Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 6, No. 1, (July 2013), pp. 151–170
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

On soft inner product spaces

Sujoy Das, S. K. Samanta

Received 14 September 2012; Revised 19 October 2012; Accepted 6 December 2012

Abstract. In the present paper an idea of soft inner product on soft linear spaces has been introduced and some of their properties are investigated. Soft Hilbert spaces, orthogonality and orthonormality in soft Hilbert spaces are also studied.

2010 AMS Classification: 03E72, 08A72

Keywords: Soft sets, Soft elements, Soft vectors, Soft linear spaces, Soft inner product spaces, Soft Hilbert spaces.

Corresponding Author: S. K. Samanta (syamal_123@yahoo.co.in)

1. Introduction

In the year 1999, Molodtsov [16] initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties. He has shown several applications of this theory in solving many practical problems in economics, engineering, social science, medical science, etc. Research works in soft set theory and its applications in various fields have been progressing rapidly since Maji et al. ([13],[14]) introduced several operations on soft sets and applied it to decision making problems. In the line of reduction and addition of parameters of soft sets some works have been done by Chen [3], Pei and Miao [17] , Kong et al. [12] , Zou and Xiao [20]. Aktas and Cagman [1] introduced the notion of soft group and discussed various properties. Jun ([10],[11]) investigated soft BCK/BCI – algebras and its application in ideal theory. Feng et al. [8] worked on soft semirings, soft ideals and idealistic soft semirings. Ali et al. [2] and Shabir and Irfan Ali ([2],[18]) studied soft semigroups and soft ideals over a semi group which characterize generalized fuzzy ideals and fuzzy ideals with thresholds of a semigroup. The idea of soft topological spaces was first given by M. Shabir, M. Naz [19] and mappings between soft sets were described by P. Majumdar, S. K. Samanta [15]. Feng et al. [9] worked on soft sets combined with fuzzy sets and rough sets. Recently in ([4],[5]) we have introduced a notion of soft real sets, soft real numbers, soft complex sets, soft complex numbers and some of their basic properties have been investigated. Some applications of soft real sets and soft real numbers have been presented in real life problems. In ([6],[7]) we introduced the concept of 'soft metric', 'soft linear spaces', 'soft norm' on a 'soft linear spaces' and studied various properties of 'soft metric spaces' and 'soft normed linear spaces' in details.

It is worth mentioning here that, in fuzzy settings, topological concepts, metric concept and norm linear space concept are extended nicely but complex valued inner product theory of vector space has not been so successful because of some inherent difficulty in the non-comparable order structure of complex number on one hand and the lattice ordering of the gradation function of fuzzy sets on the other. In this paper we have attempted to extend the inner product concept in soft set settings. We have seen that in soft settings inner product theory could be extended meaningfully.

In fact, in this paper we have introduced a notion of soft inner product on soft linear space and studied some of its properties. In section 2, some preliminary results are given. In section 3, a notion of 'soft inner product' on a 'soft linear space' is given and various properties of 'soft inner product spaces' are studied. It has been shown that every 'soft inner product space' which satisfies (I5) is also a 'soft normed linear space' ([7]) which satisfy (N5). By a counter example it has been established that the converse does not hold in general. The definition of soft Hilbert space is given in section 4 and some properties of soft Hilbert spaces are investigated. The concepts of orthogonality and orthonormality in soft Hilbert spaces and their various properties are studied in section 5 and section 6 respectively. Section 7 concludes the paper.

2. Preliminaries

Definition 2.1 ([16])**.** Let U be an universe and E be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of U and A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to \mathcal{P}(U)$. In other words, a soft set over U is a parametrized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε − approximate elements of the soft set (F, A).

Definition 2.2 ([9])**.** For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if

(1) $A \subseteq B$ and
(2) for all $e \in A$, $F(e) \subseteq G(e)$. We write $(F, A)\tilde{\subset} (G, B)$.

(F, A) is said to be a soft superset of (G, B), if (G, B) is a soft subset of (F, A). We denote it by$(F, A)\tilde{\supset} (G, B)$.

Definition 2.3 ([9])**.** Two soft sets (F, A) and (G, B) over a common universe U are said to be equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.4 ([14])**.** The union of two soft sets (F, A) and (G, B) over the common universe U is the soft set

(H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ G(e) & if\ e \in B - A \\ F(e) \cup G(e) & if\ e \in A \cap B. \end{cases}$$

We express it as $(F, A) \tilde{\cup} (G, B) = (H, C)$.

The following definition of intersection of two soft sets is given as that of the bi-intersection in [8].

Definition 2.5 ([8]). The intersection of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$. We write $(F, A) \tilde{\cap} (G, B) = (H, C)$.

Let X be an initialuniversal set and A be the non-empty set of parameters. In the above definitions the set of parameters may vary from soft set to soft set, but in our considerations, throught this paper all soft sets have the same set of parameters A. The above definitions are also valid for these type of soft sets as a particular case of those definitions.

Definition 2.6 ([9]). The complement of a soft set (F, A) is denoted by $(F, A)^c = (F^c, A)$, where $F^c : A \to \mathcal{P}(U)$ is a mapping given by $F^c(\alpha) = U - F(\alpha)$, for all $\alpha \in A$.

Definition 2.7 ([14]). A soft set (F, A) over U is said to be an *absolute* soft set denoted by $\check{U}$ if for all $\varepsilon \in A$, $F(\varepsilon) = U$.

Definition 2.8 ([14]). A soft set (F, A) over U is said to be a *null* soft set denoted by Φ if for all $\varepsilon \in A$, $F(\varepsilon) = \varnothing$.

Definition 2.9 ([19]). The difference (H, A) of two soft sets (F, A) and (G, A) over X, denoted by $(F, A) \setminus (G, A)$, is defined by $H(e) = F(e) \setminus G(e)$ for all $e \in A$.

Proposition 2.10 ([19]). *Let (F, A) and (G, A) be two soft sets over X. Then*
(i). $((F, A) \tilde{\cup} (G, A))^c = (F, A)^c \tilde{\cap} (G, A)^c$
(ii). $((F, A) \tilde{\cap} (G, A))^c = (F, A)^c \tilde{\cup} (G, A)^c$.

Definition 2.11 ([4]). Let X *be* a non-empty set and A be a non-empty parameter set. Then a function $\varepsilon : A \to X$ is said to be a soft element of X. A soft element ε of X is said to belongs to a soft set B of X, which is denoted by $\varepsilon \tilde{\in} B$, if $\varepsilon(e) \in A(e)$, $\forall e \in A$. Thus for a soft set A of X with respect to the index set A, we have $B(e) = \{\varepsilon(e), \varepsilon \tilde{\in} B\}$, $e \in A$.

It is to be noted that every singleton soft set (a soft set (F, A) for which $F(e)$ is a singleton set, $\forall e \in A$) can be identified with a soft element by simply identifying the singleton set with the element that it contains $\forall e \in A$.

Definition 2.12 ([4]). Let R be the set of real numbers and $\mathfrak{B}(R)$ the collection of all non-empty bounded subsets of R and A taken as a set of parameters. Then a mapping $F : A \to \mathfrak{B}(R)$ is called a *soft real set*. It is denoted by (F, A). If specifically (F, A) is a singleton soft set, then after identifying (F, A) with the corresponding soft element, it will be called a *soft real number*.

We use notations $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ to denote soft real numbers whereas $\overline{r}$, $\overline{s}$, $\bar{t}$ will denote a particular type of soft real numbers such that $\overline{r}(\lambda) = r$, for all $\lambda \in A$ etc. For example $\overline{0}$ is the soft real number where $\overline{0}(\lambda) = 0$, for all $\lambda \in A$.

We now introduce some definitions and prove a theorem on soft real numbers which will be used in this paper.

Definition 2.13. A sequence of soft real numbers $\{\tilde{s}_n\}$ is said to be bounded above if there is a soft real number $\tilde{M}$ such that $s_n \widetilde{\leq} \tilde{M}$ for all $n \in N$. $\{\tilde{s}_n\}$ is said to be bounded below if there is a soft real number $\tilde{m}$ such that $\tilde{m} \widetilde{\leq} \tilde{s}_n$ for all $n \in N$. $\{\tilde{s}_n\}$ is said to be bounded if it is both bounded above and bounded below.

Definition 2.14. A sequence of soft real numbers $\{\tilde{s}_n\}$ is said to be monotonically increasing if $m < n$ imply $\tilde{s}_m \widetilde{\leq} \tilde{s}_n$. $\{s_n\}$ is said to be monotonically decreasing if $m < n$ imply $\tilde{s}_m \widetilde{\geq} \tilde{s}_n$. $\{\tilde{s}_n\}$ is said to be monotene if it is either monotone increasing or monotone decreasing.

Definition 2.15. A sequence of soft real numbers $\{\tilde{s}_n\}$ is said to be convergent if for arbitrary $\widetilde{\varepsilon} \widetilde{>} \overline{0}$, there exists a natural number N such that for all $n \geq N$, $|\tilde{s} - \tilde{s}_n| \widetilde{<} \widetilde{\varepsilon}$. We denote it by $\lim_{n\to\infty} \tilde{s}_n = \tilde{s}$.

Theorem 2.16. *A sequence of soft real numbers with respect to a finite set of parameters which is monotonically increasing and bounded above is convergent.*

Proof. Let $\{\tilde{s}_n\}$ be a sequence of soft real numbers with respect to a finite set of parameters which is monotonically increasing and bounded above. Then Clearly $\{\tilde{s}_n(\lambda)\}$ is a sequence of real numbers which is monotonically increasing and bounded above for each $\lambda \in A$. Then by the property of real numbers if follows that $\{\tilde{s}_n(\lambda)\}$ is convergent for each $\lambda \in A$. Let $s_\lambda = \lim_{n\to\infty} \tilde{s}_n(\lambda)$, $\forall \lambda \in A$. Let $\tilde{s}$ be a soft real number such that $\tilde{s}(\lambda) = s_\lambda$, $\forall \lambda \in A$. Let $\widetilde{\varepsilon} \widetilde{>} \overline{0}$ be arbitrary. Then for each $\lambda \in A$ there exists $n_\lambda \in N$ such that, $|\tilde{s}(\lambda) - \tilde{s}_n(\lambda)| < \widetilde{\varepsilon}(\lambda)$, for all $n \geq n_\lambda$. Since the parameter set is finite we can choose $N(= \max\{n_\lambda; \lambda \in A\}$) such that for all $n \geq N$, $|\tilde{s}(\lambda) - \tilde{s}_n(\lambda)| < \widetilde{\varepsilon}(\lambda)$, $\forall \lambda \in A$ i.e., for all $n \geq N$, $|\tilde{s} - \tilde{s}_n| \widetilde{<} \widetilde{\varepsilon}$ i.e., $\{\tilde{s}_n\}$ is convergent. □

Theorem 2.17. *The infinite series $\sum_{k=1}^{\infty} \tilde{c}_k$ where $\tilde{c}_1, \ \tilde{c}_2, \ \ldots, \tilde{c}_n \ldots$ are soft real numbers with respect to a finite set of parameters is convergent if the sequence $\{\tilde{s}_n\}$ where $\tilde{s}_n = \sum_{k=1}^{n} \tilde{c}_k$ is monotonically increasing and bounded above.*

Proof. This proof is similar to the above theorem and hence omitted. □

Definition 2.18 ([5])**.** Let C be the set of complex numbers and $\wp(C)$be the collection of all non-empty bounded subsets of the set of complex numbers.A be a set of parameters. Then a mapping

$$F : A \to \wp(C)$$

is called a soft complex set. It is denoted by (F, A).

If in particular (F, A) is a singleton soft set, then identifying (F, A) with the corresponding soft element, it will be called a soft complex number.

The set of all soft complex numbers is denoted by $\mathbb{C}(A)$.

Definition 2.19 ([5])**.** Let (F, A) be a soft complex set. Then the complex conjugate of (F, A) is denoted by $(\overline{F}, A)$and is defined by $\overline{F}(\lambda) = \{\overline{z} : z \in F(\lambda)\}$, $\forall \lambda \in A$, where $\overline{z}$ is complex conjugate of the ordinary complex number z. The complex conjugate of a soft complex number (F, A) is $\overline{F}(\lambda) = \overline{z}; z \in F(\lambda), \forall \lambda \in A$.

Definition 2.20 ([5])**.** Let (F, A), $(G, A) \in \mathbb{C}(A)$. Then the sum, difference, product and division are defined by

$(F + G)(\lambda) = z + w; z \in F(\lambda)\,,\ w \in G(\lambda), \forall \lambda \in A;$
$(F - G)(\lambda) = z - w; z \in F(\lambda)\,,\ w \in G(\lambda), \forall \lambda \in A;$
$(FG)(\lambda) = zw; z \in F(\lambda)\,,\ w \in G(\lambda), \forall \lambda \in A;$
$(F/G)(\lambda) = z/w; z \in F(\lambda)\,,\ w \in G(\lambda)$, $\forall \lambda \in A$; provided $G(\lambda) \neq \varnothing$, $\forall \lambda \in A$.

Definition 2.21 ([5])**.** Let (F, A) be a soft complex number. Then the modulus of (F, A) is denoted by $(|F|, A)$and is defined by $|F|(\lambda) = |z|; z \in F(\lambda)$, $\forall \lambda \in A$, where z is an ordinary complex number.

Since the modulus of each ordinary complex number is a non-negative real number and by definition of soft real numbers it follows that $(|F|, A)$is a non-negative soft real number for every soft complex number (F, A).

Let X be a non-empty set. Let $\check{X}$ be the absolute soft set i.e., $F(\lambda) = X$, $\forall \lambda \in A$, where $(F, A) = \check{X}$. Let $\mathcal{S}(\check{X})$ be the collection of the null soft set Φ and those soft sets (F, A) over X for which $F(\lambda) \neq \varnothing$, for all $\lambda \in A$.

Let $(F, A)(\neq \Phi) \in \mathcal{S}(\check{X})$, then the collection of all soft elements of (F, A) will be denoted by $SE(F, A)$. For a collection $\mathfrak{B}$ of soft elements of $\check{X}$, the soft set generated by $\mathfrak{B}$ is denoted by $SS(\mathfrak{B})$.

Definition 2.22 ([6])**.** A mapping $d : SE(\check{X}) \times SE(\check{X}) \to \mathbb{R}(A)^*$, is said to be a *soft metric* on the soft set $\check{X}$ if d satisfies the following conditions:

(M1). $d(\tilde{x}, \tilde{y}) \tilde{\geq} \overline{0}$, for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$.
(M2). $d(\tilde{x}, \tilde{y}) = \overline{0}$, if and only if $\tilde{x} = \tilde{y}$.
(M3). $d(\tilde{x}, \tilde{y}) = d(\tilde{y}, \tilde{x})$for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$.
(M4). For all $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} \check{X}, d(\tilde{x}, \tilde{z}) \tilde{\leq} d(\tilde{x}, \tilde{y}) + d(\tilde{y}, \tilde{z})$

The soft set $\check{X}$ with a soft metric d on $\check{X}$ is said to be a *soft metric space* and is denoted by $(\check{X}, d, A)$ or $(\check{X}, d)$.

Definition 2.23 ([6])**.** Let $(\check{X}, d)$ be a soft metric space,$\tilde{r}$ be a non-negative soft real number and $\tilde{a} \tilde{\in} \check{X}$. By an *open ball* with centre $\tilde{a}$ and radius $\tilde{r}$, we mean the collection of soft elements of $\check{X}$ satisfying $d(\tilde{x}, \tilde{a}) \tilde{<} \tilde{r}$.
The *open ball* with centre $\tilde{a}$ and radius $\tilde{r}$ is denoted by $B(\tilde{a}, \tilde{r})$.
Thus $B(\tilde{a}, \tilde{r}) = \{\tilde{x} \tilde{\in} \check{X};; d(\tilde{x}, \tilde{a}) \tilde{<} \tilde{r}\} \subset SE(\check{X})$.
$SS(B(\tilde{a}, \tilde{r}))$ will be called a *soft open ball* with centre $\tilde{a}$ and radius $\tilde{r}$.

Definition 2.24 ([6])**.** Let (Y, A) be a soft subset in a soft metric space $(\check{X}, d)$. Then a soft element $\tilde{a}$ is said to be an *interior element* of (Y, A) if $\exists$ a positive soft real number $\tilde{r}$ such that $\tilde{a} \in B(\tilde{a}, \tilde{r}) \subset SE(Y, A)$.

Definition 2.25 ([6])**.** Let $(\check{X}, d)$ be a soft metric space and (Y, A) be a non-null soft subset belonging to $\mathcal{S}(\check{X})$ in $(\check{X}, d)$. Then (Y, A) is said to be '*soft open in* $\check{X}$ *with respect to* d' if there is a collection $\mathfrak{B}$ of soft elements of (Y, A) such that $\mathfrak{B}$ is open with respect to d and $(Y, A) = SS(\mathfrak{B})$. In a soft metric space $(\check{X}, d)$, the null soft set Φ is considered to be soft open.

Theorem 2.26 ([6]). *If a soft metric d satisfies the condition:*
(M5). For $(\xi, \eta) \in X \times X$, and $\lambda \in A$, $\{d(\tilde{x}, \tilde{y})(\lambda) : \tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \eta\}$ is a singleton set, and if for $\lambda \in A$, $d_\lambda : X \times X \to \mathbb{R}^+$ is defined by $d_\lambda(\xi, \eta) = d(\tilde{x}, \tilde{y})(\lambda)$, where $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$ such that $\tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \eta$. Then d_λ is a metric on X.

Proposition 2.27 ([6]). *Let $(\check{X}, d)$ be a soft metric space satisfying (M5). Then for every open ball $B(\tilde{a}, \tilde{r})$ in $(\check{X}, d)$, $SS(B(\tilde{a}, \tilde{r}))(\lambda) = B(\tilde{a}(\lambda), \tilde{r}(\lambda))$, an open ball in (X, d_λ), for each $\lambda \in A$.*

Proposition 2.28. *[[6]] Let $(\check{X}, d)$ be a soft metric space satisfying (M5). Then (F, A) is soft open with respect to d if and only if $(F, A)(\lambda)$ is open in (X, d_λ), with centre $\tilde{a}(\lambda)$ and radius $\tilde{r}(\lambda)$ for each $\lambda \in A$.*

Definition 2.29 ([6]). Let $(\check{X}, d)$ be a soft metric space. A soft set $(Y, A) \in \mathcal{S}(\check{X})$, is said to be '*soft closed in $\check{X}$ with respect to d*' if its complement $(Y, A)^c$ is a member of $\mathcal{S}(\check{X})$ and is soft open in $(\check{X}, d)$.

Definition 2.30 ([7]). Let V be a vector space over a field K and let A be a parameter set. Let G be a soft set over (V, A). Now G is said to be a soft vector space or soft linear space of V over K if $G(\lambda)$ is a vector subspace of V, $\forall \lambda \in A$.

Definition 2.31 ([7]). Let F be a soft vector space of V over K. Let $G : A \to \wp(V)$ be a soft set over (V, A). Then G is said to be a soft vector subspace of F if

(i) for each $\lambda \in A$, $G(\lambda)$ is a vector subspace of V over K and
(ii) $F(\lambda) \supseteq G(\lambda)$, $\forall \lambda \in A$.

Definition 2.32 ([7]). Let G be a soft vector space of V over K. Then a soft element of G is said to be a soft vector of G. In a similar manner a soft element of the soft set (K, A) is said to be a soft scalar, K being the scalar field.

Definition 2.33 ([7]). Let $\check{X}$ be the absolute soft vector space i.e., $\check{X}(\lambda) = X$, $\forall \lambda \in A$. Then a mapping $\|.\| : SE(\check{X}) \to \mathbb{R}(A)^*$ is said to be a soft norm on the soft vector space $\check{X}$ if $\|.\|$ satisfies the following conditions:
(N1). $\|\tilde{x}\| \tilde{\geq} \overline{0}$, for all $\tilde{x} \tilde{\in} \check{X}$;
(N2). $\|\tilde{x}\| = \overline{0}$ if and only if $\tilde{x} = \Theta$;
(N3). $\|\tilde{\alpha}.\tilde{x}\| = |\tilde{\alpha}| \|\tilde{x}\|$ for all $\tilde{x} \tilde{\in} \check{X}$ and for every soft scalar $\tilde{\alpha}$;
(N4). For all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$, $\|\tilde{x} + \tilde{y}\| \tilde{\leq} \|\tilde{x}\| + \|\tilde{y}\|$.
The soft vector space $\check{X}$ with a soft norm $\|.\|$ on $\check{X}$ is said to be a soft normed linear space and is denoted by $(\check{X}, \|.\|, A)$ or $(\check{X}, \|.\|)$. (N1), (N2), (N3) and (N4) are said to be soft norm axioms.

Theorem 2.34 ([7]). *If a soft norm $\|.\|$ satisfies the condition*
(N5). For $\xi \in X$, and $\lambda \in A$, $\{\|\tilde{x}\|(\lambda) : \tilde{x}(\lambda) = \xi\}$ is a singleton set.
And if for each $\lambda \in A$, $\|.\|_\lambda : X \to R^+$ be a mapping such that for each $\xi \in X$, $\|\xi\|_\lambda = \|\tilde{x}\|(\lambda)$, where $\tilde{x} \tilde{\in} \check{X}$ such that $\tilde{x}(\lambda) = \xi$. Then for each $\lambda \in A$, $\|.\|_\lambda$ is a norm on X.

Definition 2.35 ([7]). A sequence of soft elements $\{\tilde{x}_n\}$ in a soft normed linear space $(\check{X}, \|.\|, A)$ is said to be convergent and converges to a soft element $\tilde{x}$ if $\|\tilde{x}_n - \tilde{x}\| \to \overline{0}$ as $n \to \infty$. This means for every $\tilde{\varepsilon} \tilde{>} \overline{0}$, chosen arbitrarily, $\exists$ a natural number $N = N(\tilde{\varepsilon})$, such that $\overline{0} \tilde{\leq} \|\tilde{x}_n - \tilde{x}\| \tilde{<} \tilde{\varepsilon}$, whenever $n > N$.
i.e., $n > N \implies \tilde{x}_n \in B(\tilde{x}, \ \tilde{\varepsilon})$. We denote this by $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$ or by $\lim_{n\to\infty} \tilde{x}_n = \tilde{x}$. $\tilde{x}$ is said to be the limit of the sequence $\tilde{x}_n$ as $n \to \infty$.

Theorem 2.36 ([7]). *Let $(\check{X}, \|.\|, A)$ be a soft normed linear space. Then*

(i) *if $\tilde{x}_n \to \tilde{x}$ and $\tilde{y}_n \to \tilde{y}$ then $\tilde{x}_n + \tilde{y}_n \to \tilde{x} + \tilde{y}$.*
(ii) *if $\tilde{x}_n \to \tilde{x}$ and $\widetilde{\lambda}_n \to \widetilde{\lambda}$ then $\widetilde{\lambda}_n.\tilde{x}_n \to \widetilde{\lambda}.\tilde{x}$, where $\{\widetilde{\lambda}_n\}$ is a sequence of soft scalars.*
(iii) *if $\{\tilde{x}_n\}$ and $\{\tilde{y}_n\}$ are Cauchy sequences in $\check{X}$ and $\{\widetilde{\lambda}_n\}$ is a Cauchy sequence of soft scalars, then $\{\tilde{x}_n + \tilde{y}_n\}$ and $\{\widetilde{\lambda}_n.\tilde{x}_n\}$ are also Cauchy sequences in $\check{X}$.*

Definition 2.37 ([7]). Let $(\check{X}, \|.\|, A)$ be a soft normed linear space. Then $\check{X}$ is said to be complete if every Cauchy sequence in $\check{X}$ converges to a soft element of $\check{X}$. Every complete soft normed linear space is called a soft Banach Space.

Theorem 2.38 ([7]). *Every Cauchy sequence in $\mathbb{R}(A)$, where A is a finite set of parameters, is convergent, i.e., the set of all soft real numbers with its usual modulus soft norm with respect to a finite set of parameters, is a soft Banach space.*

Definition 2.39 ([7]). A soft linear space $\check{X}$ is said to be of finite dimensional if there is a finite set of linearly independent soft vectors in $\check{X}$ which also generates $\check{X}$, i.e., any soft element of $\check{X}$ can be expressed as a linear combination of those linearly independent soft vectors.

The set of those linearly independent soft vectors is said to be the basis for $\check{X}$ and the number of soft vectors of the basis is called the dimension of $\check{X}$.

Theorem 2.40 ([7]). *Every finite dimensional soft normed linear space which satisfies (N5) and have a finite set of parameters, is complete.*

Definition 2.41 ([6]). For two soft real numbers $\tilde{r}, \tilde{s}$ we define

(i) $\tilde{r} \tilde{\leq} \tilde{s}$ if $\tilde{r}(\lambda) \leq \tilde{s}(\lambda)$, for all $\lambda \in A$;
(ii) $\tilde{r} \tilde{\geq} \tilde{s}$ if $\tilde{r}(\lambda) \geq \tilde{s}(\lambda)$, for all $\lambda \in A$;
(iii) $\tilde{r} \tilde{<} \tilde{s}$ if $\tilde{r}(\lambda) < \tilde{s}(\lambda)$, for all $\lambda \in A$;
(iv) $\tilde{r} \tilde{>} \tilde{s}$ if $\tilde{r}(\lambda) > \tilde{s}(\lambda)$, for all $\lambda \in A$.

3. Soft inner product and soft inner product spaces

Let X be a vector space over a field C of complex numbers, X is also our initial universal set and A be a non-empty set of parameters. Let $\check{X}$ be the absolute soft vector space i.e., $\check{X}(\lambda) = X, \forall\lambda \in A$. We use the notation $\tilde{x}, \ \tilde{y}, \ \tilde{z}$ to denote soft vectors of a soft vector space and $\tilde{r}, \ \tilde{s}, \ \tilde{t}$ to denote soft real numbers whereas $\overline{r}, \ \overline{s}, \ \overline{t}$ will denote a particular type of soft real numbers such that $\overline{r}(\lambda) = r$, for all $\lambda \in A$ etc. For example $\overline{0}$ is the soft real number where $\overline{0}(\lambda) = 0$, for all $\lambda \in A$.

Definition 3.1. Let $\check{X}$ be the absolute soft vector space i.e., $\check{X}(\lambda) = X, \forall\lambda \in A$. Then a mapping $\langle . \rangle : SE(\check{X}) \times SE(\check{X}) \to \mathbb{C}(A)$ is said to be a soft inner product on the soft vector space $\check{X}$ if $\langle . \rangle$ satisfies the following conditions:

(I1). $\langle \tilde{x}, \tilde{x} \rangle \widetilde{\geq} \overline{0}$, for all $\tilde{x} \widetilde{\in} \check{X}$ and $\langle \tilde{x}, \tilde{x} \rangle = \overline{0}$ if and only if $\tilde{x} = \Theta$;

(I2). $\langle \tilde{x}, \tilde{y} \rangle = \overline{\langle \tilde{y}, \tilde{x} \rangle}$ where bar denote the complex conjugate of soft complex numbers;

(I3). $\langle \widetilde{\alpha}.\tilde{x}, \tilde{y} \rangle = \widetilde{\alpha}. \langle \tilde{x}, \tilde{y} \rangle$ for all $\tilde{x}, \tilde{y} \widetilde{\in} \check{X}$ and for every soft scalar $\widetilde{\alpha}$;

(I4). For all $\tilde{x}, \tilde{y} \widetilde{\in} \check{X}$, $\langle \tilde{x} + \tilde{y}, \tilde{z} \rangle = \langle \tilde{x}, \tilde{z} \rangle + \langle \tilde{y}, \tilde{z} \rangle$.

The soft vector space $\check{X}$ with a soft inner product $\langle . \rangle$ on $\check{X}$ is said to be a soft inner product space and is denoted by $(\check{X}, \langle . \rangle, A)$ or $(\check{X}, \langle . \rangle)$. (I1), (I2), (I3) and (I4) are said to be soft inner product axioms.

Example 3.2. Let $X = l_2$. Then X is an inner product space with respect to the inner product $\langle x, y \rangle = \sum_{i=1}^{\infty} \xi_i \overline{\eta_i}$ for $x = \{\xi_i\}$, $y = \{\eta_i\}$ of l_2. Let $\tilde{x}, \tilde{y}$ be soft elements of the absolute soft vector space $\check{X}$. Then $\tilde{x}(\lambda) = \{\xi_i^{\lambda}\}, \tilde{y}(\lambda) = \{\eta_i^{\lambda}\}$ are elements of l_2. The mapping $\langle . \rangle : SE(\check{X}) \times SE(\check{X}) \to \mathbb{C}(A)$ defined by $\langle \tilde{x}, \tilde{y} \rangle (\lambda) = \sum_{i=1}^{\infty} \xi_i^{\lambda} \overline{\eta_i^{\lambda}} = \langle \tilde{x}(\lambda), \tilde{y}(\lambda) \rangle$, $\forall \lambda \in A$, is a soft inner product on the soft vector space $\check{X}$.

Let us verify (I1), (I2), (I3) and (I4) for soft inner product.

(I1). We have $\langle \tilde{x}, \tilde{x} \rangle (\lambda) = \langle \tilde{x}(\lambda), \tilde{x}(\lambda) \rangle = \sum_{i=1}^{\infty} \xi_i^{\lambda} \overline{\eta_i^{\lambda}} \geq 0, \forall \lambda \in A,\ \forall \tilde{x} \widetilde{\in} \check{X}$,

$\therefore\ \langle \tilde{x}, \tilde{x} \rangle \widetilde{\geq} \overline{0}$, for all $\tilde{x} \widetilde{\in} \check{X}$.

Also, $\langle \tilde{x}, \tilde{x} \rangle = \overline{0} \iff \langle \tilde{x}, \tilde{x} \rangle (\lambda) = 0,\ \forall \lambda \in A \iff\ \tilde{x}(\lambda) = \theta,\ \forall \lambda \in A \iff\ \tilde{x} = \Theta$

$\therefore$ (I1) is satisfied.

(I2). We have, $\langle \tilde{x}, \tilde{y} \rangle (\lambda) = \langle \tilde{x}(\lambda), \tilde{y}(\lambda) \rangle = \overline{\langle \tilde{y}(\lambda), \tilde{x}(\lambda) \rangle}$

$= \overline{\langle \tilde{y}, \tilde{x} \rangle} (\lambda), \forall \lambda \in A,\ \forall \tilde{x}, \tilde{y} \widetilde{\in} \check{X}.\ \therefore \langle \tilde{x}, \tilde{y} \rangle = \overline{\langle \tilde{y}, \tilde{x} \rangle}$.

(I3) We have, $\langle \widetilde{\alpha}.\tilde{x}, \tilde{y} \rangle (\lambda) = \langle \widetilde{\alpha}(\lambda).\tilde{x}(\lambda), \tilde{y}(\lambda) \rangle$

$= \widetilde{\alpha}(\lambda). \langle \tilde{x}(\lambda), \tilde{y}(\lambda) \rangle,\ \forall \lambda \in A$

$= (\widetilde{\alpha}. \langle \tilde{x}, \tilde{y} \rangle)(\lambda),\ \forall \lambda \in A$.

$\therefore \langle \widetilde{\alpha}.\tilde{x}, \tilde{y} \rangle = \widetilde{\alpha}. \langle \tilde{x}, \tilde{y} \rangle$ for all $\tilde{x}, \tilde{y} \widetilde{\in} \check{X}$ and for every soft scalar $\widetilde{\alpha}$.

(I4). For all $\tilde{x}, \tilde{y}, \tilde{z} \widetilde{\in} \check{X}$,

$\langle \tilde{x} + \tilde{y}, \tilde{z} \rangle (\lambda) = \langle \tilde{x}(\lambda) + \tilde{y}(\lambda), \tilde{z}(\lambda) \rangle$

$= \langle \tilde{x}(\lambda), \tilde{z}(\lambda) \rangle + \langle \tilde{y}(\lambda), \tilde{z}(\lambda) \rangle$

$= \langle \tilde{x}, \tilde{z} \rangle (\lambda) + \langle \tilde{y}, \tilde{z} \rangle (\lambda)$, [by the property of $\langle . \rangle$ on X]

$= (\langle \tilde{x}, \tilde{z} \rangle + \langle \tilde{y}, \tilde{z} \rangle)(\lambda), \forall \lambda \in A$.

$\therefore\ \langle \tilde{x} + \tilde{y}, \tilde{z} \rangle = \langle \tilde{x}, \tilde{z} \rangle + \langle \tilde{y}, \tilde{z} \rangle$. Thus (I4) is satisfied.

$\therefore \langle . \rangle$ is a soft inner product on $\check{X}$ and consequently $(\check{X}, \langle . \rangle))$ is a soft inner product space.

Proposition 3.3. *Let $\{\langle . \rangle_{\lambda}, \lambda \in A\}$ be a family of crisp inner products on a crisp vector space X. Then the mapping $\langle . \rangle : SE(\check{X}) \times SE(\check{X}) \to \mathbb{C}(A)$ by $\langle \tilde{x}, \tilde{y} \rangle (\lambda) = \langle \tilde{x}(\lambda), \tilde{y}(\lambda) \rangle_{\lambda}$, $\forall \lambda \in A$, $\forall \tilde{x}, \tilde{y} \widetilde{\in} \check{X}$ is a soft inner product on the soft vector space $\check{X}$.*

Proof. The proof of this proposition is similar to the above example and hence omitted. □

Corollary 3.4. *Every crisp inner product $\langle . \rangle_X$ on a crisp vector space X can be extended to a soft inner product on the soft vector space $\check{X}$.*

Proof. First we construct the absolute soft vector space $\check{X}$ using a non-empty set of parameters A.

Let us define a mapping $\langle . \rangle : SE\left(\check{X}\right) \times SE\left(\check{X}\right) \rightarrow \mathbb{C}\left(A\right)$ by
$\langle \tilde{x}, \tilde{y} \rangle \left(\lambda\right) = \langle \tilde{x}\left(\lambda\right), \tilde{y}\left(\lambda\right) \rangle_X, \ \forall \lambda \in A, \ \forall \tilde{x}, \tilde{y} \,\tilde{\in}\, \check{X}$.
Then using the same procedure as in Example 3.2, it can be easily proved that $\langle . \rangle$ is a soft inner product on $\check{X}$.
This soft inner product is generated using the crisp inner product $\|.\|_X$ and it is said to be the soft inner product generated by $\|.\|_X$. □

Remark 3.5. The soft inner product $\langle . \rangle$ as defined in Example 3.2, satisfies the condition

(I5). For $(\xi, \eta) \in X \times X$ and $\lambda \in A$,

$$\left\{ \langle \tilde{x}, \tilde{y} \rangle \left(\lambda\right) : \tilde{x}, \tilde{y} \tilde{\in} \check{X} \text{ such that } \tilde{x}\left(\lambda\right) = \xi, \tilde{y}\left(\lambda\right) = \eta \right\}$$

is a singleton set.

Theorem 3.6. *(Decomposition Theorem). If a soft inner product $\langle . \rangle$ satisfies the condition (I5), and if for each $\lambda \in A$, $\langle . \rangle_\lambda : X \times X \rightarrow \mathbb{C}$ be a mapping such that for all $(\xi, \eta) \in X \times X$, $\langle \xi, \eta \rangle_\lambda = \langle \tilde{x}, \tilde{y} \rangle \left(\lambda\right)$, where $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$ such that $\tilde{x}\left(\lambda\right) = \xi, \tilde{y}\left(\lambda\right) = \eta$. Then for each $\lambda \in A$, $\langle . \rangle_\lambda$ is an inner product on X.*

Proof. Clearly $\langle . \rangle_\lambda : X \times X \rightarrow \mathbb{C}$ is a rule that assigns an ordered pair of X to a crisp complex number $\forall \lambda \in A$. Now the well defined property of $\langle . \rangle_\lambda$, $\forall \lambda \in A$ follows from the condition (I5) and the soft inner product axioms gives the inner product conditions of $\langle . \rangle_\lambda$, $\forall \lambda \in A$. Thus the soft inner product satisfying (I5) gives a parameterized family of crisp inner products. With this point of view, it also follows that, a soft inner product, satisfying (I5) is a particular 'soft mapping' as defined by P. Majumdar, et al. in [15] where $\langle . \rangle : A \rightarrow (\mathbb{C})^{X \times X}$. □

Proposition 3.7. *Let $(\check{X}, \langle . \rangle, A)$ be a soft inner product space, $\tilde{x}, \ \tilde{y}, \ \tilde{z} \,\tilde{\in}\, \check{X}$ and $\widetilde{\alpha}, \ \widetilde{\beta}$ etc. be soft scalars. Then*

(i) $\left\langle \widetilde{\alpha}\tilde{x} + \widetilde{\beta}\tilde{y}, \ \tilde{z} \right\rangle = \widetilde{\alpha} \langle \tilde{x}, \ \tilde{z} \rangle + \widetilde{\beta} \langle \tilde{y}, \ \tilde{z} \rangle$;

(ii) $\langle \tilde{x}, \ \widetilde{\alpha}\tilde{y} \rangle = \overline{\widetilde{\alpha}}. \langle \tilde{x}, \ \tilde{y} \rangle$;

(iii) $\left\langle \tilde{x}, \ \widetilde{\alpha}\tilde{y} + \widetilde{\beta}\tilde{z} \right\rangle = \overline{\widetilde{\alpha}}. \langle \tilde{x}, \ \tilde{y} \rangle + \overline{\widetilde{\beta}}. \langle \tilde{x}, \ \tilde{z} \rangle$.

Proof. (i) We have, $\left\langle \widetilde{\alpha}\tilde{x} + \widetilde{\beta}\tilde{y}, \ \tilde{z} \right\rangle = \langle \widetilde{\alpha}\tilde{x}, \ \tilde{z} \rangle + \left\langle \widetilde{\beta}\tilde{y}, \ \tilde{z} \right\rangle = \widetilde{\alpha} \langle \tilde{x}, \ \tilde{z} \rangle + \widetilde{\beta} \langle \tilde{y}, \ \tilde{z} \rangle$.

(ii) $\langle \tilde{x}, \ \widetilde{\alpha}\tilde{y} \rangle = \overline{\langle \widetilde{\alpha}\tilde{y}, \ \tilde{x} \rangle} = \overline{\widetilde{\alpha}. \langle \tilde{y}, \ \tilde{x} \rangle} = \overline{\widetilde{\alpha}}. \langle \tilde{x}, \ \tilde{y} \rangle$.

(iii) $\left\langle \tilde{x}, \ \widetilde{\alpha}\tilde{y} + \widetilde{\beta}\tilde{z} \right\rangle = \overline{\left\langle \widetilde{\alpha}\tilde{y} + \widetilde{\beta}\tilde{z}, \ \tilde{x} \right\rangle} = \overline{\langle \widetilde{\alpha}\tilde{y}, \ \tilde{x} \rangle} + \overline{\left\langle \widetilde{\beta}\tilde{z}, \ \tilde{x} \right\rangle} = \overline{\widetilde{\alpha}}. \overline{\langle \tilde{y}, \ \tilde{x} \rangle} + \overline{\widetilde{\beta}}. \overline{\langle \tilde{z}, \ \tilde{x} \rangle} = \overline{\widetilde{\alpha}}. \langle \tilde{x}, \ \tilde{y} \rangle + \overline{\widetilde{\beta}}. \langle \tilde{x}, \ \tilde{z} \rangle$. □

Theorem 3.8. *(Schwarz inequality). Let $(\check{X}, \langle . \rangle, A)$ be a soft inner product space satisfying (I5). Let $\tilde{x}, \ \tilde{y}, \ \tilde{z} \,\tilde{\in}\, \check{X}$ and $\widetilde{\alpha}, \ \widetilde{\beta}$ etc. be soft scalars. Then $|\langle \tilde{x}, \ \tilde{y} \rangle| \,\tilde{\leq}\, \|\tilde{x}\| . \|\tilde{y}\|$.*

Proof. (i) If $\widetilde{\alpha}$ be a soft scalar, then $\langle \tilde{x} + \widetilde{\alpha}\tilde{y}, \tilde{x} + \widetilde{\alpha}\tilde{y} \rangle \,\tilde{\geq}\, \overline{0}$ i.e.,

$$\langle \tilde{x}, \ \tilde{x} \rangle + \widetilde{\alpha} \langle \tilde{y}, \tilde{x} \rangle + \overline{\widetilde{\alpha}}. \langle \tilde{x}, \ \tilde{y} \rangle + |\widetilde{\alpha}|^2 \langle \tilde{y}, \ \tilde{y} \rangle \,\tilde{\geq}\, \overline{0} \tag{3.1}$$

If $\langle \tilde{y}, \ \tilde{y} \rangle = \overline{0}$ then $\tilde{y} = \Theta$ and so $\langle \tilde{x}, \ \tilde{y} \rangle = \langle \tilde{x}, \ \Theta \rangle = \left\langle \tilde{x}, \overline{0}. \ \Theta \right\rangle = \overline{0}$ and in this case the inequality is proved.

In case $\langle \tilde{y},\ \tilde{y}\rangle \tilde{>} \bar{0}$; Let $\widetilde{\alpha} = -\frac{\langle \tilde{x},\ \tilde{y}\rangle}{\langle \tilde{y},\ \tilde{y}\rangle}$. Then we obtain from (3.1)

$$\|\tilde{x}\|^2 - \frac{|\langle \tilde{x},\ \tilde{y}\rangle|^2}{\|\tilde{y}\|^2} - \frac{\langle \tilde{x},\ \tilde{y}\rangle . \overline{\langle \tilde{x},\ \tilde{y}\rangle}}{\|\tilde{y}\|^2} + \frac{|\langle \tilde{x},\ \tilde{y}\rangle|^2}{\|\tilde{y}\|^2} \tilde{\geq} \bar{0}$$

i.e., $\|\tilde{x}\|^2 - \frac{\langle \tilde{x},\ \tilde{y}\rangle . \overline{\langle \tilde{x},\ \tilde{y}\rangle}}{\|\tilde{y}\|^2} \tilde{\geq} \bar{0}$, i.e., $|\langle \tilde{x},\ \tilde{y}\rangle| \tilde{\leq} \|\tilde{x}\| . \|\tilde{y}\|$.

Next let $\tilde{y} \neq \Theta$ be such that $\langle \tilde{y},\ \tilde{y}\rangle \tilde{\ngtr} \bar{0}$, then $\langle \tilde{y},\ \tilde{y}\rangle \neq \bar{0}$. Let $C \subset A$ be a set of parameters such that $\langle \tilde{y},\ \tilde{y}\rangle(\lambda) = \langle \tilde{y}(\lambda), \tilde{y}(\lambda)\rangle_\lambda > 0$; for $\lambda \in C$ and $\langle \tilde{y},\ \tilde{y}\rangle(\lambda) = 0 = \langle \tilde{y}(\lambda), \tilde{y}(\lambda)\rangle_\lambda$, i.e., $\tilde{y}(\lambda) = \theta$; for $\lambda \in A \backslash C$. In this case both the sets C and $A\backslash C$ are non-empty. For $\lambda \in A\backslash C$, $\langle \tilde{x},\ \tilde{y}\rangle(\lambda) = \langle \tilde{x}(\lambda), \tilde{y}(\lambda)\rangle_\lambda = \langle \tilde{x}(\lambda), \theta\rangle_\lambda = \langle \tilde{x}(\lambda), 0.\theta\rangle_\lambda = 0 = \bar{0}(\lambda)$.
Thus,

$$|\langle \tilde{x},\ \tilde{y}\rangle|(\lambda) = (\|\tilde{x}\| . \|\tilde{y}\|)(\lambda),\ \forall \lambda \in A\backslash C. \tag{3.2}$$

Since $(\check{X}, \langle . \rangle, A)$ satisfies (I5), we get from (3.1), for all $\lambda \in C$, $\langle \tilde{x}(\lambda), \tilde{x}(\lambda)\rangle_\lambda + \widetilde{\alpha}(\lambda)\langle \tilde{y}(\lambda), \tilde{x}(\lambda)\rangle_\lambda + \overline{\widetilde{\alpha}}(\lambda)\langle \tilde{x}(\lambda), \tilde{y}(\lambda)\rangle_\lambda + |\widetilde{\alpha}|^2(\lambda)\langle \tilde{y}(\lambda), \tilde{y}(\lambda)\rangle_\lambda \geq 0$
let $\widetilde{\alpha}(\lambda) = -\frac{\langle \tilde{x}(\lambda), \tilde{y}(\lambda)\rangle_\lambda}{\langle \tilde{y}(\lambda), \tilde{y}(\lambda)\rangle_\lambda}$. Then from (3.1) we get,

$$\|\tilde{x}(\lambda)\|_\lambda{}^2 - \frac{|\langle \tilde{x}(\lambda), \tilde{y}(\lambda)\rangle_\lambda|^2}{\|\tilde{y}(\lambda)\|_\lambda{}^2} - \frac{\langle \tilde{x}(\lambda), \tilde{y}(\lambda)\rangle_\lambda . \overline{\langle \tilde{x}(\lambda), \tilde{y}(\lambda)\rangle_\lambda}}{\|\tilde{y}(\lambda)\|_\lambda{}^2} + \frac{|\langle \tilde{x}(\lambda), \tilde{y}(\lambda)\rangle_\lambda|^2}{\|\tilde{y}(\lambda)\|_\lambda{}^2} \geq 0$$

i.e., $\|\tilde{x}(\lambda)\|_\lambda{}^2 - \frac{\langle \tilde{x}(\lambda), \tilde{y}(\lambda)\rangle_\lambda . \overline{\langle \tilde{x}(\lambda), \tilde{y}(\lambda)\rangle_\lambda}}{\|\tilde{y}(\lambda)\|_\lambda{}^2} \geq 0$, i.e., $|\langle \tilde{x}(\lambda), \tilde{y}(\lambda)\rangle_\lambda|^2 \leq \|\tilde{x}(\lambda)\|_\lambda{}^2 . \|\tilde{y}(\lambda)\|_\lambda{}^2$.
i.e.,

$$|\langle \tilde{x},\ \tilde{y}\rangle|(\lambda) \leq (\|\tilde{x}\| . \|\tilde{y}\|)(\lambda),\ \forall \lambda \in C. \tag{3.3}$$

From (3.2) and (3.3), it follows that $|\langle \tilde{x},\ \tilde{y}\rangle|(\lambda) \leq (\|\tilde{x}\| . \|\tilde{y}\|)(\lambda),\ \forall \lambda \in A$.
$\therefore |\langle \tilde{x},\ \tilde{y}\rangle| \tilde{\leq} \|\tilde{x}\| . \|\tilde{y}\|$. ☐

Proposition 3.9. *Let $(\check{X}, \langle . \rangle, A)$ be a soft inner product space satisfying (I5). Let $\tilde{x},\ \tilde{y},\ \tilde{z} \tilde{\in} \check{X}$ and $\widetilde{\alpha},\ \widetilde{\beta}$ etc. be soft scalars. Then*

(i) $\|\tilde{x} + \tilde{y}\|^2 + \|\tilde{x} - \tilde{y}\|^2 = 2\|\tilde{x}\|^2 + 2\|\tilde{y}\|^2$ *(Parallelogram law);*

(ii) $\langle \tilde{x},\ \tilde{y}\rangle = \frac{1}{4}\{\|\tilde{x} + \tilde{y}\|^2 - \|\tilde{x} - \tilde{y}\|^2 + \bar{i}\|\tilde{x} + \bar{i}\tilde{y}\|^2 - \bar{i}\,\|\tilde{x} - \bar{i}\tilde{y}\|^2\}$ *(Polarization identity).*

(iii) $\lim_{n\to\infty} \tilde{x}_n = \tilde{x}$, $\lim_{n\to\infty} \tilde{y}_n = \tilde{y}$, *then* $\langle \tilde{x}_n,\ \tilde{y}_n\rangle \to \langle \tilde{x},\ \tilde{y}\rangle$ *as* $n \to \infty$.

Proof. (i) $\|\tilde{x} + \tilde{y}\|^2 + \|\tilde{x} - \tilde{y}\|^2 = \langle \tilde{x} + \tilde{y}, \tilde{x} + \tilde{y}\rangle + \langle \tilde{x} - \tilde{y}, \tilde{x} - \tilde{y}\rangle$
$= \langle \tilde{x},\ \tilde{x}\rangle + \langle \tilde{y}, \tilde{x}\rangle + \langle \tilde{x},\ \tilde{y}\rangle + \langle \tilde{y},\ \tilde{y}\rangle + \langle \tilde{x},\ \tilde{x}\rangle - \langle \tilde{y}, \tilde{x}\rangle - \langle \tilde{x},\ \tilde{y}\rangle + \langle \tilde{y},\ \tilde{y}\rangle$
$= 2\langle \tilde{x},\ \tilde{x}\rangle + 2\langle \tilde{y},\ \tilde{y}\rangle = 2\|\tilde{x}\|^2 + 2\|\tilde{y}\|^2$.

(ii) We have

$$\|\tilde{x} + \tilde{y}\|^2 = \langle \tilde{x} + \tilde{y}, \tilde{x} + \tilde{y}\rangle = \|\tilde{x}\|^2 + \|\tilde{y}\|^2 + \langle \tilde{y}, \tilde{x}\rangle + \langle \tilde{x},\ \tilde{y}\rangle \tag{3.4}$$

In (3.4) replace y by $-y$, $\bar{i}y$, $-\bar{i}y$ then we obtain
$\|\tilde{x} - \tilde{y}\|^2 = \|\tilde{x}\|^2 + \|\tilde{y}\|^2 - \langle \tilde{y}, \tilde{x}\rangle - \langle \tilde{x},\ \tilde{y}\rangle$
$\|\tilde{x} + \bar{i}\tilde{y}\|^2 = \|\tilde{x}\|^2 + \|\tilde{y}\|^2 + \bar{i}\langle \tilde{y}, \tilde{x}\rangle - \bar{i}\langle \tilde{x},\ \tilde{y}\rangle$ and

$\left\|\tilde{x}-\bar{i}\tilde{y}\right\|^2=\|\tilde{x}\|^2+\|\tilde{y}\|^2-\bar{i}\langle\tilde{y},\tilde{x}\rangle+\bar{i}\langle\tilde{x},\ \tilde{y}\rangle$ Or what are same as

$$-\|\tilde{x}-\tilde{y}\|^2=-\|\tilde{x}\|^2-\|\tilde{y}\|^2+\langle\tilde{y},\tilde{x}\rangle+\langle\tilde{x},\ \tilde{y}\rangle \tag{3.5}$$

$$\bar{i}\left\|\tilde{x}+\bar{i}\tilde{y}\right\|^2=\bar{i}\|\tilde{x}\|^2+\bar{i}\|\tilde{y}\|^2-\langle\tilde{y},\tilde{x}\rangle+\langle\tilde{x},\ \tilde{y}\rangle \tag{3.6}$$

and

$$\bar{i}\left\|\tilde{x}-\bar{i}\tilde{y}\right\|^2=-\bar{i}\|\tilde{x}\|^2-\bar{i}\|\tilde{y}\|^2-\langle\tilde{y},\tilde{x}\rangle+\langle\tilde{x},\ \tilde{y}\rangle \tag{3.7}$$

After adding (3.4), (3.5), (3.6) and (3.7) the right hand side becomes $4\langle\tilde{x},\ \tilde{y}\rangle$ and that proves the identity.

(iii) If $\lim_{n\to\infty}\tilde{x}_n=\tilde{x}$, then the sequence $\{\|\tilde{x}_n\|\}$ is bounded. Let $\|\tilde{x}_n\|\,\tilde{\leq}\tilde{M}$ for $n=1,2,3,\ldots$

Now, $|\langle\tilde{x}_n,\ \tilde{y}_n\rangle-\langle\tilde{x},\ \tilde{y}\rangle|\,\tilde{\leq}\,|\langle\tilde{x}_n,\ \tilde{y}_n\rangle-\langle\tilde{x}_n,\ \tilde{y}\rangle+\langle\tilde{x}_n,\ \tilde{y}\rangle-\langle\tilde{x},\ \tilde{y}\rangle|$
$=|\langle\tilde{x}_n,\ \tilde{y}_n-\tilde{y}\rangle-\langle\tilde{x}_n-\tilde{x},\ \tilde{y}\rangle|\,\tilde{\leq}\,\|\tilde{x}_n\|\,.\,\|\tilde{y}_n-\tilde{y}\|+\|\tilde{x}_n-\tilde{x}\|\,.\,\|\tilde{y}\|$
$\tilde{\leq}\tilde{M}.\,\|\tilde{y}_n-\tilde{y}\|+\|\tilde{x}_n-\tilde{x}\|\,.\,\|\tilde{y}\|\to\bar{0}$ as $n\to\infty$. So, $\langle\tilde{x}_n,\ \tilde{y}_n\rangle\to\langle\tilde{x},\ \tilde{y}\rangle$. □

Theorem 3.10. *Let $(\check{X},\langle.\rangle,A)$ be a soft inner product space satisfying (I5). Let us define $\|.\|:\ \check{X}\to R(A)^*$ by $\|\tilde{x}\|=\sqrt{\langle\tilde{x},\ \tilde{x}\rangle}$, for all $\tilde{x}\tilde{\in}\ \check{X}$. Then $\|.\|$ is a soft norm on $\check{X}$ satisfying (N5).*

Proof. We have, (N1, N2). $\|\tilde{x}\|=\sqrt{\langle\tilde{x},\ \tilde{x}\rangle}\tilde{\geq}\bar{0}$, for all $\tilde{x}\tilde{\in}\check{X}$, and $\|\tilde{x}\|=\bar{0}\iff\sqrt{\langle\tilde{x},\ \tilde{x}\rangle}=\bar{0}\iff\tilde{x}=\Theta$ [using (I1)]

(N3). $\|\tilde{\alpha}.\tilde{x}\|=\sqrt{\langle\tilde{\alpha}.\tilde{x},\tilde{\alpha}.\ \tilde{x}\rangle}=\sqrt{\tilde{\alpha}.\langle\tilde{x},\tilde{\alpha}.\ \tilde{x}\rangle}=\sqrt{\tilde{\alpha}.\bar{\tilde{\alpha}}.\langle\tilde{x},\ \tilde{x}\rangle}=|\tilde{\alpha}|\,.\sqrt{\langle\tilde{x},\ \tilde{x}\rangle}=|\tilde{\alpha}|\,.\,\|\tilde{x}\|$ for all $\tilde{x}\,\tilde{\in}\,\check{X}$ and for every soft scalar $\tilde{\alpha}$. [using (I2)].

(N4). We have, $\|\tilde{x}+\tilde{y}\|^2=\langle\tilde{x}+\tilde{y},\ \tilde{x}+\tilde{y}\rangle=\langle\tilde{x},\ \tilde{x}\rangle+\langle\tilde{x},\ \tilde{y}\rangle+\langle\tilde{y},\ \tilde{x}\rangle+\langle\tilde{y},\ \tilde{y}\rangle$

$$=\|\tilde{x}\|^2+\|\tilde{y}\|^2+\langle\tilde{y},\tilde{x}\rangle+\langle\tilde{x},\ \tilde{y}\rangle \tag{3.8}$$

Now, since $\langle.\rangle$ satisfies (I5), we have $|\langle\tilde{x},\ \tilde{y}\rangle|\,\tilde{\leq}\,\|\tilde{x}\|\,.\,\|\tilde{y}\|$ and so

$$|\langle\tilde{y},\ \tilde{x}\rangle|=\overline{|\langle\tilde{x},\ \tilde{y}\rangle|}\tilde{\leq}\,\|\tilde{x}\|\,.\,\|\tilde{y}\|$$

So, $|\langle\tilde{x},\ \tilde{y}\rangle+\langle\tilde{y},\ \tilde{x}\rangle|\,\tilde{\leq}\,|\langle\tilde{x},\ \tilde{y}\rangle|+|\langle\tilde{y},\ \tilde{x}\rangle|\tilde{\leq}2\,\|\tilde{x}\|\,.\,\|\tilde{y}\|$. But

$$\langle\tilde{x},\ \tilde{y}\rangle+\langle\tilde{y},\ \tilde{x}\rangle=\overline{\langle\tilde{x},\ \tilde{y}\rangle}+\langle\tilde{x},\ \tilde{y}\rangle$$

is a soft real number. Hence, $\langle\tilde{x},\ \tilde{y}\rangle+\langle\tilde{y},\ \tilde{x}\rangle\,\tilde{\leq}2\,\|\tilde{x}\|\,.\,\|\tilde{y}\|$. So, from (3.8),

$$\|\tilde{x}+\tilde{y}\|^2\tilde{\leq}\|\tilde{x}\|^2+\|\tilde{y}\|^2+2\,\|\tilde{x}\|\,.\,\|\tilde{y}\|=(\|\tilde{x}\|+\|\tilde{y}\|)^2.$$

Hence, $\|\tilde{x}+\tilde{y}\|\,\tilde{\leq}\,\|\tilde{x}\|+\|\tilde{y}\|$. Hence $\|.\|$ is a soft norm on $\check{X}$. Further since $\langle.\rangle$ satisfies (I5). For $(\xi,\eta)\in X\times X$ and $\lambda\in A$, $\{\langle\tilde{x},\tilde{y}\rangle(\lambda):\tilde{x}(\lambda)=\xi,\tilde{y}(\lambda)=\eta\}$ is a singleton set. Therefore, For $\xi\in X$ and $\lambda\in A$, $\{\|\tilde{x}\|(\lambda)=\langle\tilde{x},\tilde{x}\rangle(\lambda):\tilde{x}(\lambda)=\xi\}$ is a singleton set. Thus, $\|.\|$ satisfies (N5). □

Remark 3.11. From the above theorem it follows that every 'soft inner product space' which satisfies (I5) is also a 'soft normed linear space' which satisfies (N5) with the soft norm defined as above. With the help of this soft norm, we can introduce a 'soft metric' on $\check{X}$ by the formula $d(\tilde{x}, \tilde{y}) = \|\tilde{x} - \tilde{y}\| = \sqrt{\langle \tilde{x} - \tilde{y},\ \tilde{x} - \tilde{y} \rangle}$.In a similar way as above, it can be proved that, this soft metric satisfies (M5).

Remark 3.12. The converse of the above theorem is however not true. For example consider $X = l_p (p \geq 1, p \neq 2)$. Then X is a normed linear space with respect to the norm $\|x\| = (\sum_{i=1}^{\infty} |\xi_i|^p)^{1/p}$ for $x = \{\xi_i\}$ of l_p. Let $\tilde{x}$ be a soft element of the absolute soft vector space $\check{X}$. Then $\tilde{x}(\lambda) = \{\xi_i^\lambda\}$ is an element of l_p. Then it is easy to verify that the mapping $\|.\| : SE(\check{X}) \to \mathbb{R}(A)$ defined by $\|\tilde{x}\|(\lambda) = (\sum_{i=1}^{\infty} |\xi_i|^p)^{1/p} = \|\tilde{x}(\lambda)\|,\ \forall \lambda \in A$, is a soft norm on the soft vector space $\check{X}$. Obviously this soft norm satisfy (N5). Let us define $d : SE(\check{X}) \times SE(\check{X}) \to \mathbb{R}(A)$ by $d(\tilde{x}, \tilde{y}) = \|\tilde{x} - \tilde{y}\|$, $\forall \tilde{x}, \tilde{y} \tilde{\in} \check{X}$. Then it is simple to verify that $(\check{X}, d)$ is a soft metric space satisfying (M5).

We now show that the soft norm $\|.\|$ of $\check{X}$ cannot be obtained from a soft inner product. We verify this by showing that such a norm does not satisfy parallelogram law. Let us consider soft elements $\tilde{x}_1, \tilde{y}_1$ of $\check{X}$ such that for each $\lambda \in A$, $\tilde{x}_1(\lambda) = \{1, 1, 0, 0, \ldots.\} \in l_p$, $\tilde{y}_1(\lambda) = \{1, -1, 0, 0, \ldots.\} \in l_p$. Then $\|\tilde{x}_1\| = \|\tilde{y}_1\| = \overline{2}^{1/p}$ and $\|\tilde{x}_1 + \tilde{y}_1\| = \|\tilde{x}_1 - \tilde{y}_1\| = \overline{2}$. We see that if $p \neq 2$, then the parallelogram law does not hold. Hence for $p \neq 2$, $\check{X}$ is not a soft inner product space.

4. Soft Hilbert Space and its properties

Definition 4.1. A soft inner product space which satisfies (I5), is said to be complete if it is complete with respect to the soft metric defined by soft inner product. A complete soft inner product space is said to be a soft Hilbert space.

Example 4.2. The soft inner product space $(\check{X}, \langle . \rangle, A)$ defined as in Example 3.2, with the parameter set A being a finite set, is a soft Hilbert space. Obviously $(\check{X}, \langle . \rangle, A)$ satisfies (I5). We recall that l_2 is a complete metric space with respect to the metric $d(x, y) = (\sum_{i=1}^{\infty} |\xi_i - \eta_i|^2)^{\frac{1}{2}}$ for $x = \{\xi_i\},\ y = \{\eta_i\}$ of l_2. Let $\tilde{x}, \tilde{y}$ be any two soft elements of $\check{X}$, then $\tilde{x}(\lambda) = \{\xi_i^\lambda\}, \tilde{y}(\lambda) = \{\eta_i^\lambda\}$ are elements of l_2. It can be easily verified that the mapping $d : SE(\check{X}) \times SE(\check{X}) \to \mathbb{R}(A)$ defined by $d(\tilde{x}, \tilde{y})(\lambda) = (\sum_{i=1}^{\infty} |\xi_i^\lambda - \eta_i^\lambda|^2)^{\frac{1}{2}} = d(\tilde{x}(\lambda), \tilde{y}(\lambda)),\ \forall \lambda \in A$, is a soft metric on the absolute soft set $\check{X}$.

Let $\{\tilde{x}_n\}$ be a Cauchy sequence of soft elements in $(\check{X}, d)$. Then corresponding to every $\tilde{\varepsilon} \tilde{>} \overline{0}$, $\exists\ m \in N$ such that $d(\tilde{x}_i, \tilde{x}_j) \tilde{\leq} \tilde{\varepsilon}$, $\forall\ i, j \geq m$; i.e., $d(\tilde{x}_i, \tilde{x}_j)(\lambda) = d(\tilde{x}_i(\lambda), \tilde{x}_j(\lambda)) \leq \tilde{\varepsilon}(\lambda)$, $\forall \lambda \in A$ and $\forall\ i, j \geq m$. Thus $\{\tilde{x}_n(\lambda)\}$ is a Cauchy sequence of elements in X. Since X is complete, $\{\tilde{x}_n(\lambda)\}$ is convergent for each $\lambda \in A$. Let $\tilde{\varepsilon} \tilde{>} \overline{0}$ be arbitrary. Then corresponding to every $\lambda \in A$ there are elements $x_\lambda \in X$ and a positive integer N_λ such that $d(\tilde{x}_n(\lambda), x_\lambda) < \tilde{\varepsilon}(\lambda)$ for $n \geq N_\lambda$. Let us consider the soft element $\tilde{x}$ of $\check{X}$ such that $\tilde{x}(\lambda) = x_\lambda$, $\forall \lambda \in A$ and a positive integer N such that $N = \max\{N_\lambda,\ \lambda \in A\}$. Such N exists, since A is finite. Then $(\tilde{x}_n, \tilde{x}) \tilde{<} \tilde{\varepsilon}$, whenever $n > N$. This proves that $\{\tilde{x}_n\}$ is convergent and hence $(\check{X}, d)$ is complete.

Now we have $\sqrt{\langle \tilde{x}-\tilde{y},\ \tilde{x}-\tilde{y}\rangle}\,(\lambda) = \left(\sum_{i=1}^{\infty}(\xi_i^{\lambda}-\eta_i^{\lambda})(\overline{\xi_i^{\lambda}-\eta_i^{\lambda}})\right)^{1/2}$
$= \left(\sum_{i=1}^{\infty}\left|\xi_i^{\lambda}-\eta_i^{\lambda}\right|^2\right)^{\frac{1}{2}} = d\,(\tilde{x},\tilde{y})\,(\lambda)\,,\ \forall\lambda\in A.$
Therefore, $d\,(\tilde{x},\tilde{y}) = \sqrt{\langle \tilde{x}-\tilde{y},\ \tilde{x}-\tilde{y}\rangle}$, $\forall \tilde{x},\tilde{y}\tilde{\in}\check{X}$. Thus the soft metric d is induced by soft inner product $\langle . \rangle$ and the soft metric space $\left(\check{X},d\right)$ is complete. Hence $(\check{X},\langle . \rangle\,,A)$ is a soft Hilbert space.

Theorem 4.3. *A soft Banach space which satisfies (N5) is a soft Hilbert space if and only if the parallelogram law holds.*

Proof. For simplicity, we consider the underlying soft linear space of a real vector space. We have already noted that in every soft inner product space which satisfies (I5), parallelogram law holds and also every soft inner product space which satisfies (I5), is also a soft normed linear space which satisfies (N5) and in both spaces completeness is based on the completeness of the soft metric obtain from them. Thus, it is clear that a soft Hilbert space is a soft Banach space which satisfies (N5), where the parallelogram law holds.
We now suppose that $\check{X}$ be a soft Banach space which satisfies (N5), where the parallelogram law holds. We introduce a soft inner product on $\check{X}$ by $\langle \tilde{x},\ \tilde{y}\rangle = \frac{1}{4}\left\{\|\tilde{x}+\tilde{y}\|^2 - \|\tilde{x}-\tilde{y}\|^2\right\}$.
Clearly, $\langle \tilde{x},\ \tilde{x}\rangle\,\tilde{\geq}\overline{0}$, and $\langle \tilde{x},\ \tilde{x}\rangle = \overline{0}$ if and only if $\tilde{x}{=}\Theta$; Also, $\langle \tilde{x},\ \tilde{x}\rangle = \|\tilde{x}\|^2$ and $\langle \tilde{x},\ \tilde{y}\rangle = \langle \tilde{y},\tilde{x}\ \rangle$; therefore we should verify the other two axioms
(I3). $\langle \tilde{\alpha}.\tilde{x},\tilde{y}\rangle = \tilde{\alpha}.\langle \tilde{x},\tilde{y}\rangle$ for all $\tilde{x},\tilde{y}\tilde{\in}\ \check{X}$
and for every soft scalar $\tilde{\alpha}$;
(I4). For all $\tilde{x},\tilde{y}\tilde{\in}\ \check{X}$, $\langle \tilde{x}+\tilde{y},\tilde{z}\rangle = \langle \tilde{x},\tilde{z}\rangle + \langle \tilde{y},\tilde{z}\rangle$.
By parallelogram law we obtain,
$\|\tilde{u}+\tilde{v}+\tilde{w}\|^2 + \|\tilde{u}+\tilde{v}-\tilde{w}\|^2 = 2\|\tilde{u}+\tilde{v}\|^2 + 2\|\tilde{w}\|^2$ and
$\|\tilde{u}-\tilde{v}+\tilde{w}\|^2 + \|\tilde{u}-\tilde{v}-\tilde{w}\|^2 = 2\|\tilde{u}-\tilde{v}\|^2 + 2\|\tilde{w}\|^2$. On subtraction, we obtain

$$\|\tilde{u}+\tilde{v}+\tilde{w}\|^2+\|\tilde{u}+\tilde{v}-\tilde{w}\|^2-\|\tilde{u}-\tilde{v}+\tilde{w}\|^2-\|\tilde{u}-\tilde{v}-\tilde{w}\|^2 = 2\|\tilde{u}+\tilde{v}\|^2-2\|\tilde{u}-\tilde{v}\|^2$$

By the definition of soft inner product we have

$$\langle \tilde{u}+\tilde{v},\tilde{w}\rangle + \langle \tilde{u}-\tilde{v},\tilde{w}\rangle = 2\,\langle \tilde{u},\tilde{v}\rangle\,.$$

If we set $\tilde{x} = \tilde{u}+\tilde{v}, \tilde{y} = \tilde{u}-\tilde{v}$ and $\tilde{z} = \tilde{w}$, then we obtain

$$\langle \tilde{x},\tilde{z}\rangle + \langle \tilde{y},\tilde{z}\rangle = \langle \tilde{x}+\tilde{y},\tilde{z}\rangle\,.$$

The relation $\langle \tilde{\alpha}.\tilde{x},\tilde{y}\rangle = \tilde{\alpha}.\langle \tilde{x},\tilde{y}\rangle$ for all $\tilde{x},\tilde{y}\tilde{\in}\ \check{X}$ and for every soft scalar $\tilde{\alpha}$; may be proved similarly. This proves the theorem.
In the complex case, we take the inner product of $\tilde{x},\tilde{y}$ as

$$\langle \tilde{x},\ \tilde{y}\rangle = \frac{1}{4}\{\|\tilde{x}+\tilde{y}\|^2 - \|\tilde{x}-\tilde{y}\|^2 + \bar{i}\left\|\tilde{x}+\bar{i}\tilde{y}\right\|^2 - \bar{i}\left\|\tilde{x}-\bar{i}\tilde{y}\right\|^2\}.$$

□

5. Orthogonality in soft Hilbert spaces

In the rest of this paper the symbol $\tilde{H}$ stands for a soft Hilbert space.

Definition 5.1. Let $\tilde{L}$ be a non-null soft subset of $\tilde{H}$ such that $\tilde{L}(\lambda) \neq \varnothing, \forall\lambda \in A$. Two soft vectors $\tilde{x},\ \tilde{y}$ of $\tilde{H}$ are said to be *orthogonal* if $\langle \tilde{x},\ \tilde{y}\rangle = \overline{0}$. In symbol, we write $\tilde{x}\perp\tilde{y}$. If $\tilde{x}$ is orthogonal to every soft vectors of $\tilde{L}$ then we say that $\tilde{x}$ is orthogonal to $\tilde{L}$ and we write $\tilde{x}\perp\tilde{L}$.

Example 5.2. Consider the soft Hilbert space $(\check{X}, \langle . \rangle, A)$ as in Example 4.2. Let us consider soft elements $\tilde{x}_1,\ \tilde{x}_2, \dots \tilde{x}_n, \dots\ \tilde{\in}\ \check{X}$ such that for each $\lambda \in A$, $\tilde{x}_1(\lambda) = \{2, 0, 0, \dots\},\ \tilde{x}_2(\lambda) = \{0, 2, 0, \dots\},\ \dots, \tilde{x}_n(\lambda) = \{0, 0, 0, \dots, 2, \dots\}, \dots$
Then we have $\langle \tilde{x}_i,\ \tilde{x}_j\rangle = \overline{0}$ for $i \neq j$.
So, $\tilde{x}_i$ is an orthogonal to $\tilde{x}_j$ for $i \neq j$.

Theorem 5.3. *(Pythagorean theorem). If $\tilde{x},\ \tilde{y}\ \tilde{\in}\ \tilde{H}$ and $\tilde{x}\perp\tilde{y}$, then $\|\tilde{x}+\tilde{y}\|^2 = \|\tilde{x}\|^2 + \|\tilde{y}\|^2$ and $\|\tilde{x}-\tilde{y}\|^2 = \|\tilde{x}\|^2 + \|\tilde{y}\|^2$.*

Proof. We have $\|\tilde{x}+\tilde{y}\|^2 = \langle \tilde{x}+\ \tilde{y}, \tilde{x}+\ \tilde{y}\rangle = \langle \tilde{x}, \tilde{x}\rangle + \langle \tilde{x},\ \tilde{y}\rangle + \langle\ \tilde{y}, \tilde{x}\rangle + \langle \tilde{y},\ \tilde{y}\rangle = \langle \tilde{x}, \tilde{x}\rangle + \langle \tilde{y},\ \tilde{y}\rangle = \|\tilde{x}\|^2 + \|\tilde{y}\|^2$ and $\|\tilde{x}-\tilde{y}\|^2 = \langle \tilde{x}-\ \tilde{y}, \tilde{x}-\ \tilde{y}\rangle = \langle \tilde{x}, \tilde{x}\rangle - \langle \tilde{x},\ \tilde{y}\rangle - \langle\ \tilde{y}, \tilde{x}\rangle + \langle \tilde{y},\ \tilde{y}\rangle = \langle \tilde{x}, \tilde{x}\rangle + \langle \tilde{y},\ \tilde{y}\rangle = \|\tilde{x}\|^2 + \|\tilde{y}\|^2$. □

Definition 5.4. Let $\tilde{S}$ be a non-null soft subset of $\tilde{H}$ such that $\tilde{S}(\lambda) \neq \varnothing, \forall\lambda \in A$. Then the set of all soft vectors of $\tilde{H}$ orthogonal to $\tilde{S}$ is called the *orthogonal complement* of $\tilde{S}$ and is denoted by $\tilde{S}^{\perp}$. $SS(\tilde{S}^{\perp})$ will be called the *soft orthogonal complement* of $\tilde{S}$.

Theorem 5.5.
(i) $\{\Theta\}^{\perp} = \tilde{H}, \left\{\tilde{H}\right\}^{\perp} = \Theta$;
(ii) *If $\tilde{S}$ be a non-null soft subset of $\tilde{H}$ such that $\tilde{S}(\lambda) \neq \varnothing, \forall\lambda \in A$, then $SE\left(\tilde{S}\right) \cap \tilde{S}^{\perp} = \{\Theta\}$;*
(iii) *If $\tilde{S}_1 \tilde{\subset} \tilde{S}_2 \tilde{\subset} \tilde{H}$ then $\tilde{S}_1^{\perp} \supset \tilde{S}_2^{\perp}$ and hence $SS(\tilde{S}_1^{\perp}) \tilde{\supset} SS(\tilde{S}_2^{\perp})$.*

Proof. We prove only (iii). For $\tilde{x} \tilde{\in} \tilde{S}_2^{\perp} \implies \tilde{x}$ is orthogonal to every soft vectors of $\tilde{S}_2 \implies \tilde{x}$ is orthogonal to every soft vectors of $\tilde{S}_1$(since $\tilde{S}_1 \tilde{\subset} \tilde{S}_2 \tilde{\subset} \tilde{H}$) $\implies \tilde{x} \tilde{\in} \tilde{S}_1^{\perp}$. Hence $\tilde{S}_1^{\perp} \supset \tilde{S}_2^{\perp}$and consequently $SS(\tilde{S}_1^{\perp}) \tilde{\supset} SS(\tilde{S}_2^{\perp})$. □

To prove the next proposition we first prove the following lemma.

Lemma 5.6. *Let $(\check{X}, d)$ be a soft metric space satisfying (M5). Then for every soft closed set $(Y, A) \in \mathcal{S}(\check{X})$, the sets $Y(\lambda)$ is closed in (X, d_λ) for each $\lambda \in A$.*

Proof. Let $(Y, A) \in \mathcal{S}(\check{X})$ be soft closed in $(\check{X}, d)$ satisfying (M5).Then $(Y, A)^c$ is a member of $\mathcal{S}(\check{X})$ and is soft open in $(\check{X}, d)$. Now by Proposition 2.28, it follows that $(Y, A)^c(\lambda) = Y^c(\lambda)$ is open in (X, d_λ) for each $\lambda \in A$. Then
$[Y^c(\lambda)]^{\mathbf{c}} = Y(\lambda)$ is closed in (X, d_λ) for each $\lambda \in A$. □

Proposition 5.7. *Let $\tilde{C}$ be a soft closed subset of a soft inner product space $(\check{X}, \langle . \rangle, A)$ satisfying (I5). Then $\tilde{C}(\lambda)$ is a closed subset of the inner product space $(X, \langle . \rangle_\lambda)$, $\forall\lambda \in A$.*

Proof. From the Remark 3.11, we know that Every soft inner product space satisfying (I5) is also a soft metric space satisfying (M5). Also using the above lemma it follows that for a soft closed set $\tilde{C}$ of a soft inner product space $(\check{X}, \langle . \rangle)$ satisfying (I5), $\tilde{C}(\lambda)$ is a closed set for each $\lambda \in A$. □

Theorem 5.8. *Let $\tilde{C}$ be a soft closed convex soft subset of a soft Hilbert space $\tilde{H}$ and having a finite set of parameters A. Then $\tilde{C}$ contains a unique soft vector of smallest soft norm.*

Proof. Since $\tilde{H}$ is a soft Hilbert space, it satisfies (I5) and hence $\tilde{C}(\lambda)$ is a closed convex subset of the Hilbert space $\tilde{H}(\lambda)$, $\forall\lambda \in A$. Let us consider the soft real number $\tilde{d}$ defined by $\tilde{d}(\lambda) = \inf\{\|\tilde{x}\|(\lambda)(= \|\tilde{x}(\lambda)\|_\lambda), \tilde{x}\tilde{\in}\tilde{C}\}$, $\forall\lambda \in A$. Then for each $\lambda \in A$, there exists a sequence $\{\tilde{x}_{n,\lambda}\}$ of soft vectors of $\tilde{C}$ such that $\|\tilde{x}_{n,\lambda}\|(\lambda) = \|\tilde{x}_{n,\lambda}(\lambda)\|_\lambda \to \tilde{d}(\lambda)$. Let us consider the sequence $\{\tilde{y}_n\}$ of soft vectors of $\tilde{C}$ such that $\tilde{y}_n(\mu) = \tilde{x}_{n,\mu}(\mu)$ $\forall\mu \in A$. Let $\tilde{\varepsilon}\tilde{>}\bar{0}$ be chosen arbitrarily. Then $\tilde{\varepsilon}(\lambda) > 0$, $\forall\lambda \in A$. Since $\|\tilde{x}_{n,\lambda}(\lambda)\|_\lambda \to \tilde{d}(\lambda)$, $\exists$ a natural number $N_\lambda = N_\lambda(\tilde{\varepsilon}(\lambda))$, such that $0 \le |\|\tilde{x}_{n,\lambda}(\lambda)\|_\lambda - \tilde{d}(\lambda)| < \tilde{\varepsilon}(\lambda)$, whenever $n > N_\lambda$ i.e., $0 \le |\|\tilde{y}_n\|(\lambda) - \tilde{d}(\lambda)| < \tilde{\varepsilon}(\lambda)$ i.e., $|\|\tilde{y}_n\| - \tilde{d}|(\lambda) < \tilde{\varepsilon}(\lambda)$, whenever $n > N_\lambda$. Let N be a positive integer such that $N > N_\lambda$, $\forall\lambda \in A$, (such an N exists because A is a finite set) then $|\|\tilde{y}_n\| - \tilde{d}|(\lambda) < \tilde{\varepsilon}(\lambda)$, whenever $n > N$, for all $\lambda \in A$ i.e., $|\|\tilde{y}_n\| - \tilde{d}|\tilde{<}\tilde{\varepsilon}$ whenever $n > N$. Thus $\|\tilde{y}_n\| \to \tilde{d}$ as $n \to \infty$. Now $\tilde{y}_n$, $\tilde{y}_m\tilde{\in}\tilde{C}$ and since $\tilde{C}$ is convex, $(\tilde{y}_n + \tilde{y}_m)/2\tilde{\in}\tilde{C}$ and therefore $\|(\tilde{y}_n + \tilde{y}_m)/2\| \tilde{\geq}\tilde{d}$ i.e.,

$$\|\tilde{y}_n + \tilde{y}_m\| \tilde{\geq} 2\tilde{d} \tag{5.1}$$

Using parallelogram law, we get

$$\|\tilde{y}_n - \tilde{y}_m\|^2 = 2\|\tilde{y}_n\|^2 + 2\|\tilde{y}_m\|^2 - \|\tilde{y}_n + \tilde{y}_m\|^2 \tilde{\leq} 2\|\tilde{y}_n\|^2 + 2\|\tilde{y}_m\|^2 - 4\tilde{d}^2 \tag{5.2}$$

from (5.1).
Since $\|\tilde{y}_n\| \to \tilde{d}$ as $n \to \infty$, $\|\tilde{y}_m\| \to \tilde{d}$ as $m \to \infty$, the right hand side of (5.2) tends to $2\tilde{d}^2 + 2\tilde{d}^2 - 4\tilde{d}^2 = \bar{0}$ as $m, n \to \infty$. So $\{\tilde{y}_n\}$ is a Cauchy sequence. Since $\tilde{H}$ is complete, $\exists\ \tilde{y}\tilde{\in}\tilde{H}$ such that $\tilde{y}_n \to \tilde{y}$. Then $\tilde{y}_n(\lambda) \to \tilde{y}(\lambda)$, $\forall\lambda \in A$. Since $\tilde{C}(\lambda)$ is a closed, $\tilde{y}(\lambda) \in \tilde{C}(\lambda)$, $\forall\lambda \in A$. i.e., $\tilde{y}\tilde{\in}\tilde{C}$. So, $\|\tilde{y}\| = \lim_{n\to\infty}\|\tilde{y}_n\| = \tilde{d}$.
$\therefore$ $\tilde{y}$ is a soft vector of $\tilde{C}$ with the smallest norm.
Let $\tilde{y}'$ be a soft vector of $\tilde{C}$ such that $\|\tilde{y}'\| = \tilde{d}$. Then $\tilde{C}$ being convex, we have, $(\tilde{y} + \tilde{y}')/2\tilde{\in}\tilde{C}$ and also,

$$\|(\tilde{y} + \tilde{y}')/2\| \tilde{\geq} \tilde{d} \tag{5.3}$$

Applying the parallelogram law, we obtain $\|(\tilde{y} + \tilde{y}')/2\|^2 = \|\tilde{y}\|^2/2 + \|\tilde{y}'\|^2/2 - \|(\tilde{y} - \tilde{y}')/2\|^2 \tilde{\ngeq} \|\tilde{y}\|^2/2 + \|\tilde{y}'\|^2/2 = \tilde{d}^2$ (if $\tilde{y} \neq \tilde{y}'$)
Giving that, $\|(\tilde{y} + \tilde{y}')/2\| \tilde{\ngeq} \tilde{d}$. This contradicts (5.3) and so $\tilde{y} = \tilde{y}'$. This proves the theorem. □

6. Orthonormality in soft Hilbert spaces

Definition 6.1. Let $\tilde{H}$ be a Hilbert space. Then a collection $\mathfrak{B}$ of soft vectors of $\tilde{H}$ is said to be *orthonormal* if for all $\tilde{x}$, $\tilde{y} \in \mathfrak{B}$

$$\langle \tilde{x}, \tilde{y}\rangle = \begin{cases} \bar{0} & \text{if } \tilde{x} \neq \tilde{y} \\ \bar{1} & \text{if } \tilde{x} = \tilde{y}, \end{cases}$$

If the soft set $\mathfrak{B}$ contains only a countable number of soft vectors then we can arrange it in a sequence of soft vectors and call it an *orthonormal sequence*.

Example 6.2. Consider the soft Hilbert space $(\check{X}, \langle . \rangle, A)$ as in Example 4.2. Let us consider soft elements $\tilde{x}_1,\ \tilde{x}_2, \ldots \tilde{x}_n, \ldots\ \tilde{\in}\ \check{X}$ such that for each $\lambda \in A,\ \tilde{x}_1(\lambda) = \{1, 0, 0, \ldots\},\ \tilde{x}_2(\lambda) = \{0, 1, 0, \ldots\},\ \ldots, \tilde{x}_n(\lambda) = \{0, 0, 0, \ldots, 1, \ldots\}, \ldots$
Then we have $\langle \tilde{x}_i,\ \tilde{x}_i \rangle = \overline{1}$ and $\langle \tilde{x}_i,\ \tilde{x}_j \rangle = \overline{0}$ for $i \neq j$.
So, $\{\tilde{x}_i\}$ is an orthogonal sequence in $(\check{X}, \langle . \rangle, A)$.

Theorem 6.3. *An orthonormal set of soft vectors is linearly independent.*

Proof. Let $\mathfrak{B}$ be an orthonormal set and $\mathfrak{B} = \{\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\}$. Suppose that $\tilde{c}_1.\ \widetilde{\alpha}_1 + \tilde{c}_2.\ \widetilde{\alpha}_2 + .. + \tilde{c}_n.\ \widetilde{\alpha}_n = \Theta$, where $\tilde{c}_1,\ \tilde{c}_2,\ \ldots, \tilde{c}_n$ are soft scalars. Then if $1 \leq j \leq n$

$$\overline{0} = \langle \Theta,\ \widetilde{\alpha}_j \rangle = \left\langle \sum_{k=1}^{n} \tilde{c}_k.\ \widetilde{\alpha}_k,\ \widetilde{\alpha}_j \right\rangle = \sum_{k=1}^{n} \tilde{c}_k \langle \widetilde{\alpha}_k,\ \widetilde{\alpha}_j \rangle = \tilde{c}_j \langle \widetilde{\alpha}_j,\ \widetilde{\alpha}_j \rangle = \tilde{c}_j.$$

So $\tilde{c}_j = \overline{0}$ for each $j = 1, 2, .., n$. This proves that any orthonormal set consists of a finite number of soft vectors is linearly independent. If $\mathfrak{B}$ consists of an infinite number of soft vectors then from definition of linearly independence of soft vectors it follows that $\mathfrak{B}$ is linearly independent. □

Theorem 6.4. *(Bessel's inequality). Let* $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n, \ldots.$ *be an orthonormal sequence in a soft Hilbert space* $\tilde{H}$*. Then for every* $\tilde{x}\ \tilde{\in}\ \tilde{H}$, $\sum_{i=1}^{\infty} |\langle \tilde{x},\ \widetilde{\alpha}_i \rangle|^2 \tilde{\leq} \|\tilde{x}\|^2$.

Proof. Let n be any positive integer and let $\widetilde{\xi}_i = \langle \tilde{x},\ \widetilde{\alpha}_i \rangle$. Then
$\overline{0} \tilde{\leq} \left\langle \tilde{x} - \sum_{i=1}^{n} \widetilde{\xi}_i \widetilde{\alpha}_i,\ \tilde{x} - \sum_{j=1}^{n} \widetilde{\xi}_j \widetilde{\alpha}_j \right\rangle$
$= \langle \tilde{x},\ \tilde{x} \rangle - \left\langle \tilde{x},\ \sum_{j=1}^{n} \widetilde{\xi}_j \widetilde{\alpha}_j \right\rangle - \left\langle \sum_{i=1}^{n} \widetilde{\xi}_i \widetilde{\alpha}_i,\ \tilde{x} \right\rangle + \left\langle \sum_{i=1}^{n} \widetilde{\xi}_i \widetilde{\alpha}_i,\ \sum_{j=1}^{n} \widetilde{\xi}_j \widetilde{\alpha}_j \right\rangle$
$= \langle \tilde{x},\ \tilde{x} \rangle - \sum_{j=1}^{n} \overline{\widetilde{\xi}_j} \langle \tilde{x}, \widetilde{\alpha}_j \rangle - \sum_{i=1}^{n} \widetilde{\xi}_i \langle \widetilde{\alpha}_i,\ \tilde{x} \rangle + \sum_{i=1}^{n} \widetilde{\xi}_i \left\langle \widetilde{\alpha}_i,\ \sum_{j=1}^{n} \widetilde{\xi}_j \widetilde{\alpha}_j \right\rangle$
$= \langle \tilde{x},\ \tilde{x} \rangle - \sum_{j=1}^{n} \overline{\widetilde{\xi}_j} \widetilde{\xi}_j - \sum_{i=1}^{n} \widetilde{\xi}_i \overline{\widetilde{\xi}_i} + \sum_{i=1}^{n} \sum_{j=1}^{n} \widetilde{\xi}_i \overline{\widetilde{\xi}_j} \langle \widetilde{\alpha}_i,\ \widetilde{\alpha}_j \rangle$
$= \|\tilde{x}\|^2 - \sum_{j=1}^{n} \left|\widetilde{\xi}_j\right|^2 - \sum_{i=1}^{n} \left|\widetilde{\xi}_i\right|^2 + \sum_{i=1}^{n} \left|\widetilde{\xi}_i\right|^2 = \|\tilde{x}\|^2 - \sum_{j=1}^{n} \left|\widetilde{\xi}_j\right|^2$ So,
$\sum_{j=1}^{n} \left|\widetilde{\xi}_j\right|^2 \tilde{\leq} \|\tilde{x}\|^2$ i.e., $\sum_{i=1}^{n} |\langle \tilde{x},\ \widetilde{\alpha}_i \rangle|^2 \tilde{\leq} \|\tilde{x}\|^2$. Since this is true for any positive integer n, the infinite series
$\sum_{i=1}^{\infty} |\langle \tilde{x},\ \widetilde{\alpha}_i \rangle|^2$ is convergent and $\sum_{i=1}^{\infty} |\langle \tilde{x},\ \widetilde{\alpha}_i \rangle|^2 \tilde{\leq} \|\tilde{x}\|^2$. □

Lemma 6.5. *If* $\{\tilde{x}_1, \tilde{x}_2, \ldots.., \tilde{x}_n\}$ *be an orthogonal set of soft vectors in* $\tilde{H}$*, i.e.,* $\tilde{x}_i \perp \tilde{x}_j, i \neq j,\ i, j = 1, 2, .., n,$ *then*

$$\|\tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_n\|^2 = \|\tilde{x}_1\|^2 + \|\tilde{x}_2\|^2 + \cdots + \|\tilde{x}_n\|^2.$$

Proof. We have $\langle \tilde{x}_i, \tilde{x}_j \rangle = \overline{0}$ for $i \neq j$ and so $\|\sum_{i=1}^{n} \tilde{x}_i\|^2 = \left\langle \sum_{i=1}^{n} \tilde{x}_i, \sum_{j=1}^{n} \tilde{x}_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \tilde{x}_i, \tilde{x}_j \rangle = \sum_{i=1}^{n} \|\tilde{x}_i\|^2$. □

Definition 6.6. Let $\{\tilde{x}_n\}$ be a sequence of soft vectors in a soft normed linear space $(\check{X}, \|.\|, A)$. If $\tilde{s}_n = \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_n$ and if there exists $\tilde{s} \tilde{\in} \check{X}$ such that $\|\tilde{s}_n - \tilde{s}\| \to \overline{0}$ as $n \to \infty$, then the infinite series $\tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_n + \cdots = \sum_{k=1}^{\infty} \tilde{x}_k$ is said to converge to $\tilde{s}$ in the soft norm of $\check{X}$ and we write $\tilde{s} = \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_n + \cdots = \sum_{k=1}^{\infty} \tilde{x}_k$.

Theorem 6.7. *If $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n, \ldots$ be an orthonormal sequence in a soft Hilbert space $\tilde{H}$ having a finite set of parameters. Then the infinite series*

$$\sum_{k=1}^{\infty} \tilde{c}_k \widetilde{\alpha}_k \text{ ———-(*)}$$

where $\tilde{c}_1,\ \tilde{c}_2,\ \ldots, \tilde{c}_n \ldots$ are soft scalars, is convergent if and only if the series $\sum_{k=1}^{\infty} |\tilde{c}_k|^2$ is convergent.

Proof. Let $\tilde{s}_n = \tilde{c}_1\widetilde{\alpha}_1 + \ \tilde{c}_2\widetilde{\alpha}_2 + \ \cdots + \tilde{c}_n\widetilde{\alpha}_n$ and $\widetilde{\sigma}_n = \sum_{i=1}^{n} |\tilde{c}_i|^2$. Now, $\{\widetilde{\alpha}_n\}$ is orthonormal, so using the technique of Lemma 6.5, we obtain for $n > m$

$$\|\tilde{s}_n - \tilde{s}_m\|^2 = \|\tilde{c}_{m+1}\widetilde{\alpha}_{m+1} + \ \tilde{c}_{m+2}\widetilde{\alpha}_{m+2} + \ \cdots + \tilde{c}_n\widetilde{\alpha}_n\|^2 = \sum_{i=m+1}^{n} |\tilde{c}_i|^2 = \widetilde{\sigma}_n - \widetilde{\sigma}_m.$$

So, $\{\tilde{s}_n\}$ is a Cauchy sequence in $\tilde{H}$ if and only if $\{\widetilde{\sigma}_n\}$ is a Cauchy sequence of soft real numbers. Since the set of all soft real number with a finite set of parameters is Complete and $\tilde{H}$ is also complete, it follows that $\{\tilde{s}_n\}$ is convergent if and only if $\{\widetilde{\sigma}_n\}$ is convergent. □

Theorem 6.8. *If the series (*) converges to $\tilde{x}$ then $\tilde{c}_k = \langle \tilde{x},\ \widetilde{\alpha}_k \rangle,\ k = 1, 2, .., n.$*

Proof. For any positive integer n, let $\tilde{s}_n = \tilde{c}_1\widetilde{\alpha}_1 + \ \tilde{c}_2\widetilde{\alpha}_2 + \ \cdots + \tilde{c}_n\widetilde{\alpha}_n$. We take the soft inner product of $\tilde{s}_n$ and $\widetilde{\alpha}_j$. Then using the orthonormality of $\{\widetilde{\alpha}_j\}$, we get $\langle \tilde{s}_n,\ \widetilde{\alpha}_j\rangle = \tilde{c}_j$ for $j = 1, 2, ., k$ and $k \leq n$. Since $\tilde{s}_n \to \tilde{x}$ as $n \to \infty$ and using the properties of soft inner product, we have $\tilde{c}_j = \langle \tilde{s}_n,\ \widetilde{\alpha}_j\rangle \to \langle \tilde{x},\ \widetilde{\alpha}_j\rangle$.
So, $\widetilde{\alpha}_j = \langle \tilde{x},\ \widetilde{\alpha}_j\rangle$, for $j = 1, 2, ., k$. Since k may be taken as large as we please because $n \to \infty$, we have $\widetilde{\alpha}_j = \langle \tilde{x},\ \widetilde{\alpha}_j\rangle$, for $j = 1, 2, \ldots$. □

Theorem 6.9. *If the series (*) converges to $\tilde{x}$ then $\|\tilde{x}\|^2 = \sum_{k=1}^{\infty} |\tilde{c}_k|^2$.*

Proof. Let $\tilde{s}_n = \tilde{c}_1\widetilde{\alpha}_1 + \tilde{c}_2\widetilde{\alpha}_2 + \cdots + \tilde{c}_n\widetilde{\alpha}_n$. Then, by assumption, $\tilde{s}_n \to \tilde{x}$ as $n \to \infty$. So, $\langle \tilde{s}_n,\ \tilde{s}_n\rangle \to \langle \tilde{x},\ \tilde{x}\rangle = \|\tilde{x}\|^2$. But

$$\begin{aligned}
\langle \tilde{s}_n,\ \tilde{s}_n\rangle &= \left\langle \sum_{i=1}^{n} \tilde{c}_i\widetilde{\alpha}_i,\ \sum_{j=1}^{n} \tilde{c}_j\widetilde{\alpha}_j \right\rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_i\overline{\tilde{c}_j}\,\langle \widetilde{\alpha}_i,\ \widetilde{\alpha}_j\rangle \\
&= \sum_{i=1}^{n} \tilde{c}_i\overline{\tilde{c}_j} \\
&= \sum_{i=1}^{n} |\tilde{c}_i|^2.
\end{aligned}$$

Therefore we have $\|\tilde{x}\|^2 = \sum_{k=1}^{\infty} |\tilde{c}_k|^2$. □

Theorem 6.10. *If $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n, \ldots$ be an orthonormal sequence in a soft Hilbert space $\tilde{H}$ having a finite set of parameters.*
Then for any $\tilde{x} \mathrel{\tilde{\in}} \tilde{H}$, $(\tilde{x} - \sum_{i=1}^{\infty} \langle \tilde{x},\ \widetilde{\alpha}_i\rangle\, \widetilde{\alpha}_i) \perp \widetilde{\alpha}_j$ for each j.

Proof. By Theorem 2.17 and Theorem 6.7, the series $\sum_{i=1}^{\infty}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i$ is convergent. Let $\tilde{s}=\sum_{i=1}^{\infty}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i$ and $\tilde{s}_n=\sum_{i=1}^{n}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i$.
Then if $1\leq j\leq n$ $\langle \tilde{x},\ \widetilde{\alpha}_j\rangle-\langle \tilde{s}_n,\ \widetilde{\alpha}_j\rangle=\langle \tilde{x},\ \widetilde{\alpha}_j\rangle-\langle\sum_{i=1}^{n}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i,\ \widetilde{\alpha}_j\rangle=\langle \tilde{x},\ \widetilde{\alpha}_j\rangle-\langle \tilde{x},\ \widetilde{\alpha}_j\rangle=\overline{0}$, giving that $\langle \tilde{x},\ \widetilde{\alpha}_j\rangle=\langle \tilde{s}_n,\ \widetilde{\alpha}_j\rangle$. Now we have

$$\begin{aligned}\left\langle \tilde{x}-\sum_{i=1}^{\infty}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i,\ \widetilde{\alpha}_j\right\rangle &= \langle \tilde{x}-\tilde{s},\ \widetilde{\alpha}_j\rangle=\langle \tilde{x},\ \widetilde{\alpha}_j\rangle-\langle \tilde{s},\ \widetilde{\alpha}_j\rangle\\ &=\langle \tilde{x},\ \widetilde{\alpha}_j\rangle-\langle lim\tilde{s}_n,\ \widetilde{\alpha}_j\rangle=\langle \tilde{x},\ \widetilde{\alpha}_j\rangle-lim\,\langle \tilde{s}_n,\ \widetilde{\alpha}_j\rangle\\ &=\langle \tilde{x},\ \widetilde{\alpha}_j\rangle-\langle \tilde{x},\ \widetilde{\alpha}_j\rangle=\overline{0}.\end{aligned}$$

Thus the result follows. □

Definition 6.11. Let $\mathfrak{B}$ be a non-null collection of orthonormal soft elements of $\tilde{H}$. Then $\mathfrak{B}$ is said to be *complete orthonormal* if there exists no orthonormal set $\mathfrak{D}$ such that $\mathfrak{D}$ is a proper subset of $\mathfrak{B}$.
If the set $\mathfrak{B}$ contains only a countable number of soft elements then we call it an *complete orthonormal sequence.*
Let S be the collection of all soft vectors $\tilde{x}$ of $\tilde{H}$ such that $\tilde{x}(\lambda)\neq\theta,\ \forall\lambda\in A$, together with the null soft vector Θ.
Then for a non-null soft vector $\tilde{y}\in S$ implies $\tilde{y}(\lambda)\neq\theta,\ \forall\lambda\in A$.

Theorem 6.12. *Let $\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n,\ldots.$ be an orthonormal sequence in a soft Hilbert space $\tilde{H}$ having a finite set of parameters. Then the following conditions are equivalent:*

(i) *$\{\widetilde{\alpha}_n\}$ is complete;*
(ii) *$\forall\tilde{x}\in S,\ \tilde{x}\perp\widetilde{\alpha}_i$ for $i=1,2,..$ implies $\tilde{x}=\Theta$;*
(iii) *For all $\tilde{x}\,\tilde{\in}\,\tilde{H}$, with $(\tilde{x}-\sum_{i=1}^{\infty}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i)\in S$, $\tilde{x}=\sum_{i=1}^{\infty}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i$;*
(iv) *For all $\tilde{x}\,\tilde{\in}\,\tilde{H}$, with $(\tilde{x}-\sum_{i=1}^{\infty}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i)\in S$, $\|\tilde{x}\|^2=\sum_{i=1}^{\infty}|\langle \tilde{x},\ \widetilde{\alpha}_i\rangle|^2$.*

Proof. (i) ⇒ (ii). If (ii) is not true, then there exists a soft element $\tilde{x}\in S$ such that $\tilde{x}\neq\Theta$ and $\tilde{x}\perp\widetilde{\alpha}_i$ for $i=1,2,..$ Then $\tilde{x}(\lambda)\neq\theta,\ \forall\lambda\in A$. Since $\tilde{H}$ satisfies (I5), we must have $\|\tilde{x}(\lambda)\|_{\lambda}=\|\tilde{x}\|(\lambda)\neq 0,\ \forall\lambda\in A$, i.e., $\|\tilde{x}\|\,\tilde{>}\,\overline{0}$. Let $\widetilde{\alpha}=\tilde{x}/\|\tilde{x}\|$. Then $\|\widetilde{\alpha}\|=\overline{1}$ and $\widetilde{\alpha}\perp\widetilde{\alpha}_i$ for $=1,2,\ldots.$. So $\{\widetilde{\alpha},\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n,\ldots.\}$ is a orthonormal set such that $\{\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n,\ldots.\}\subset\{\widetilde{\alpha},\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n,\ldots.\}$ and $\{\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n,\ldots.\}\neq\{\widetilde{\alpha},\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n,\ldots.\}$. That contradicts that $\{\widetilde{\alpha}_n\}$ is complete. Hence (ii) holds.

(ii) ⇒ (iii). By Theorem 6.10, $(\tilde{x}-\sum_{i=1}^{\infty}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i)\perp\widetilde{\alpha}_j$ for each j and by the given condition $(\tilde{x}-\sum_{i=1}^{\infty}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i)\in S$, so by (ii) $\tilde{x}-\sum_{i=1}^{\infty}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i=\Theta$ i.e., $\tilde{x}=\sum_{i=1}^{\infty}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i$.

(iii) ⇒ (iv). Since $\{\widetilde{\alpha}_n\}$ is orthonormal and $(\tilde{x}-\sum_{i=1}^{\infty}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i)\in S$, we have
$\|\tilde{x}\|^2=\langle \tilde{x},\ \tilde{x}\rangle=\left\langle\sum_{i=1}^{\infty}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i,\ \sum_{j=1}^{\infty}\langle \tilde{x},\ \widetilde{\alpha}_j\rangle \widetilde{\alpha}_j\right\rangle$
$=\left\langle lim\sum_{i=1}^{n}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i,\ lim\sum_{j=1}^{n}\langle \tilde{x},\ \widetilde{\alpha}_j\rangle \widetilde{\alpha}_j\right\rangle$
$=lim\left\langle\sum_{i=1}^{n}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle \widetilde{\alpha}_i,\ \sum_{j=1}^{n}\langle \tilde{x},\ \widetilde{\alpha}_j\rangle \widetilde{\alpha}_j\right\rangle$
$=lim\sum_{i=1}^{n}\langle \tilde{x},\ \widetilde{\alpha}_i\rangle\overline{\langle \tilde{x},\ \widetilde{\alpha}_i\rangle}=lim\sum_{i=1}^{n}|\langle \tilde{x},\ \widetilde{\alpha}_i\rangle|^2=\sum_{i=1}^{\infty}|\langle \tilde{x},\ \widetilde{\alpha}_i\rangle|^2$.

(iv) ⇒ (i). If $\{\widetilde{\alpha}_n\}$ is not complete, then $\{\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n,\ldots.\}$ is a subset of an orthonormal subset say $\{\widetilde{\alpha},\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n,\ldots.\}$ such that $\{\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n,\ldots.\}\neq$

$\{\widetilde{\alpha}, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n, \ldots\}$, where $\|\widetilde{\alpha}\| = \overline{1}$. Now $\widetilde{\alpha} \perp \widetilde{\alpha}_i$ for $= 1, 2, \ldots$. So $\langle \widetilde{\alpha},\ \widetilde{\alpha}_i \rangle = \overline{0}$, i.e., $\langle \widetilde{\alpha},\ \widetilde{\alpha}_i \rangle\, \widetilde{\alpha}_i = \Theta$, for $i = 1, 2, \ldots$ .and hence $\sum_{i=1}^{\infty} \langle \widetilde{\alpha},\ \widetilde{\alpha}_i \rangle\, \widetilde{\alpha}_i = \Theta$ but since $\tilde{H}$ satisfy (I5) and $\|\widetilde{\alpha}\| = \overline{1}$ so $\widetilde{\alpha}\,(\lambda) \neq \theta$ for any $\lambda \in A$. Thus $(\widetilde{\alpha} - \sum_{i=1}^{\infty} \langle \widetilde{\alpha},\ \widetilde{\alpha}_i \rangle\, \widetilde{\alpha}_i) \in S$ and then from (iv) we get $\|\widetilde{\alpha}\|^2 = \sum_{i=1}^{\infty} |\, \langle \widetilde{\alpha},\ \widetilde{\alpha}_i \rangle\, |^2 = \overline{0} \Rightarrow \widetilde{\alpha} = \Theta$ and that contradicts $\|\widetilde{\alpha}\| = \overline{1}$. So (i) is true. □

Definition 6.13. Let $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n, \ldots$ be a complete orthonormal sequence of soft vectors in a soft Hilbert space $\tilde{H}$. Then $\langle \tilde{x},\ \widetilde{\alpha}_i \rangle$ are called the *Fourier coefficients* of $\tilde{x}$, the expansion $\tilde{x} = \sum_{i=1}^{\infty} \langle \tilde{x},\ \widetilde{\alpha}_i \rangle\, \widetilde{\alpha}_i$ is called the *Fourier expansion* of $\tilde{x}$ and $\|\tilde{x}\|^2 = \sum_{i=1}^{\infty} |\, \langle \tilde{x},\ \widetilde{\alpha}_i \rangle\, |^2$ is called *Parseval's equation*, all with respect to the sequence $\{\widetilde{\alpha}_n\}$.

7. Conclusions

In this paper we have introduced a concept of soft inner product on a soft linear space. Some basic properties of soft inner product spaces has been investigated with examples and counter examples. Soft Hilbert spaces has been defined and some basic properties of soft Hilbert spaces are also investigated. Properties of orthogonal and orthonormal sets in soft Hilbert spaces are studied. There is an ample scope for further research on soft vector spaces and soft inner product spaces.

Acknowledgements. The authors express their sincere thanks to the Referees for their valuable suggestions in rewriting the paper in the present form. The authors are also very much thankful to the Editors-in-Chief and the Managing Editors for their valuable advice.

This work is partially supported by the Department of Mathematics, Visva Bharati, under the project UGC SAP (DRS) Phase – II.

References

[1] H. Aktas and N. Cagman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2226–2735.

[2] M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (2009) 1547–1553.

[3] D. Chen, The parametrization reduction of soft sets and its applications, Comput. Math. Appl. 49 (2005) 757–763.

[4] Sujoy Das and S. K. Samanta, Soft real sets, soft real numbers and their properties, J. Fuzzy Math. 20(3) (2012) 551–576.

[5] Sujoy Das and S. K. Samanta, On soft complex sets and soft complex numbers, J. Fuzzy Math. 21(1) (2013) 195–216.

[6] Sujoy Das and S. K. Samanta, On soft metric spaces, J. Fuzzy Math. Accepted.

[7] Sujoy Das, Pinaki Majumdar and S. K. Samanta, On soft linear spaces and soft Normed linear spaces, Communicated to Ukrainian Math. J.

[8] F. Feng, Y. B. Jun and X. Zhao, Soft semirings, Comput. Math. Appl. 56 (2008) 2621–2628.

[9] F. Feng, C. X. Li, B. Davvaz and M. I. Ali, Soft sets combined with fuzzy sets and rough sets: a tentative approach, Soft Computing 14 (2010) 8999–9911.

[10] Y. B. Jun, Soft BCK/BCI-algebras, Comput. Math. Appl. 56 (2008) 1408–1413.

[11] Y. B. Jun and C. H. Park, Applications of soft sets in ideal theory of BCK/BCI-algebras, Inform. Sci. 178 (2008) 2466–2475.

[12] Z. Kong, L. Gao, L. Wong and S. Li, The normal parameter reduction of soft sets and its algorithm, Comput. Math. Appl. 56 (2008) 3029–3037.

[13] P. K. Maji, R. Biswas and A. R. Roy, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077–1083.

[14] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[15] P. Majumdar and S. K. Samanta, On soft mappings, Comput. Math. Appl. 60 (2010) 2666–2672.
[16] D. Molodtsov, Soft set theory first results, Comput. Math. Appl. 37 (1999) 19–31.
[17] D. Pie and D. Miao, From soft sets to information systems, Granular Computing, 2005, IEEE International Conference, Volume 2, 617–622.
[18] M. Shabir and M. Irfan Ali, Soft ideals and generalized fuzzy ideals in semigroups, New Math. Nat. Comput. 5 (2009) 599–615.
[19] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (2011) 1786–1799.
[20] Y. Zou and Z. Xiao, Data analysis approaches of soft sets under incomplete information, Knowledge-Based Systems 21 (2008) 941–945.

Sujoy Das (sujoy_math@yahoo.co.in)
Assistant Professor, Department of Mathematics, Bidhan Chandra College, Asansol-4, West Bengal, India

Syamal Kumar Samanta (syamal_123@yahoo.co.in)
Professor, Department of Mathematics, Visva Bharati, Santiniketan-731235, West Bengal, India