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An application of compatibility and weak compatibility for fixed point theorems in fuzzy metric spaces

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Abstract. The object of this paper is to establish a unique common fixed point theorem for six self-mappings satisfying a contractive conditions through compatibility and weak compatibility with different pairs of continuities in a fuzzy metric space. The established results generalize, extend,unify and fuzzify several existing fixed point results in metric spaces and fuzzy metric spaces.

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1. Introduction

The theory of fuzzy sets was first introduced by Zadeh [16], after that a lot of research papers have been published on fuzzy sets. The motivation of introducing fuzzy metric space is the fact that in many situations the distance between two points is inexact due to fuzziness rather than randomness. Kramosil and Michalek [7] introduced the concept of fuzzy metric space by generalizing the concept of probabilistic metric space to fuzzy situation. George and Veeramani [3] modified this concept of fuzzy metric space and obtain a Hausdroff topology for this kind of fuzzy metric spaces. It appears that the study of Kramosil and Michalek [7] of fuzzy metric spaces paves the way for developing the smooth machinery in the field of fixed point theory for the study of contractive maps.

Sessa [10] initiated the tradition of improving commutativity condition in fixed point theorems by introducing the notion of weakly commuting maps in metric spaces. Jungck [5] soon enlarged this concept to compatible maps. The concepts of

R-weakly commuting maps and compatible maps in fuzzy metric space have been introduced by Vasuki [14] and Mishra et al [9]respectively. In [6] Jungck and Rhoades termed a pair of self-map to be coincidentally commuting or equivalently weak compatible if they commute at their coincidence points. This concept is most general among all the commutativity concepts in this field as every pair of commuting maps or of compatible maps is weak compatible but the reverse is not true always. In this paper we establish the existence of unique common fixed point of six selfmaps through compatibility and weak compatibility satisfying a contraction adopted in [8]. Our results generalize, extend, unify and fuzzify [11, 12] and several existing fixed point results in metric spaces and fuzzy metric spaces.

2. Preliminaries

Definition 2.1. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if ([0, 1], *) is an abelian topological monoid with unit 1 such that $a*b \leq c*d$, whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.
Examples of t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2 ([7])**.** The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$

(FM-1) $M(x, y, 0) = 0$;
(FM-2) $M(x, y, t) = 1$,for all $t > 0$ iff $x = y$;
(FM-3) $M(x, y, t) = M(y, x, t)$;
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$, for all $t > 0$. The following example shows that every metric space induces a fuzzy metric space.

Example 2.3 ([3])**.** Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$. Let for all $x, y \in X$, $M(x, y, t) = t/(t + d(x, y))$, for all $t > 0$ and $M(x, y, 0) = 0$. Then $(X, M, *)$ is a fuzzy metric space. It is called the fuzzy metric space induced by the metric space (X, d).

Lemma 2.4 ([4])**.** *For all $x, y \in X$, $M(x, y, .)$ is a non-decreasing function.*

Definition 2.5 ([4])**.** Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$, for all $t > 0$. Further, sequence $\{x_n\}$ is said to be a Cauchy sequence if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$, for all $t > 0$ and for all p. The space is said to be complete if every Cauchy sequence in it converges to a point of it.

Remark 2.6. Since * is continuous, it follows from F.M-4 that in a fuzzy metric space the limit of a sequence is unique, if it exists.

In this paper $(X, M, *)$ will be considered to be the fuzzy metric space with condition (F.M-6) $\lim_{n\to\infty} M(x, y, t) = 1$, for all $x, y \in X$ and $t > 0$.

Definition 2.7 ([15]). Two maps A and S from a fuzzy metric space $(X, M, *)$ into itself are said to be R-weakly commuting if there exists a positive real number R such that for each $x \in X$, $M(ASx, SAy, kt) \geq M(Ax, Sy, t)$, $\forall t > 0$.

Definition 2.8 ([9]). A pair (A, B) of self mappings of a fuzzy metric space is said to be compatible (or asymptotically commuting) maps of type if $\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1$, for all $t > 0$ and when ever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 2.9 ([1]). Two maps A and B from a fuzzy metric space $(X, M, *)$ into itself are said to be compatible of type (α) if $M(ABx_n, BBx_n, t) = 1$, $M(BBx_n, AAx_n, t) = 1$, for all $t > 0$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some $x \in X$.

Definition 2.10 ([2]). Two maps A and B from a fuzzy metric space $(X, M, *)$ into itself are said to be compatible of type (β) if $M(A^2x_n, B^2x_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 2.11. Two maps A and B from a fuzzy metric space $(X, M, *)$ whenever into itself are said to be weak-compatible if they commute at their coincidence points, i.e., $Ax = Bx$ implies $ABx = BAx$.

Definition 2.12. A pair (A, S) of self-maps of a fuzzy metric space $(X, M, *)$ is said to be semi-compatible if $\lim_{n\to\infty} ASx_n = Sx$ whenever $\{x_n\}$ is a sequence such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X.$$

It follows that (A, S) is semi-compatible and $Ay = Sy$ then $ASy = SAy$.

Remark 2.13. Let (A, S) be a pair of self-maps on a fuzzy metric space $(X, M, *)$ Then (A, S) is R-weakly commuting implies that (A, S) is compatible, which implies that (A, S) is weak-compatible. But the converse is not true. The following is an example of a pair of self-maps which is weakly compatible, but not compatible. Hence it is not R-weakly commuting.

Example 2.14. Let $(X, M, *)$ be a fuzzy metric space, where $X = [0, 3]$, t-norm is defined by $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $M(x, y, t) = e^{\frac{-|x-y|}{t}}$ for all $x, y \in X$ and all $t > 0$. Define self-maps A and S on X as follows:

$$A(x) = \begin{pmatrix} 3-x & if\ 0 \leq x < 1, \\ 3 & if 1 \leq x \leq 3, \end{pmatrix}$$

Take $x_n = 2 - \frac{1}{n}$. Then $x_n \to 2$, $x_n < 2$ and $3 - x_n > 1$ for all n.
Also $Ax_n, Sx_n \to 2$ and $n \to \infty$.
Now $M(ASx_n, SAx_n, t) = e^{\frac{-|ASx_n - SAx_n|}{t}} \to e^{\frac{-1}{t}} \neq 2$ as $n \to \infty$. So A and S are not compatible. The set of coincident points of A and S is 1,3.
For any $x \in 1, 3$, $Ax = Sx = 3$ and $ASx = A(3) = 3 = S(3) = SAx$. Thus A and S are weak-compatible but not compatible.

Proposition 2.15 ([13]). *Let A and S be selfmaps on a fuzzy metric space $(X, M, *)$ Assume that S is continuous. Then (A, S) is semi-compatible if and only if (A, S) is compatible.*

Proposition 2.16. *Let A and S be continuous self-maps on a fuzzy metric space $(X, M, *)$ If (A, S) is semi-compatible, then (A, S) is compatible of type (β).*

Proposition 2.17 ([13])**.** *Let A and S be self-maps on a fuzzy metric space $(X, M, *)$ .If S is continuous and (A, S) is compatible of type (α), then (A, S) is semi-compatible.*

Proposition 2.18 ([13])**.** *Let A and S be continuous selfmaps on a fuzzy metric space $(X, M, *)$ Then (A, S) is semi-compatible if and only if (A, S) is compatible of type (β).*

The following is an example of a pair (S, T) of self-maps, which is semi-compatible, but not compatible. Further, it is shown that the semi-compatibility of the pair (S, T) need not imply the semi-compatibility of (T, S).

Example 2.19. Let $X = [0, 1]$ and (X, M, t) be the induced fuzzy metric space with $M(x, y, t) = \frac{t}{t+|x-y|}$. Define a self-map S on X as follows:

$$S(x) = \begin{cases} x & if\ 0 \leq x < 1/3, \\ 1 & if \quad x \geq 1/3, \end{cases}$$

Let I be the identity map on X and $x_n = \frac{1}{3} - \frac{1}{n}$.Then $\{Ix_n\} = \{x_n\} \to \frac{1}{3}$ and $\{Sx_n\} \to \frac{1}{3} \neq S\frac{1}{3}$. Thus (I, S) is not semi-compatible though it is compatible. For a sequence $\{x_n\}$ in X such that $\{x_n\} \to x$ and $\{Sx_n\} \to x$,we have $\{SIx_n\} = \{Sx_n\} \to x = Ix$. Thus (S, I) is semi-compatible.

Remark 2.20. The above example gives an important aspect of semi-compatibility as the pair (I, S) is commuting, weakly commuting, compatible, and weak-compatible, but it is not semi-compatible.

Example 2.21. Let $(X, M, *)$ be the fuzzy metric space as defined in Example 2.14. Define self-maps A and S on X as follows:

$$A(x) = \begin{cases} 3 & if\ 0 \leq x \leq 1, \\ \frac{x}{3} & if\ 1 \leq x, \leq 3 \end{cases}$$

$$S(x) = \begin{cases} 1 & if\ 0 \leq x < 1, \\ 3 & if \quad x = 1, \\ \frac{x+2}{5} & if\ 1 \leq x \leq 3, \end{cases}$$

$\{x_n\} = 3 - \frac{1}{2n}$. Then we have $S(1) = A(1) = 3$ and $S(3) = A(3) = 1$.
$SA(1) = AS(1) = 1$ and $SA(3) = AS(3) = 3$.
Hence $\{Ax_n\} \to 1$ and $\{Sx_n\} \to 3$ and $\{SAx_n\} \to 1$ as $n \to \infty$.
Now $\lim_{n\to\infty} M(ASx_n, Sy, t) = M(3, 3, t) = 1$.
$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = M(3, 1, t) = \frac{t}{2+t} < 1$.
Hence (A, S) is semi-compatible but not compatible. In [14], Vasuki proved the following theorem for R-weakly commuting pair of self-maps.

Theorem 2.22 ([15])**.** *Let f and g be R-weakly commuting self-maps on a complete fuzzy metric space $(X, M, *)$ such that $M(fx, fy, t) \geq r(M(gx, gy, t))$ where $r : [0, 1] \to [0, 1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1$. If $f(X) \subset g(X)$ and either f or g is continuous then f and g have a unique common fixed point.*

Remark 2.23. If self-mappings A and S of a fuzzy metric space $(X, M, *)$ are compatible maps of type (α), then they are weak compatible.

Lemma 2.24 ([1])**.** *Let $\{x_n\}$ be a sequence in a Fuzzy metric space $(X, M, *)$ with condition $(FM-6)$. If there exists a number $k \in (0,1)$ such that $M(x_n, x_{n+1}, kt) \geq M(x_{n-1}, x_n, t)$ $\forall t > 0$ and $n \in N$. Then $\{x_n\}$ is a Cauchy sequence in X.*

Lemma 2.25 ([9])**.** *Let $(X, M, *)$ be a Fuzzy metric space. If there exists $k \in (0,1)$ such that for all $x, y \in X$, $M(x, y, kt) \geq M(x, y, t)$ $\forall t > 0$ then $x = y$.*

3. Main results

Theorems on compatibility and weak compatibility on fuzzy metric spaces

Theorem 3.1. *Let $(X, M, *)$ be a complete metric space with $r * r \geq r$, $\forall r \in [0,1]$ and let A, B, S, T, P, Q be mappings from X into itself such that following conditions are satisfied*

(i) *$A(X) \subset ST(X)$ and $B(X) \subset PQ(X)$.*
(ii) *Either A or PQ is continuous .*
(iii) *(A, PQ) is compatible and (B, ST) is weakly compatible.*
(iv) *$PQ = QP$, $ST = TS$, $AQ = QA$ and $BT = TB$.*
(v) *There exist a constant $k \in (0,1)$, such that for every $x, y \in X$ and $t > 0$,*

$$\begin{aligned} M(Ax, By, kt) \geq \min\{&M(PQx, STy, t), M(Ax, PQx, t), M(By, STy, 2t), \\ &M(Ax, STy, t) * M(By, PQx, 2t), \\ &\frac{M(By, PQx, 2t)}{M(PQx, STy, t) * M(By, PQx, 2t)}\} \end{aligned}$$

Then A, B, S, T, P and Q had a unique common fixed point in X.

Proof. Let x_0 be an arbitrarily point in X as $A(X) \subset ST(X)$ and $B(X) \subset PQ(X)$ then there exists $x_1, x_2 \in X$ such that $Ax_0 = STx_1 = y_0$ and $Bx_1 = PQx_2 = y_1$. We construct sequence $\{x_n\}$, $\{y_n\}$ in X. Such that
$y_{2n} = STx_{2n+1} = Ax_{2n}$ and $y_{2n+1} = Bx_{2n+1} = PQx_{2n+2}$ for $n = 0, 1, 2, ...$
Now, we first show that $\{y_n\}$ is a Cauchy sequence in X, From (v) we have

$$\begin{aligned} M(y_{2n}, y_{2n+1}, kt) &= M(Ax_{2n}, Bx_{2n+1}, kt) \\ &\geq \min\{M(PQx_{2n}, STx_{2n+1}, t), M(Ax_{2n}, PQx_{2n}, t), \\ &M(Bx_{2n+1}, STx_{2n+1}, t), \\ &M(Ax_{2n}, STx_{2n+1}, t) * M(Bx_{2n+1}, PQx_{2n}, 2t), \\ &\frac{M(Bx_{2n+1}, PQx_{2n}, 2t)}{M(PQx_{2n}, STx_{2n+1}, t) * M(Bx_{2n+1}, PQx_{2n}, 2t)}\} \\ &\geq \min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t), M(y_{2n+1}, y_{2n}, t), \\ &M(y_{2n}, y_{2n}, t) * M(y_{2n+1}, y_{2n-1}, 2t), \\ &\frac{M(y_(2n+1), y_(2n-1), 2t)}{M(y_{2n-1}, y_{2n}, 2t) * M(y_{2n+1}, y_{2n-1}, 2t)}\} \end{aligned}$$

$$\geq \min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n+1}, y_{2n}, t),$$
$$M(y_{2n}, y_{2n}, t) * M(y_{2n+1}, y_{2n-1}, 2t),$$
$$\frac{M(y_{2n+1}, y_{2n-1}, 2t)}{M(y_{2n-1}, y_{2n}, t) * M(y_{2n+1}, y_{2n-1}, 2t)}\}$$

$$\geq \min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n+1}, y_{2n-1}, t), \}$$

which implies that $M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t)$, $\forall t > 0$.
In general $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$, $\forall t > 0$.
Thus from Lemma 2.24 $\{y_n\}$ is a Cauchy sequence in X. By completeness of $(X, M, *)$, $\{y_n\}$ converges to some point z in X. The subsequence $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{STx_{2n+1}\}$ and $\{PQx_{2n+2}\}$ of sequence $\{y_n\}$ also converges to z in X.

Case I:- Suppose A is continuous, we have $APQx_{2n} \to Az$. The compatibility of the pair (A, PQ) gives that $A^2x_{2n} \to Az$, $A(PQ)x_{2n} \to Az$ and $(PQ)Ax_{2n} \to Az$. We know that the limit in a fuzzy metric space is unique therefore $Az = PQz$

Step 1: Putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in I we get

$$M(AAx_{2n}, Bx_{2n+1}, kt) \geq \phi\{M(PQAx_{2n}, STx_{2n+1}, t), M(AAx_{2n}, PQz, t),$$
$$M(Bx_{2n+1}, STx_{2n+1}, 2t),$$
$$M(AAx_{2n}, STx_{2n+1}, t) * M(Bx_{2n+1}, PQAx_{2n}, t),$$
$$\frac{M(Bx_{2n+1}, PQAx_{2n}, 2t)}{M(PQAx_{2n}, STx_{2n+1}, t) * M(Bx_{2n+1}, PQAx_{2n}, 2t)}\},$$

Letting $n \to \infty$ and using above result we get

$$M(Az, z, kt) \geq \min\{M(Az, z, t), M(Az, Az, t), M(z, z, t), M(Az, z, t) * M(z, Az, t),$$
$$\frac{M(z, Az, 2t)}{M(Az, z, t) * M(z, Az, 2t)}\}$$

Therefore $M(Az, z, kt) \geq M(Az, z, t)$
Now by lemma 2.25 $Az = z$, hence $Az = z = PQz$.

Step 2: putting $x = Qz$ and $y = x_{2n+1}$ in (v) we have

$$M(AQz, Bx_{2n+1}, kt) \geq \min\{M(PQ(Qz), STx_{2n+1}, t), M(AQz, PQ.Qz, t),$$
$$M(Bx_{2n+1}, STx_{2n+1}, 2t),$$
$$M(Az, STx_{2n+1}, t) * M(Bx_{2n+1}, PQ.Qz, t),$$
$$\frac{M(Bx_{2n+1}, PQ.Qz, 2t)}{M(PQ.Qz, STx_{2n+1}, t) * M(Bx_{2n+1}, PQ.Qz, 2t)}\},$$

As $AQ = QA$ and $PQ = QP$, $A(Qz) = Qz$ and $PQ(Qz) = Qz$
Letting $n \to \infty$ and using above results we get

$$M(Qz, z, kt) \geq \min\{M(Qz, z, t), M(Qz, Qz, t), M(z, z, t),$$
$$M(Qz, z, t) * M(z, Qz, t), \frac{M(z, Qz, zt)}{M(Qz, z, t) * M(z, Qz, 2t)}\}.$$

$M(Qz, z, kt) \geq M(Qz, z, t)$.
By lemma 2.25 we get $Qz = z$.

Now $PQz = z$. Which implies that $Pz = z$.
Therefore $Az = Pz = Qz = z$.

Step 3: Since $A(X) \subset ST(X)$, $there exist u \in X$, such that $z = Az = STu$.
Putting $x = x_{2n}$ and $y = u$ in (v) we get,

$$M(Ax_{2n}, Bu, kt) \geq \min\{M(PQx_{2n}, STu, t), M(Ax_{2n}, PQx_{2n}, t),$$
$$M(Bu, STu, 2t), M(Ax_{2n}, STu, t) * M(Bu, PQx_{2n}, t),$$
$$\frac{M(Bu, PQx_{2n}, 2t)}{M(PQx_{2n}, STu, t) * M(Bu, PQx_{2n}, 2t)}\},$$

Letting $n \to \infty$ and using above results we get

$$M(z, Bu, kt) \geq \min\{M(z, z, t), M(z, z, t), M(Bu, z, 2t), M(z, z, t), M(Bu, z, t),$$
$$\frac{M(Bu, z, 2t)}{M(z, z, t) * M(Bu, z, 2t)}\},$$

$M(z, Bu, kt) \geq M(Bu, z, t)$.
Using lemma 2.25 we get $z = Bu = STu$.
Which implies that u is a coincidence point of (B, ST). The weak compatibility of the pair (B, ST) gives that $STBu = TSBu$ $STz = Bz$.

Step 4: Putting $x = x_{2n}$ and $y = z$ in (v) we get

$$M(Ax_{2n}, Bz, kt) \geq \min\{M(PQx_{2n}, STz, t), M(Ax_{2n}, PQx_{2n}, t), M(Bz, STz, 2t),$$
$$M(Ax_{2n}, STz, t) * M(Bz, PQx_{2n}, t),$$
$$\frac{M(Bz, PQx_{2n}, 2t)}{M(PQx_{2n}, STz, t) * M(Bz, PQx_{2n}, 2t)}\},$$

Letting $n \to \infty$ and using above result we get.

$$M(z, Bz, kt) \geq \min\{M(z, Bz, t), M(z, z, t), M(Bz, z, 2t), M(z, Bz, t) * M(Bz, z, t),$$
$$\frac{M(Bz, z, 2t)}{M(z, Bz, t) * M(z, Bz, 2t)}\}$$

$M(z, Bz, kt) \geq M(z, Bz, t)$. Using lemma 2.25 $Bz = z$. Thus $STz = Bz = z$.

Step 5: Putting $x = x_{2n}$ and $y = Tz$ in (v) we get

$$M(Ax_{2n}, BTz, kt) \geq \min\{M(PQx_{2n}, STTz, t), M(Ax_{2n}, PQx_{2n}, t),$$
$$M(BTz, STTz, 2t), M(Ax_{2n}, STTz, t) * M(BTz, PQx_{2n}, t),$$
$$\frac{M(BTz, PQx_{2n}, 2t)}{M(PQx_{2n}, STTz, t) * M(BTz, PQx_{2n}, 2t)}\}.$$

Since $BT = TB$ and $ST = TS$, we have $BTz = Tz$ and $ST(Tz) = Tz$
Letting $n \to \infty$ and using above results we get.

$$M(z, Tz, kt) \geq \min\{M(z, Tz, t), M(z, z, t), M(Tz, Tz, 2t), M(z, Tz, t) * M(Tz, z, t),$$
$$\frac{M(Tz, z, 2t)}{M(z, Tz, t) * M(z, Tz, 2t)}\}.$$

We obtain $\to M(z, Tz, Kt) \geq M(z, Tz, t)$.
Using lemma 2.25 we get $Tz = z$.
Now $STz = z$ which implies that $Sz = z$.

Hence $Az = Bz = Sz = Tz = Pz = Qz = z$.
Thus z is a common fixed point of A, B, S, T, P and Q.

Case II:- suppose PQ is continuous and the pair (A, PQ) is compatible, we have $(PQ)Ax_{2n} \to PQz$, $(PQ)^2x_{2n} \to PQz$. and $APQx_{2n}PQz$.

Step 6: Putting $x = PQx_{2n}$ and $y = x_{2n+1}$ in (v) we get

$$M(APQx_{2n}, Bx_{2n+1}, kt) \geq \min\{M(PQ(PQx_{2n}), STx_{2n+1}, t), M(APQx_{2n}, PQPQx_{2n}, t), M(Bx_{2n+1}, STx_{2n+1}, 2t), M(Ax_{2n}, STx_{2n+1}, t) * M(Bx_{2n+1}, PQPQx_{2n}, t), \frac{M(Bx_{2n+1}, PQPQx_{2n}, 2t)}{M(PQPQx_{2n}, STx_{2n+1}, t) * M(Bx_{2n+1}, PQPQx_{2n}, 2t)}\}.$$

Letting $n \to \infty$ and using above results we get

$$M(PQz, z, kt) \geq \min\{M(PQz, z, t), M(PQz, PQz, t), M(z, z, 2t), M(z, z, t) * M(z, PQz, t), \frac{M(z, PQz, 2t)}{M(PQz, z, t) * M(z, PQz, 2t)}\}.$$

implies $M(PQz, z, Kt) \geq M(z, PQz, t)$
Using lemma 2.25 we get $PQz = z$.
Putting $x = z$ and $y = x_{2n+1}$ in (i) we get

$$M(APQx_{2n}, Bx_{2n+1}, kt) \geq \min\{M(PQ(PQx_{2n}), STx_{2n+1}, t), M(APQx_{2n}, PQPQx_{2n}, t), M(Bx_{2n+1}, STx_{2n+1}, 2t), M(Ax_{2n}, STx_{2n+1}, t) * M(Bx_{2n+1}, PQPQx_{2n}, t), \frac{M(Bx_{2n+1}, PQPQx_{2n}, 2t)}{M(PQPQx_{2n}, STx_{2n+1}, t) * M(Bx_{2n+1}, PQPQx_{2n}, 2t)}\}.$$

Letting $n \to \infty$ and using above results we get

$$M(Az, z, kt) \geq \min\{M(PQz, STx_{2n+1}, t), M(Az, PQz, t), M(Bx_{2n+1}, STx_{2n+1}, 2t), M(Az, STx_{2n+1}, t) * M(Bx_{2n+1}, PQz, t), \frac{M(Bx_{2n+1}, PQz, 2t)}{M(PQz, STx_{2n+1}, t) * M(Bx_{2n+1}, PQz, 2t)}\}.$$

Letting $n \to \infty$ and using above results we get

$$M(Az, z, kt) \geq \min\{M(z, z, t), M(Az, z, t), M(z, z, 2t), M(Az, z, t) * M(z, z, t), \frac{M(z, z, 2t)}{M(z, z, t) * M(z, z, 2t)}\}.$$

implies $M(Az, z, kt) \geq M(Az, z, t)$
Using lemma 2.25 we get
$Az = z$. Using step 2 we get $Qz = z$.
Now $PQz = z \Rightarrow Pz = z$ This implies that $Az = Qz = Pz = z$,
Hence $Az = Bz = Sz = Tz = Pz = Qz = z$.
Thus z is a common fixed point of A, B, S, T, P and Q.

Uniqueness:- let v be another common fixed point of A, B, S, T, P and Q, then $v = Av = Bv = Sv = Tv = Pv = Qv$.
Putting $x = z$ and $y = v$ in (i) we get

$$M(Az, Bv, kt) \geq \min\{M(PQz, STv, t), M(Az, PQz, t), M(Bv, STv, 2t), M(Az, STv, t) * M(Bv, PQz, t), \frac{M(Bv, PQz, 2t)}{M(PQz, STv, t) * M(Bv, PQz, 2t)}\}.$$

$$M(z, v, kt) \geq \min\{M(z, v, t), M(z, z, t), M(v, v, 2t), M(z, v, t) * M(v, z, t), \frac{M(v, z, 2t)}{M(z, v, t) * M(v, z, 2t)}\},$$

implies $M(z, v, kt) \geq M(z, v, t)$
Using lemma 2.25 we get $z = v$. There fore z is common fixed point of A, B, S, T, P and Q. □

Remark 3.2. If we take $a * b = \min\{a, b\}$ where $a, b \in [0, 1]$ in Corollary 3.7,then this is generalization of the result of singh and chouhan[16],as only one mapping of the first pair needed to be continuous and the second pair of mapping is weakly compatible in (3.7).

If we take $Q = T = 1$ in Theorem 3.1 then conditions (iv) is satisfied trivially and we get the following result.

Theorem 3.3. *Let $(X, M, *)$ be a complete Fuzzy metric space and let A, B, S, P and Q be mappings from X into itself such that following conditions are satisfied*
(i) $A(X) \subset S(X)$*and* $B(X) \subset P(X)$.
(ii) *Either A or P is continuous.*
(iii) *(A, P) and (B, S) is compatible.*
(iv) *There exist a constant $k \in (0, 1)$, such that for every $x, yinX$ and $t > 0$,.*

$$M(Ax, By, kt) \geq \min\{M(Px, Sy, t), M(Ax, Px, t), M(By, Sy, 2t), M(Ax, Sy, t) * M(By, PQx, 2t), \frac{M(By, Px, 2t)}{M(Px, Sy, t) * M(By, Px, 2t)}\}.$$

Then A, B, S and P had a unique common fixed point in X.

Proof. As compatibility implies weak compatibility.The proof follows from Theorem 3.1. □

If we take $A = PQ = f$ and $B = ST = g$ in Theorem 3.1.

Theorem 3.4. *Let $(X, M, *)$ be a complete fuzzy metric space and let f and g be mappings from X into itself such that following conditions are satisfied*
(i) $f(X) \subset g(X)$.
(ii) *Either f or g is continuous,*
(iii) *(f, g) is compatible,*
(iv) *There exist a constant $k \in (0, 1)$, such that for every $x, y \in X$ and $t > 0$.*

$$M(fx, gy, kt) \geq \min\{M(fx, gy, t), M(fx, fx, t), M(gy, gy, 2t), M(fx, gy, t), * M(gy, fx, t), \frac{M(gy, fx, 2t)}{M(fx, gy, t) * M(gy, fx, 2t)}\}.$$

Then f and g had a unique common fixed point in X.

Remark 3.5. Theorem 3.4 generalizes Theorem of Vasuki [15] by assuming only compatibility of the pair (f, g) in place of its being R-weakly commuting. Thus Theorem 3.3 is a still better generalization of a result of [15] for four self-maps.

Theorem 3.6. *Let $(X, M, *)$ be a complete Fuzzy metric space and let A, B, S and P be mappings from X into itself such that following conditions are satisfied*
(i) *$A(X) \subset S(X)$ and $B(X) \subset P(X)$.*
(ii) *Either A or P is continuous.*
(iii) *(A, P) is compatible of type(α) and (B, S) is weak compatible.*
(iv) *There exist a constant $k \in (0, 1)$, such that for every $x, y \in X$ and $t > 0$,.*

$$M(Ax, By, kt) \geq \min\{M(Px, Sy, t), M(Ax, Px, t), M(By, Sy, 2t), M(Ax, Sy, t) * M(By, Px, 2t), \frac{M(By, Px, 2t)}{M(Px, Sy, t) * M(By, Px, 2t)}\}.$$

Then A, B, S and P had a unique common fixed point in X.

Proof. The proof follows from Theorem 3.1 and Proposition 2.17. □

Theorem 3.7. *Let $(X, M, *)$ be a complete Fuzzy metric space and let A, B, S and P be mappings from X into itself such that following conditions are satisfied*
(i) *$A(X) \subset S(X)$ and $B(X) \subset P(X)$.*
(ii) *Either A or P is continuous.*
(iii) *(A, P) is compatible of type(α) and (B, S) is weak compatible.*
(iv) *There exist a constantk $\in (0, 1)$, such that for every $x, y \in X$ and $t > 0$.*

$$M(Ax, By, kt) \geq \min\{M(Px, Sy, t), M(Ax, Px, t), M(By, Sy, 2t), M(Ax, Sy, t) * M(By, Px, 2t), \frac{M(By, Px, 2t)}{M(Px, Sy, t) * M(By, Px, 2t)}\}.$$

Then A, B, S and P had a unique common fixed point in X.

Proof. The proof follows from Theorem 2.22 and Proposition 2.18.

□

Taking $A = I$ in Theorem 3.1, we have another result for three self-maps, none of which are continuous and just a pair of them is needed to be weak-compatible only.

Corollary 3.8. *Let B, S and P be self-maps on a complete fuzzy metric space $(X, M, *)$ satisfying*
(i) *$B(X) \subset P(X)$ and S is surjective.*
(ii) *(B, S) is weak-compatible*
(iii) *There exist a constantk $\in (0, 1)$, such that for every $x, y \in X$ and $t > 0$.*

$$M(x, By, kt) \geq \min\{M(Px, Sy, t), M(Ax, Px, t), M(By, Sy, 2t), M(Ax, Sy, t) * M(By, Px, 2t), \frac{M(By, Px, 2t)}{M(Px, Sy, t) * M(By, Px, 2t)}\}.$$

Then B, S and T have a unique common fixed point.

Theorem 3.9. *Let $(X, M, *)$ be a complete fuzzy metric space with $r * r \geq r$, $\forall r \in [0, 1]$ and let A, B, S, T, P, Q be mappings from X into itself such that following conditions are satisfied*

(i) $A(X) \subseteq ST(X)$ *and* $B(X) \subseteq PQ(X)$.
(ii) *Either A or PQ is continuous.*
(iii) (A, PQ) *is compatible* (B, ST) *is weakly compatible.*
(iv) $PQ = QP$, $ST = TS$, $AQ = QA$ *and* $BT = TB$.
(v) *There exist a constantk* $\in (0, 1)$, *such that for every $x, y \in X$ and $t > 0$*

$$\begin{aligned} M(Ax, By, kt) \geq \phi\{ & M(PQx, STy, t) * M(Ax, PQx, t) * M(By, STy, 2t) \\ & * M(Ax, STy, t), *M(By, PQx, t) \\ & * \frac{M(By, PQx, 2t)}{M(PQx, STy, t) * M(By, PQx, 2t)} \Big\} \end{aligned}$$

Where $\phi : (0, 1) \to (0, 1)$, is a continuous function such that $\phi(c) > c$ for each $0 < t < 1$ such that for every $x, y \in X$ and $c > 0$ and $k \in (0, 1)$. Then A, B, S, T, P and Q had a unique common fixed point in X.

Proof. The proof follows in similar lines as done in Theorem 3.1 and the fact, $\phi(c) > c$. □

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