vee (\mu(y) \wedge 0.5) = [\mu(x) \vee \mu(y)] \wedge 0.5$. Then by Theorem 3.2 and 3.8, μ is a $(\in, \in \vee q)$-fuzzy ideal of S. □

Corollary 3.12. *Let μ be a weakly convex fuzzy set of S. Then μ is a $(\in, \in \vee q)$-fuzzy ideal of S if and only if $\chi_S \circ \mu \subseteq \vee q\mu$ and $\mu \circ \chi_S \subseteq \vee q\mu$.*

Proof. Straight forward. □

Remark 3.13. Let μ be a fuzzy set of S and $|Im\ \mu \cap [0, 0.5]| = \varnothing$. Then by Theorem 3.3, μ is a $(\in, \in \vee q)$-fuzzy ideal. These fuzzy sets are called trivial $(\in, \in \vee q)$-fuzzy ideals.

Remark 3.14. (1) Every fuzzy ideal is a $(\in, \in \vee q)$-fuzzy ideal of S.

(2) Every $(\in, \in \vee q)$-fuzzy ideal need not be a fuzzy ideal of S as shown by the following example.

Example 3.15. Consider the ordered semigroup $(S, ., \leq)$ as in the Example 3.10. Now, we define a fuzzy set μ as follows:

$$\mu(x) = \begin{cases} 0.7 & \text{if } \ x = a \\ 0.5 & \text{if } \ x = 0 \\ 0.3 & \text{otherwise} \end{cases}$$

Clearly μ is a $(\in, \in \vee q)$-fuzzy ideal of S but μ is not a fuzzy ideal of S, since $0.5 = \mu(0.a) < \mu(a) \vee \mu(0) = 0.7 \vee 0.5 = 0.7$.

4. $(\in, \in \vee q)$-fuzzy prime ideals

Definition 4.1 ([9])**.** Let μ be a fuzzy ideal of S. Then μ is a fuzzy prime ideal of S if $\lambda \circ \sigma \subseteq \mu$ implies $\lambda \subseteq \mu$ or $\sigma \subseteq \mu$, for the fuzzy ideals λ and σ of S.

We generalize fuzzy prime ideal of an ordered semigroup to $(\in, \in \vee q)$-fuzzy prime ideal. Now, we characterize $(\in, \in \vee q)$-fuzzy prime ideals through prime ideals and level sets.

Definition 4.2. Let μ be a $(\in, \in \vee q)$-fuzzy ideal of S. Then μ is a $(\in, \in \vee q)$-fuzzy prime ideal of S if $\lambda \circ \sigma \subseteq \mu$ implies $\lambda \subseteq \vee q\mu$ or $\sigma \subseteq \vee q\mu$, for the $(\in, \in \vee q)$-fuzzy ideals λ and σ of S.

Theorem 4.3 ([9]). *Let μ be a fuzzy set of S. Then μ is a fuzzy prime ideal of S if and only if μ satisfies the following conditions:*
(1) $|Im\ \mu| \leq 2$.
(2)$\mu_1 \neq \varnothing$ *and* μ_1 *is a prime ideal.*

Theorem 4.4. *Let μ be a non-constant fuzzy set of S and $|Im\ \mu \cap [0, 0.5]| = 1$. Then μ is a $(\in, \in \vee q)$-fuzzy prime ideal if and only if $\mu_{0.5}$ is a prime ideal of S.*

Proof. Let μ be a $(\in, \in \vee q)$-fuzzy prime ideal of S and $|Im\ \mu \cap [0, 0.5]| = 1$. If $Im\ \mu \cap [0, 0.5] = \{0.5\}$, then $\mu_{0.5} = S$, which is a prime ideal. Let $Im\ \mu \cap [0, 0.5) = \{t\}$ and A and B be the fuzzy ideals in S such that $AB \subseteq \mu_{0.5}$. Let us define λ and σ as follows:

$$\lambda(x) = \begin{cases} 0.5 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \qquad \sigma(x) = \begin{cases} 0.7 & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}$$

By Theorem 3.3, λ and σ are $(\in, \in \vee q)$-fuzzy ideals of S.

$$(\lambda \circ \sigma)(x) = \begin{cases} 0.5 & \text{if } x \in (AB] \\ 0 & \text{otherwise} \end{cases}$$

Clearly $\lambda \circ \sigma \subseteq \mu$. Then $\lambda \subseteq \vee q\mu$ or $\sigma \subseteq \vee q\mu$. If $\sigma \subseteq \vee q\mu$, then for all $x \in B$, by Lemma 3.5, $\mu(x) \geq \sigma(x) \wedge 0.5 = 0.7 \wedge 0.5 = 0.5$. Thus $x \in \mu_{0.5}$. Therefore $B \subseteq \mu_{0.5}$. Similarly, $\lambda \subseteq \vee q\mu$ implies $A \subseteq \mu_{0.5}$. Hence $\mu_{0.5}$ is a prime ideal.

Conversely, by Theorem 3.3, μ is a $(\in, \in \vee q)$-fuzzy ideal. If there exists $(\in, \in \vee q)$-fuzzy ideals α and β of S such that $\alpha \circ \beta \subseteq \mu$ with $\alpha \not\subseteq \vee q\mu$ and $\beta \not\subseteq \vee q\mu$. Then by Lemma 3.5, there exists $x, y \in S$ such that $\mu(x) < \alpha(x) \wedge 0.5$ and $\mu(y) < \beta(y) \wedge 0.5$. Let $\alpha(x) \wedge 0.5 = s_1$ and $\beta(y) \wedge 0.5 = s_2$. Thus $\alpha(x) \wedge 0.5 = s_1 > \mu(x) = t$ and $\beta(y) \wedge 0.5 = s_2 > \mu(x) = t$. By Theorem 3.3, α_{s_1} and β_{s_2} are ideals in S. Then $x \in \alpha_{s_1}, y \in \beta_{s_2}, x \notin \mu_{0.5}$ and $y \notin \mu_{0.5}$. Now, $\alpha_{s_1} \not\subseteq \mu_{0.5}$ and $\beta_{s_2} \not\subseteq \mu_{0.5}$ imply $\alpha_{s_1}\beta_{s_2} \not\subseteq \mu_{0.5}$ Then there exists $a \in \alpha_{s_1}, b \in \beta_{s_2}$, such that $ab \notin \mu_{0.5}$. Thus $\mu(ab) = t$.

$$\begin{aligned} (\alpha \circ \beta)(ab) &= \bigvee_{ab \leq yz} \{\alpha(y) \wedge \beta(z)\} \\ &\geq \alpha(a) \wedge \beta(b) \\ &= s_1 \wedge s_2 \\ &> t = \mu(ab). \end{aligned}$$

This is a contradiction to $\alpha \circ \beta \subseteq \mu$. Hence μ is a $(\in, \in \vee q)$-fuzzy prime ideal of S. $\square$

Lemma 4.5. *Let μ be a $(\in, \in \vee q)$-fuzzy prime ideal of S and $|Im\ \mu \cap [0, 0.5]| > 1$, then $|Im\ \mu \cap [0, 0.5]| = 2$.*

Proof. If $|Im\ \mu \cap [0, 0.5]| > 2$, then there exists t_1, t_2, t_3 such that $0.5 \geq t_1 > t_2 > t_3 \geq 0$ with $\mu(a) = t_1; \mu(b) = t_2; \mu(c) = t_3$ for some $a, b, c \in S$. Choose $s_1, s_2 \in [0, 0.5]$ such that $0.5 \geq t_1 > s_1 > t_2 > s_2 > t_3 \geq 0$. Let us define

$$\lambda(x) = \begin{cases} s_1 & \text{if } x \in \mu_{t_2} \\ 0 & \text{otherwise} \end{cases}$$

and $\sigma(x) = s_2$, for all $x \in S$. By Theorem 3.3, λ and σ are $(\in, \in \vee q)$-fuzzy ideals of S.

$$(\lambda \circ \sigma)(x) = \begin{cases} s_2 & \text{if } \ x \in (\mu_{t_2} S] \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $\lambda \circ \sigma \subseteq \mu$. By Lemma 3.5, $t_2 = \mu(b) < \lambda(b) \wedge 0.5 = s_1$ and $t_3 = \mu(c) < \sigma(c) \wedge 0.5 = s_2$ imply $\lambda \not\subseteq \vee q\mu$ and $\sigma \not\subseteq \vee q\mu$ which is a contradiction. Therefore $|Im\ \mu \cap [0, 0.5]| = 2$. □

Lemma 4.6. *Let μ be a $(\in, \in \vee q)$-fuzzy prime ideal of S and $|Im\ \mu \cap [0, 0.5]| > 1$, then $Im\ \mu \cap [0, 0.5] = \{0.5, t\}$ where $0 \leq t < 0.5$ and $\mu_{0.5}$ is a prime ideal.*

Proof. Let μ be a $(\in, \in \vee q)$-fuzzy prime ideal of S. By Lemma 4.5, $|Im\ \mu \cap [0, 0.5]| = 2$. Let $Im\ \mu \cap [0, 0.5] = \{t_1, t_2\}$ where $0.5 > t_1 > t_2 \geq 0$. Then $\mu(a) = t_1; \mu(b) = t_2$ for some $a, b \in S$. We choose $s_1, s_2 \in (0, 0.5)$ such that $0.5 > s_1 > t_1 > s_2 > t_2 \geq 0$ and define fuzzy sets λ and σ as follows:

$$\lambda(x) = \begin{cases} s_1 & \text{if } \ x \in \mu_{t_1} \\ t_2 & \text{otherwise} \end{cases} \qquad \sigma(x) = s_2 \text{ for all } x.$$

By Theorem 3.3, λ, σ are $(\in, \in \vee q)$-fuzzy ideals of S.

$$(\lambda \circ \sigma)(x) = \begin{cases} s_2 & \text{if } \ x \in (\mu_{t_1} S] \\ t_2 & \text{if } \ x \in (SS] - (\mu_{t_1} S] \\ 0 & \text{otherwise} \end{cases}$$

Then $(\lambda \circ \sigma) \subseteq \mu$ but $0.5 > \lambda(a) = s_1 > t_1 = \mu(a)$ implies $a_{s_1} \in \lambda$ but $a_{s_1} \overline{\in \vee q} \mu$. Similarly, $b_{s_2} \in \sigma$ but $b_{s_2} \overline{\in \vee q} \mu$. Thus $\lambda \not\subseteq \vee q\mu$ and $\sigma \not\subseteq \vee q\mu$, which is a contradiction. Therefore $|Im\ \mu \cap [0, 0.5]| = \{0.5, t\}$.

Let I_1 andI_2 be any two ideals of S such that $I_1 I_2 \subseteq \mu_{0.5}$. Let us define α and β as follows:

$$\alpha(x) = \begin{cases} 0.7 & \text{if } \ x \in I_1 \\ 0 & \text{otherwise} \end{cases} \qquad \beta(x) = \begin{cases} 0.5 & \text{if } \ x \in I_2 \\ 0 & \text{otherwise} \end{cases}$$

$$(\alpha \circ \beta)(x) = \begin{cases} 0.5 & \text{if } \ x \in (I_1 I_2] \\ 0 & \text{otherwise} \end{cases}$$

By Theorem 3.3, α and β are $(\in, \in \vee q)$-fuzzy ideals. Clearly $\alpha \circ \beta \subseteq \mu$. Then $\alpha \subseteq \vee q\mu$ or $\beta \subseteq \vee q\mu$. If $\alpha \subseteq \vee q\mu$, then for all $x \in I_1$, by Lemma 3.5, $\mu(x) \geq \alpha(x) \wedge 0.5 = 0.7 \wedge 0.5 = 0.5$. Thus $x \in \mu_{0.5}$. Therefore $I_1 \subseteq \mu_{0.5}$. Similarly, $\beta \subseteq \vee q\mu$ implies $I_2 \subseteq \mu_{0.5}$. Hence $\mu_{0.5}$ is a prime ideal. □

Theorem 4.7. *Let μ be a non-constant fuzzy set of S and $|Im\ \mu \cap [0, 0.5]| > 1$. Then μ is a $(\in, \in \vee q)$-fuzzy prime ideal if and only if*

(1)*$Im\ \mu \cap [0, 0.5] = \{0.5, t\}$ where $0 \leq t < 0.5$.*

(2) *$\mu_{0.5}$ is a prime ideal.*

Proof. Let μ be a $(\in, \in \vee q)$-fuzzy prime ideal of S. The result (1) and (2) follows from Lemma 4.6.

Conversely, by Theorem 3.3, μ is a $(\in, \in \vee q)$-fuzzy prime ideal of S. If there exists two $(\in, \in \vee q)$-fuzzy ideals λ, σ of S such that $(\lambda \circ \sigma) \subseteq \mu$ with $\lambda \nsubseteq \vee q\mu$ and $\sigma \nsubseteq \vee q\mu$, then by Lemma 3.5, there exists $a, b \in S$ such that $\mu(a) < \lambda(a) \wedge 0.5 = s_1$ and $\mu(b) < \sigma(b) \wedge 0.5 = s_2$. Then $\mu(a) = t < \lambda(a) \wedge 0.5 = s_1$ and $\mu(b) = t < \sigma(b) \wedge 0.5 = s_2$. By Theorem 3.3, λ_{s_1} and σ_{s_2} are ideals in S. Then $a \in \lambda_{s_1}, b \in \sigma_{s_2}, a \notin \mu_{0.5}$ and $b \notin \mu_{0.5}$. Now,$\lambda_{s_1} \nsubseteq \mu_{0.5}$ and $\sigma_{s_2} \nsubseteq \mu_{0.5}$ imply $\lambda_{s_1} \cdot \sigma_{s_2} \nsubseteq \mu_{0.5}$. Then there exists $x \in \lambda_{s_1}, y \in \sigma_{s_2}$, such that $xy \notin \mu_{0.5}$. Thus $\mu(xy) = t$.

$$\begin{aligned}
(\lambda \circ \sigma)(xy) &= \bigvee_{xy \leq ab} \{\lambda(a) \wedge \sigma(b)\} \\
&\geq \lambda(x) \wedge \sigma(y) \\
&\geq s_1 \wedge s_2 \\
&> t = \mu(xy).
\end{aligned}$$

This contradicts to $\lambda \circ \sigma \subseteq \mu$. Therefore μ is a $(\in, \in \vee q)$-fuzzy prime ideal of S. □

Theorem 4.8. *Let μ be a non-constant fuzzy set of S. Then μ is a $(\in, \in \vee q)$-fuzzy prime ideal of S if and only if*

(1) $|Im\ \mu \cap [0, 0.5]| \leq 2$.

(2) $\mu_{0.5} \neq \varnothing$ *and* $\mu_{0.5}$ *is a prime ideal.*

Proof. Let μ be a $(\in, \in \vee q)$-fuzzy prime ideal of S. If $|Im\ \mu \cap [0, 0.5]| \neq 0$, then the result follows from Theorem 4.4 and 4.7. If $|Im\ \mu \cap [0, 0.5]| = 0$, then $\mu_{0.5} = S$, which is a prime ideal.

Conversely, if $|Im\ \mu \cap [0, 0.5]| = 0$, then μ is a trivial $(\in, \in \vee q)$-fuzzy ideal. By Lemma 3.5, $\lambda \subseteq \vee q\mu$ for every fuzzy set λ of S. Thus μ is a $(\in, \in \vee q)$-fuzzy prime ideal of S. If $|Im\ \mu \cap [0, 0.5]| \neq 0$, then the result follows from Theorem 4.4 and 4.7. □

Remark 4.9. (1) Every fuzzy prime ideal is a $(\in, \in \vee q)$-fuzzy prime ideal of S.

(2) Every $(\in, \in \vee q)$-fuzzy prime ideal need not be a fuzzy prime ideal of S by the following example:

Example 4.10. Consider the ordered semigroup $(S, ., \leq)$ as in the Example 3.10. Now, we define a fuzzy set μ as follows:

$$\mu(x) = \begin{cases} 0.7 & \text{if } \ x \in \{0, a\} \\ 0.3 & \text{otherwise} \end{cases}$$

Clearly $\{0, a\}$ is a prime ideal in S. By Theorem 3.3 and 4.4, μ is a $(\in, \in \vee q)$-fuzzy prime ideal. By Theorem 4.3, μ is not a fuzzy prime ideal.

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