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A new method for system of fully fuzzy linear equations based on a certain decomposition of its coefficient matrix

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ABSTRACT. A certain decomposition of the coefficient matrix of the Fully Fuzzy Linear System (FFLS) is given to obtain a simple method for solving these systems. The new method can solve FFLS in a smaller computing process. Moreover, the method will be useful when the system is in the rectangular form. The method is described in detail and is illustrated by solving some examples.

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1. Introduction

One major application of the fuzzy number arithmetic is treating linear systems whose parameters are all or partially represented by fuzzy numbers. The term fuzzy matrix, which is the most important concept in this paper, has various meanings. For definition of a fuzzy matrix we follow the definition of Dubois and Prade, i.e. a matrix with fuzzy numbers as its elements [5]. This class of fuzzy matrices consist of applicable matrices, which can model uncertain aspects and the works on them are too limited. Some of the most interesting works on these matrices can be seen in [2, 3, 4, 5]. A general model for solving a fuzzy linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy vector, first proposed by Friedman et al. [6]. Friedman and his colleagues used the embedding method and replaced the original fuzzy linear system by a crisp linear system and then they solved it. A review of some methods for solving these systems can be found in [5]. In addition, an other important kind of fuzzy linear systems are including fuzzy numbers in whose all parameters and is named fully fuzzy linear systems (see in [3, 4, 7]). Nevertheless, there is just a few computational methods for solving a fully fuzzy linear systems until now. For example, recently Dehghan and his colleagues in [3] and [4] proposed two numerical methods for solving these kind of systems. In [7], authors proposed a new method for solving these systems based on QR decomposition. Here, we are going to give a new method for solving $\widetilde{A}\otimes\widetilde{x}=\widetilde{b}$, where $\widetilde{A}$ is a fuzzy matrix and $\widetilde{x}$ and $\widetilde{b}$ are fuzzy vectors with appropriate sizes. This paper is organized in 5 section. In next section, we give some basic backgrounds of fuzzy arithmetic and then define a fully fuzzy linear system of equations. In Section 3 we design a numerical method for computing the solution of FFLS where the mean value matrix of the coefficient matrix is in rectangular form. Numerical examples are given in Section 4 to examine our method. We conclude in Section 5.

2. Preliminaries

In this section, we review some necessary backgrounds and notions of fuzzy arithmetic which is useful throughout the paper and taken from [5, 7, 8].

Definition 2.1. A fuzzy subset $\widetilde{A}$ of $\mathbb{R}$ is defined by its membership function

$$\mu_{\widetilde{A}}:\mathbb{R}\to[0,1],$$

which assigns a real number $\mu_{\widetilde{A}}$ in the interval $[0,1]$, to each element $x\in\mathbb{R}$, where the value of $\mu_{\widetilde{A}}$ at x shows the grade of membership of x in $\widetilde{A}$. Indeed, a fuzzy subset $\widetilde{A}$ can be characterized as a set of ordered pairs of element x and grade $\mu_{\widetilde{A}}$ and is often written

$$\widetilde{A}=\{(x,\mu_{\widetilde{A}}(x))|x\in\mathbb{R}\}.$$

Definition 2.2. A fuzzy set with the following membership function is named a triangular fuzzy number and in this paper we will use these fuzzy numbers.

$$\mu_{\widetilde{A}}(x)=\begin{cases}1-\frac{m-x}{\alpha}, & m-\alpha\leqslant x<m,\ \alpha>0,\\ 1-\frac{x-m}{\beta}, & m\leqslant x\leqslant m+\beta,\ \beta>0,\\ 0, & \text{otherwise}.\end{cases}\tag{2.1}$$

Definition 2.3. A fuzzy number $\tilde{A}$ is called positive (negative), denoted by $\tilde{A}>0\,(\tilde{A}<0)$, if its membership function $\mu_{\tilde{A}}(x)$ satisfies $\mu_{\tilde{A}}(x)=0$, $\forall x\leq 0$ $(\forall x\geq 0)$. Using its mean value and left and right spreads, and shape functions, such a triangular fuzzy number $\tilde{A}$ is symbolically written

$$\tilde{A}=(m,\alpha,\beta).$$

Clearly, $\tilde{A}=(m,\alpha,\beta)$ is positive, if and only if $m-\alpha\geq 0$.

Remark 2.1. We consider $\widetilde{0}=(0,0,0)$ as zero fuzzy number.

Remark 2.2. We show the set of all triangular fuzzy numbers by $F(\mathbb{R})$.

Definition 2.4. (Equality in fuzzy numbers). Two triangular fuzzy numbers $M=(m,\alpha,\beta)$ and $N=(n,\gamma,\delta)$ are said to be equal, if and only if $m=n$, $\alpha=\gamma$ and $\beta=\delta$.

Definition 2.5. For two triangular fuzzy numbers $M = (m, \alpha, \beta)$ and $N = (n, \gamma, \delta)$ the formula for the extended addition becomes:

$$(m, \alpha, \beta) \oplus (n, \gamma, \delta) = (m + n, \alpha + \gamma, \beta + \delta). \tag{2.2}$$

The formula for the extended opposite becomes:

$$-M = -(m, \alpha, \beta) = (-m, \beta, \alpha). \tag{2.3}$$

Let $M = (m, \alpha, \beta)$ and $N = (n, \gamma, \delta)$ be two LR and RL fuzzy numbers, respectively.

$$M \ominus N = (m, \alpha, \beta) \ominus (n, \gamma, \delta) = (m - n, \alpha + \delta, \beta + \gamma). \tag{2.4}$$

The approximate formulas for the extended multiplication of two fuzzy numbers can be summarized as follows as given in [5]:
If $M > 0$ and $N > 0$, then

$$(m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, m\gamma + n\alpha, m\delta + n\beta). \tag{2.5}$$

For scalar multiplication:

$$\lambda \otimes M = \lambda \otimes (m, \alpha, \beta) = \begin{cases} (\lambda m, \lambda\alpha, \lambda\beta), & \lambda \geqslant 0, \\ (\lambda m, -\lambda\beta, -\lambda\alpha), & \lambda < 0. \end{cases} \tag{2.6}$$

Definition 2.6. A matrix $\widetilde{A} = (\widetilde{a_{ij}})$ is called a fuzzy matrix, if each element of $\widetilde{A}$ is a fuzzy number.
A fuzzy matrix $\widetilde{A}$ will be positive and denoted by $\widetilde{A} > \widetilde{0}$, if each element of $\widetilde{A}$ be positive. We may represent $n \times n$ fuzzy matrix $\widetilde{A} = (\widetilde{a_{ij}})_{n\times n}$, such that $\widetilde{a_{ij}} = (a_{ij}, \alpha_{ij}, \beta_{ij})$, with the new notation $\widetilde{A} = (A, M, N)$, where $A = (a_{ij})$, $M = (\alpha_{ij})$ and $N = (\beta_{ij})$ are three $n \times n$ crisp matrices.

Definition 2.7. A square fuzzy matrix $\widetilde{A} = (\widetilde{a_{ij}})$ will be an upper triangular fuzzy matrix, if

$$\widetilde{a_{ij}} = \widetilde{0} = (0, 0, 0), \qquad \forall i > j,$$

and a square fuzzy matrix $\widetilde{A} = (\widetilde{a_{ij}})$ will be a lower triangular fuzzy matrix, if

$$\widetilde{a_{ij}} = \widetilde{0} = (0, 0, 0), \qquad \forall i < j.$$

Definition 2.8. Consider the $m \times n$ fuzzy linear system of equations [3, 10]:

$$\begin{cases} (\widetilde{a_{11}} \otimes \widetilde{x_1}) \oplus (\widetilde{a_{12}} \otimes \widetilde{x_2}) \oplus ... \oplus (\widetilde{a_{1n}} \otimes \widetilde{x_n}) = \widetilde{b_1}, \\ (\widetilde{a_{21}} \otimes \widetilde{x_1}) \oplus (\widetilde{a_{22}} \otimes \widetilde{x_2}) \oplus ... \oplus (\widetilde{a_{2n}} \otimes \widetilde{x_n}) = \widetilde{b_2}, \\ \quad . \\ \quad . \\ \quad . \\ (\widetilde{a_{m1}} \otimes \widetilde{x_1}) \oplus (\widetilde{a_{m2}} \otimes \widetilde{x_2}) \oplus ... \oplus (\widetilde{a_{mn}} \otimes \widetilde{x_n}) = \widetilde{b_m}. \end{cases} \tag{2.7}$$

The matrix form of the above equations is

$$\widetilde{A} \otimes \widetilde{x} = \widetilde{b},$$

where the coefficient matrix $\widetilde{A} = (\widetilde{a_{ij}}), 1 \leq i \leq m$ and $1 \leq j \leq n$ is an $m \times n$ fuzzy matrix such that $\widetilde{a_{ij}} \in F(\mathbb{R})$ and $\widetilde{x_j}, \widetilde{b_i} \in F(\mathbb{R})$, for all $j = 1, ..., n$ and $i = 1, ..., m$. This system is called a Fully Fuzzy Linear System (FFLS).

In this paper, we are going to obtain a positive solution of the FFLS $\widetilde{A} \otimes \widetilde{x} = \widetilde{b}$, where $\widetilde{A} = (A, M, N) > \widetilde{0}$, $\widetilde{b} = (b, h, g) > \widetilde{0}$ and $\widetilde{x} = (x, y, z) > \widetilde{0}$. So we have

$$(A, M, N) \otimes (x, y, z) = (b, h, g). \tag{2.8}$$

Then by using Eq.(2.5) we have

$$(Ax, Ay + Mx, Az + Nx) = (b, h, g). \tag{2.9}$$

Therefore, Definition 2.4 concludes that

$$\begin{cases} Ax = b, \\ Ay + Mx = h, \\ Az + Nx = g. \end{cases} \tag{2.10}$$

So, by assuming that A be a nonsingular matrix we have

$$\begin{cases} Ax = b & \Rightarrow x = A^{-1}b, \\ Ay = h - Mx & \Rightarrow y = A^{-1}(h - Mx), \\ Az = g - Nx & \Rightarrow z = A^{-1}(g - Nx). \end{cases}$$

3. New method for solving FFLS

Theorem 3.1. If A is an $m \times k$ matrix with full column rank, then A can be factored as $A = QR$ where Q is an $m \times k$ matrix whose column vectors form an orthonormal basis for the column space of A and R is a $k \times k$ invertible upper triangular matrix [1].

Now again consider the fully fuzzy linear system as defined in Eq.(2.8). Assuming that, in the fully fuzzy coefficient matrix $\widetilde{A} = (A, M, N)$, A is a full rank crisp matrix and its QR-decomposition is $A = Q_1R_1$. Then we have:

$$\widetilde{A} = (A, M, N) = (Q_1, 0, Q_3) \otimes (R_1, R_2, 0) = (Q_1R_1, Q_1R_2, Q_3R_1) \tag{3.1}$$

Thus,

$$\begin{cases} Q_1R_1 = A & \Rightarrow & R_1 = Q_1^T A, \\ Q_1R_2 = M & \Rightarrow & R_2 = Q_1^T M, \\ Q_3R_1 = N & \Rightarrow & Q_3 = NR_1^{-1}, \end{cases} \tag{3.2}$$

where matrix Q_1 is an orthonormal crisp matrix and matrix R_1 is an upper triangular crisp matrix.

Now we are a place to apply the above idea for solving FFLS, $\widetilde{A} \otimes \widetilde{x} = \widetilde{b}$, where $\widetilde{A} = (A, M, N)$, $\widetilde{x} = (x, y, z)$, $\widetilde{b} = (b, g, h)$,i.e.

$$(A, M, N) \otimes (x, y, z) = (b, g, h)$$

Therefore, Eq.(3.1) yields that

$$(Q_1R_1, Q_1R_2, Q_3R_1) \otimes (x, y, z) = (b, g, h),$$

or equivalently,

$$(Q_1R_1x, Q_1R_2x + Q_1R_1y, Q_3R_1x + Q_1R_1z) = (b, g, h).$$

Therefore, the original fully fuzzy linear system can be solved as follows:

$$\left\{\begin{array}{lll} Q_1R_1x = b & \Rightarrow & x = R_1^{-1}Q_1^T b \\ Q_1R_2x + Q_1R_1y = g & \Rightarrow & y = R_1^{-1}Q_1^T(g - Mx) \\ Q_3R_1x + Q_1R_1z = h & \Rightarrow & z = R_1^{-1}Q_1^T(h - Nx) \end{array}\right. \tag{3.3}$$

4. Numerical examples

Example 4.1. Consider the following FFLS:

$$\begin{pmatrix} (19,10,21) & (26,15,14) \\ (45,20,15) & (75,30,20) \end{pmatrix} \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = \begin{pmatrix} (150,100,550) \\ (220,150,720) \end{pmatrix}$$

First we obtain QR-decomposition for matrix A, as $A = Q_1R_1$, such that

$$A = \begin{pmatrix} 19 & 26 \\ 45 & 75 \end{pmatrix} = Q_1R_1 = \begin{pmatrix} -0.5900 & -0.8073 \\ -0.8073 & -0.5900 \end{pmatrix} \begin{pmatrix} -32.2024 & -87.105 \\ 0 & 7.9186 \end{pmatrix}.$$

So we use Eq.(3.2) to obtain the matrices R_2, Q_3 as follows:

$$R_2 = Q_1^T M = \begin{pmatrix} -18.0110 & -36.0220 \\ 0.7763 & 1.5526 \end{pmatrix},$$

$$Q_3 = NR_1^{-1} = \begin{pmatrix} -0.6521 & -5.2790 \\ -0.4347 & -2.2565 \end{pmatrix}.$$

Therefore, by using Eq.(3.3) we have

$$\widetilde{x} = (5.2941, 0.7381, 141.1176), \quad \widetilde{y} = (1.0980, 0.2460, 3.4248).$$

Example 4.2. Consider the following FFLS:

$$\begin{pmatrix} (5,2,4) & (5,4,4) & (5,4,5) \\ (8,4,5) & (6,3,8) & (3,2,7) \\ (4,2,4) & (8,5,4) & (3,3,8) \end{pmatrix} \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix} = \begin{pmatrix} (500,400,600) \\ (500,435,800) \\ (300,300,700) \end{pmatrix}$$

First we obtain QR-decomposition for matrix A as:

$$A = Q_1R_1,$$

$$\begin{pmatrix} 5 & 5 & 5 \\ 8 & 6 & 3 \\ 4 & 8 & 3 \end{pmatrix} = \begin{pmatrix} -0.5773 & 0.6556 & 0.4866 \\ -0.5773 & 0.0936 & -0.8111 \\ -0.5773 & -0.7492 & 0.3244 \end{pmatrix} \begin{pmatrix} -8.6602 & -9.8149 & -8.6602 \\ 0 & 3.5590 & 1.1239 \\ 0 & 0 & -3.5688 \end{pmatrix}.$$

So by using Eq.(3.2) we obtain the matrices R_2, Q_3 as follows:

$$R_2 = Q_1^T M = \begin{pmatrix} -5.7735 & -5.1961 & -5.7735 \\ -1.3112 & 1.4048 & -0.4682 \\ -0.9733 & 0.1622 & -2.1088 \end{pmatrix},$$

$$Q_3 = NR_1^{-1} = \begin{pmatrix} -0.4618 & 0.1311 & 0.0412 \\ -0.4618 & 0.9740 & 0.3067 \\ -0.5773 & 0.3746 & -0.7226 \end{pmatrix}.$$

Finally, by using Eq(3.3) we obtain the solution as follows:

$\widetilde{x} = (27.272, 0.479, 10.909),\ \widetilde{y} = (36.363, 19.896, 9.090),\ \widetilde{z} = (180.181, 0.516, 27.272).$

5. Conclusions

We used a certain decomposition of the coefficient matrix of the system of fully fuzzy linear equations to obtain a new method for solving these systems. The numerical examples illustrated our method.

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