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Coupled coincidence point results in fuzzy metric spaces without completeness

Binayak S. Choudhury, D. Gopal, Pradyut Das

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Abstract. In the present paper we prove a coupled coincidence point theorem for two pairs of coincidentally commuting mappings in fuzzy metric spaces in the sense of George and Veeramani. We define (E.A.)- property in this context and use it in our theorem. Several corollaries are also given. The results are illustrated with an example.

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Corresponding Author: Pradyut Das (pradyutdas747@gmail.com)

1. Introduction

Coupled fixed point problems belong to a category of problems of fixed point theory in which much interest has been generated recently after the publication of a coupled contraction mapping theorem by Bhaskar and Lakshmikantham [1]. One of the reasons for this interest is the application of these results, especially to boundary value problems [1, 14, 18]. The result proved by Bhaskar and Lakshmikantham in [1] was generalised to coupled coincidence point results in [2] and [13] under two separate sets of conditions. Some other works on this topic may be found in [4, 14, 18]. Coupled fixed point problems have also been studied in structures which are generalizations of metric spaces as, for instances, in probabilistic metric spaces [20], in cone metric spaces [11, 17] and in G- metric spaces [3].

Coupled fixed point results were proved in fuzzy metric spaces by Zhu et al [21] in which they obtained a fuzzy version of the result of Bhaskar et al [1]. After that common coupled fixed point results in fuzzy metric spaces were established by Hu [9].

In the present work we prove a coupled coincidence point theorem for two pairs of coincidentally commuting mappings in fuzzy metric spaces. We define (E.A.)-property in this context and use it in our theorem. Several corollaries are also given. The results are illustrated with an example.

2. Preliminaries

The following are necessary notions for the discussion in this paper.

Definition 2.1 ([8, 19]). A binary operation $* : [0,1]^2 \longrightarrow [0,1]$ is called a t-norm if the following properties are satisfied:
(i) $*$ is associative and commutative,
(ii) $a * 1 = a$ for all $a \in [0,1]$,
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Generic examples of $t-$norm are $a *_1 b = \min\{a, b\}$, $a *_2 b = \frac{ab}{max\{a,b,\lambda\}}$ for $0 < \lambda < 1$, $a *_3 b = ab$, $a *_4 b = max\{a + b - 1, 0\}$ etc.

Kramosil and Michalek defined fuzzy metric space in the following way.

Definition 2.2 ([12]). The 3-tuple $(X, M, *)$ is called a fuzzy metric space in the sense of Kramosil and Michalek if X is a non-empty set, $*$ is a t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:
(i) $M(x, y, 0) = 0$,
(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and
(v) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is left-continuous,
where $t, s > 0$ and $x, y, z \in X$.

George and Veeramani in their paper introduced a modification of the above definition. The motivation was to make the corresponding induced topology necessarily into a Hausdroff topology.

Definition 2.3 ([5]). The 3-tuple $(X, M, *)$ is called a fuzzy metric space in the sense of George and Veeramani if X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:
(i)$M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and
(v) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

The following details of this space are described in the introductory paper [5]. Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, $0 < r < 1$, the open ball $B(x, t, r)$ with center $x \in X$ is defined by

$$B(x, t, r) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, t, r) \subset A$. Let τ denote the family of all open subsets of X. Then τ is a topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.

In the present work we will only consider the space as described in definition 2.3 and refer this space simply as fuzzy metric space.

Example 2.4 ([5]). Let $X = \mathbb{R}$. Let $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ $x, y \in X$, let

$$M(x, y, t) = \frac{t}{t + |x - y|}.$$

Then $(X, M, *)$ is a fuzzy metric space.

Lemma 2.5 ([7]). *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, .)$ is nondecreasing for all $x, y \in X$.*

Definition 2.6 ([5]). Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

Lemma 2.7 ([16]). *M is a continuous function on $X^2 \times (0, \infty)$.*

Definition 2.8 ([1]). Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x, \;\; F(y, x) = y.$$

Further Lakshmikantham and Ćirić have introduced the concept of coupled coincidence point.

Definition 2.9 ([13]). Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $h : X \to X$ if

$$hx = F(x, y), \;\; hy = F(y, x).$$

Definition 2.10 ([13]). Let X be a nonempty set and the mappings $F : X \times X \to X$ and $h : X \to X$ are commuting if for all $x, y \in X$

$$hF(x, y) = F(hx, hy).$$

We next give the following definition.

Definition 2.11. Let X be a non-empty set and $F : X \times X \longrightarrow X$ and $h : X \longrightarrow X$ be two mappings. F and h are said to be coincidentally commuting if they commute at their coupled coincidence points, that is, if $hx = F(x,\ y)$ and $hy = F(y,\ x)$, for some $(x,\ y) \in X \times X$, then

$hF(x,\ y) = F(hx,\ hy)$ and $hF(y,\ x) = F(hy,\ hx)$.

Example 2.12. Let $X = [0,\ \infty)$. Let $F : X \times X \to X$ and Let $h : X \to X$ be defined respectively as follows:

$$F(x,\ y) = \begin{cases} \frac{1}{3}, & \text{if } x > 1 \text{ and } 0 < y < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$hx = \begin{cases} 0, & \text{if } x = 0, \\ 100, & \text{if } 0 < x < 1, \\ 1, & \text{if } x = 1, \\ 20, & \text{if } x > 1. \end{cases}$$

Here, the functions h and F commute at their only coupled coincidence point $(0,\ 0)$. Therefore, the pair of functions (h, F) is coincidentally commuting. But the pair of functions (h, F) is not commuting.

In view of the above example we have the observation that every commuting pair is a coincidentally commuting pair but the converse is not true.

Next we define (E.A.)- property.

Definition 2.13. Two maps $F : X \times X \to X$ and $h : X \to X$, where X is a nonempty set, are said to satisfy (E.A.)- property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for some $x, y \in X$,

$$F(x_n, y_n) = hx_n \to x \text{ as } n \to \infty$$

and

$$F(y_n, x_n) = hy_n \to y \text{ as } n \to \infty.$$

This definition is the counterpart of the notion of (E.A.)-property for mapping $f, g : X \to X$. It is utilized in a number of works to obtain fixed point results, for instances [6, 10, 15].

We will use the following class of real mappings.

Definition 2.14 ([15])**.** Let ϕ be the class of mappings $\psi : [0, 1] \to [0, 1]$ satisfying the following properties:
(i) ψ continuous and monotone increasing [0,1],
(ii) $\psi(t) > t$ for all $0 < t < 1$.

We note that if $\psi \in \phi$, then $\psi(1) = 1$.

3. Major Section

Theorem 3.1. *Let $(X, M, *)$ be a fuzzy metric space. Let $F : X \times X \to X$, $G : X \times X \to X$, $h : X \to X$ and $g : X \to X$ be four mappings satisfies the following conditions:*
(i) $M(F(x, y), G(u, v), s) \geq \psi(\min\{M(hx, gu, s), M(hy, gv, s), M(hx, F(x, y), s),$
$M(hx, G(u, v), s), M(gu, G(u, v), s), M(gu, F(x, y), s)\}),$
(ii) $F(X \times X) \subseteq g(X), G(X \times X) \subseteq h(X)$ *and* $h(X), g(X)$ *are two closed subsets of* X*,*
(iii) (h, F) *and* (g, G) *are coincidentally commuting pairs,*
for every $x, y, u, v \in X$, for $s > 0$ and $\psi \in \phi$. If (h, F) and (g, G) satisfy the (E.A.)- property, then there exist $x, y \in X$ such that $hx = F(x, y)$, $hy = F(y, x)$, $gx = G(x, y)$ and $gy = G(y, x)$, that is, the pair of mappings (h, F) and (g, G) have common coupled coincidence point in X.

Proof. Since (h, F) and (g, G) satisfy the (E.A.)- property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X, such that

$$F(x_n, y_n) = hx_n \to x \text{ as } n \to \infty,$$
$$F(y_n, x_n) = hy_n \to y \text{ as } n \to \infty$$

and

$G(x_n, y_n) = gx_n \to x$ as $n \to \infty$,
$G(y_n, x_n) = gy_n \to y$ as $n \to \infty$.

Also $x, y \in h(X) \bigcap g(X)$. Since, $G(X \times X) \subseteq h(X)$, there exists $u \in X$ such that $hu = x$ and also there exists $v \in X$ such that $hv = y$. Now for all $s > 0$, we have

$$M(F(u,v), G(x_n, y_n), s) \geq \psi(\min\{M(hu, gx_n, s), M(hv, gy_n, s), M(hu, F(u,v), s), M(gx_n, G(x_n, y_n), s), M(hu, G(x_n, y_n), s), M(gx_n, F(u,v), s)\}).$$

Taking $n \to \infty$ on the both sides, for all $s > 0$, we have

$$M(F(u,v), x, s) \geq \psi(\min\{M(x,x,s), M(y,y,s), M(x, F(u,v), s), M(x,x,s), M(x,x,s), M(x, F(u,v), s)\})$$
$$\geq \psi(\min\{1, 1, M(x, F(u,v), s), 1, 1, M(x, F(u,v), s)\}).$$

Now, if $x \neq F(u,v)$, then $0 < M(x, F(u,v), s) < 1$, therefore, $\psi(M(x, F(u,v), s)) > M(x, F(u,v), s)$, which is a contradiction of the above inequality. This prove that $M(x, F(u,v), s) = 1$, which implies that $x = F(u,v)$. Therefore, $x = hu = F(u,v)$. Similarly, we can prove $y = hv = F(v,u)$. Since, $F(X \times X) \subseteq g(X)$, there exists $r \in X$ such that $gr = x$ and also there exists $z \in X$ such that $gz = y$. Now for all $s > 0$, we have

$$M(F(x_n, y_n), G(r,z), s)$$
$$\geq \psi(\min\{M(hx_n, gr, s), M(hy_n, gz, s), M(hx_n, F(x_n, y_n), s), M(gr, G(r,z), s), M(hx_n, G(r,z), s), M(gr, F(x_n, y_n), s)\}).$$

Taking $n \to \infty$ on the both sides, for all $s > 0$, we have

$$M(x, G(r,z), s) \geq \psi(\min\{M(x,x,s), M(y,y,s), M(x,x,s), M(x, G(r,z), s), M(x, G(r,z), s), M(x,x,s)\})$$
$$M(x, G(r,z), s) \geq \psi(\min\{1, 1, 1, M(x, G(r,z), s), M(x, G(r,z), s), 1\})$$

Now, if $x \neq G(r,z)$, then $0 < M(x, G(r,z), s) < 1$, therefore, $\psi(M(x, G(r,z), s) > M(x, G(r,z), s)$, which is a contradiction of the above inequality. This prove that $M(x, G(r,z), s) = 1$, which implies that $x = G(r,z)$. Therefore, $x = gr = G(r,z)$. Similarly, we can prove $y = gz = G(z,r)$. Therefore, $x = gr = hu = G(r,z) = F(u,v)$ and $y = gz = hv = F(v,u) = G(z,r)$. Since, (h, F) is coincidentally commuting, therefore $hF(u,v) = F(hu, hv)$ and $hF(v,u) = F(hv, hu)$, which implies $hx = F(x,y)$ and $hy = F(y,x)$. Since, (g, G) is coincidentally commuting, therefore $gG(r,z) = G(gr, gz)$ and $gG(z,r) = G(gz, gr)$, which implies $gx = G(x,y)$ and $gy = G(y,x)$, that is, (x,y) is the common coupled coincidence point of the pair of mappings (h, F) and (g, G). This is the proof of the theorem. □

Corollary 3.2. *Let $(X, M, *)$ be a fuzzy metric space. Let $F : X \times X \to X$ and $h : X \to X$ be two mappings satisfies the following conditions:*
(i) $M(F(x,y), F(u,v), s) \geq \psi(\min\{M(hx, hu, s), M(hy, hv, s), M(hx, F(x,y), s), M(hu, F(u,v), s), M(hx, F(u,v), s), M(hu, F(x,y), s)\})$,
(ii) $F(X \times X) \subseteq h(X)$ *and* $h(X)$ *is a closed subset of* X,
(iii) (h, F) *is coincidentally commuting pair,*
for every $x, y, u, v \in X$, for $s > 0$ and $\psi \in \phi$. If (h, F) satisfy the (E.A.)- property, then there exist $x, y \in X$ such that $hx = F(x,y)$ and $hy = F(y,x)$, that is, (h, F) has coupled coincidence point in X.

Proof. The proof follows by putting $F = G$, $h = g$ in Theorem 3.1. □

Corollary 3.3. *Let $(X, M, *)$ be a fuzzy metric space. Let $F : X \times X \to X$ be a mapping satisfies the following condition:*

$$M(F(x,y), F(u,v), s) \geq \psi(\min\{M(x,u,s), M(y,v,s), M(x, F(x,y), s), \\ M(u, F(u,v), s), M(x, F(u,v), s), M(u, F(x,y), s)\}),$$

for every $x, y, u, v \in X$, for $s > 0$ and $\psi \in \phi$. If F satisfy the (E.A.)- property, then there exist $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$, that is, F has fixed point in X.

Proof. The proof follows by putting $F = G$, $h = g = I$, the identity function, in Theorem 3.1. □

Example 3.4. Let $X = [0,1]$. Let for all $s > 0$, $M(x,y,s) = e^{-\frac{|x-y|}{s}}$, for all $x, y \in X$ and $a * b = \min\{a, b\}$. Then $(X, M, *)$ is a fuzzy metric space. Let the mappings $F : X \times X \to X$ and $G : X \times X \to X$ be defined as follows:

$$F(x,y) = G(x,y) = 1$$

and the mappings $h : X \to X$ and $g : X \to X$ be defined as follows:

$$hx = \frac{3x-1}{2} \text{ and } gx = \frac{x+1}{2}, \text{ for all } x, y \in X.$$

Let $\psi(t) = \sqrt{t}$. Then all the conditions of theorem 3.1 are satisfied. Here $(1,1)$ is a common coupled coincidence point.

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Binayak S. Choudhury (binayak12@yahoo.co.in)
Bengal Engineering and Science University, Shibpur, P.O.: B. Garden, Shibpur, Howrah - 711103, West Bengal, India

D. Gopal (gopal.dhananjay@rediffmail.com)
Department of Applied Mathematics and Humanities, SV National Institute of Technology, Surat,Gujarat, India

Pradyut Das (pradyutdas747@gmail.com)
Bengal Engineering and Science University, Shibpur, P.O.: B. Garden, Shibpur, Howrah - 711103, West Bengal, India