Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 6, No. 1, (July 2013), pp. 1–15
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

Neighbourhood properties of soft topological spaces

Sk. Nazmul, S. K. Samanta

Received 3 July 2012; Revised 29 October 2012; Accepted 11 November 2012

Abstract. In the present paper we study the neighbourhood properties in a soft topological space. For that we introduce the definitions of soft element, soft interior element, soft limiting element, soft neighbourhood operator, soft closure operstor etc. and study their properties in terms of neighbourhood system.

2010 AMS Classification: 54A40, 03E72, 20N25, 22A99, 06D72

Keywords: Soft sets, Soft topology, Soft element, Soft limiting element, Soft interior element, Soft open sets, Soft closed sets.

Corresponding Author: Sk. Nazmul (sk.nazmul_math@yahoo.in)

1. Introduction

In dealing with the uncertainties inherent in the problems of physical science, biological science, engineering, economics, social science, medical science etc., various theories such as probability theory, fuzzy set theory, and rough set theory were developed. In 1999, Molodtsov [11] proposed a new approach, viz. soft set theory, for modeling vagueness and uncertainties, and in 2001-03, Maji et al. [7, 8, 9] worked on some mathematical aspects of soft sets, fuzzy soft sets, and intuitionistic fuzzy soft sets. After that, many researchers have implemented different algebraic structures on soft sets and shown several applications in the process. Aktas and Cagman [1] in 2007 introduced a basic version of soft group theory which was further extended to fuzzy soft group by us [12]. Feng [3] in 2008, introduced soft semirings; Shabir and Ali (2009) [13] studied soft semigroups and soft ideals. Kharal and Ahmed [6] as well as Majumdar and Samanta [10] defined soft mappings etc. Recently in 2011, Shabir and Naz [14] gave a definition of soft topological spaces on the collection of soft sets. Later on, Cagman et al. in [2] and Hazra et al. in [4] defined soft topologies in different manners. In 2011, Hussain and Ahmad [5] studied some properties of soft topological spaces, and, in 2012, Zorlutuna et al. [15] gave some remarks on soft topological spaces. As a continuation of this, it is natural to insvestigate the behaviour of topological structures in soft set-theoretic form. To this end, we introduce in this paper a notion of neighbourhoods in Shabir and Naz-type [14] soft topological spaces, and study some of its properties. The organization of the paper is as follows:

Section 2 is the preliminary part where definitions and some properties of soft sets (in our form) have been given and features of soft topologies have been studied. In section 3, we have defined soft elements, soft neighbourhood system, soft interior operators, soft closure operators etc., and have studied their properties. Section 4 concludes the paper. For the sake of economy, a few straightforward proofs have been omitted throughout this paper.

2. Preliminaries

Following Molodtsov [11] and Maji et al. [7], some definitions and preliminary results on soft sets are presented in this section in our form. Unless otherwise stated, U will be assumed to be an initial universal set and A will be taken to be a set of parameters. Let $P(U)$ denote the power set of U and $S(U)$ denote the set of all soft sets over U.

Soft sets:

Definition 2.1. A pair (F, A), where F is a mapping from A to $P(U)$, is called a *soft set* over U.

Let (F_1, A) and (F_2, A) be two soft sets over a common universe U. Then (F_1, A) is said to be a *soft subset* of (F_2, A) if $F_1(x) \subseteq F_2(x), \forall x \in A$. This relation is denoted by $(F_1, A) \widetilde{\subseteq} (F_2, A)$. (F_1, A) is said to be *soft equal* to (F_2, A) if $F_1(x) = F_2(x), \forall x \in A$. This relation is denoted by $(F_1, A) = (F_2, A)$.

The *complement* of a soft set (F, A) is defined as $(F, A)^c = (F^c, A)$, where $F^c(x) = (F(x))^c = U - F(x), \forall x \in A$.

A soft set (F, A) over U is said to be *a null soft set* $\Big($*an absolute soft set*$\Big)$ if $F(x) = \phi \ \Big(F(x) = U\Big), \forall x \in A$. This is denoted by $\widetilde{\Phi}\Big(\widetilde{A}\Big)$.

The *difference* of two soft sets (F, A) and (G, A) is defined by $(F, A) - (G, A) = (F - G, A)$, where $(F - G)(x) = F(x) - G(x),\ \forall x \in A$.

Definition 2.2. Let $\{(F_i, A); i \in I\}$ be a nonempty family of soft sets over a common universe U. Then their

(a) *Intersection*, denoted by $\widetilde{\cap}_{i\in I}$, is defined by $\widetilde{\cap}_{i\in I}(F_i, A) = (\widetilde{\cap}_{i\in I} F_i, A)$, where $(\widetilde{\cap}_{i\in I} F_i)(x) = \cap_{i\in I}(F_i(x)), \forall x \in A$.

(b) *Union*, denoted by $\widetilde{\cup}_{i\in I}$, is defined by $\widetilde{\cup}_{i\in I}(F_i, A) = (\widetilde{\cup}_{i\in I} F_i, A)$, where $(\widetilde{\cup}_{i\in I} F_i)(x) = \cup_{i\in I}(F_i(x)),\ \forall x \in A$.

Definition 2.3. Let X, Y be two non-empty sets and $f : X \to Y$ be a mapping. Then

(i) the *image* of a soft set $(F, A) \in S(X)$ under the mapping f is defined by $f(F, A) = (f(F), A)$, where $[f(F)](x) = f[F(x)],\ \forall x \in A$;

(ii) the *inverse image* of a soft set $(G, A) \in S(Y)$ under the mapping f is defined by $f^{-1}(G, A) = (f^{-1}(G), A)$, where $[f^{-1}(G)](x) = f^{-1}[G(x)],\ \forall x \in A$.

Proposition 2.4. *Let X, Y be two non-empty sets and $f : X \to Y$ be a mapping. If (F_1, A), $(F_2, A) \in S(X)$, then*
(i) $(F_1, A) \,\widetilde{\subset}(F_2, A) \Rightarrow f[(F_1, A)] \,\widetilde{\subset} f[(F_2, A)]$;
(ii) $f[(F_1, A) \,\widetilde{\cup}(F_2, A)] = f[(F_1, A)] \,\widetilde{\cup} f[(F_2, A)]$;
(iii) $f[(F_1, A) \,\widetilde{\cap}(F_2, A)] \,\widetilde{\subset} f[(F_1, A)] \,\widetilde{\cap} f[(F_2, A)]$;
(iv) $f[(F_1, A) \,\widetilde{\cap}(F_2, A)] = f[(F_1, A)] \,\widetilde{\cap} f[(F_2, A)]$, *if f is injective.*

Proposition 2.5. *Let X, Y be two non-empty sets and $f : X \to Y$ be a mapping. If (G_1, A), $(G_2, A) \in S(Y)$, then*
(i) $(G_1, A) \,\widetilde{\subset}(G_2, A) \Rightarrow f^{-1}[(G_1, A)] \,\widetilde{\subset} f^{-1}[(G_2, A)]$;
(ii) $f^{-1}[(G_1, A) \,\widetilde{\cup}(G_2, A)] = f^{-1}[(G_1, A)] \,\widetilde{\cup} f^{-1}[(G_2, A)]$;
(iii) $f^{-1}[(G_1, A) \,\widetilde{\cap}(G_2, A)] = f^{-1}[(G_1, A)] \,\widetilde{\cap} f^{-1}[(G_2, A)]$.

Proposition 2.6. *Let X, Y be two non-empty sets and $f : X \to Y$ be a mapping. If $(G, A) \in S(Y)$, then*
(i) $ff^{-1}(G, A) \,\widetilde{\subset}(G, A)$;
(ii) $ff^{-1}(G, A) = (G, A)$, *if f is surjective.*

Proposition 2.7. *Let X, Y be two non-empty sets and $f : X \to Y$ be a mapping. If $(F, A) \in S(X)$, then*
(i) $(F, A) \,\widetilde{\subset} f^{-1}f(F, A)$;
(ii) $f^{-1}f(F, A) = (F, A)$, *if f is injective.*

Soft topology:

In this section some properties of soft topologies are studied using the definition of a soft topology given by Shabir and Naz [14]. Unless mentioned otherwise, X is an initial universal set; A is the non-empty set of parameters; $S(X)$ denotes the collection of all soft sets over X under the parameter set A; $P(S(X))$ denotes the power set of $S(X)$.

Definition 2.8 ([14]). Let τ be the collection of soft sets over X. Then τ is said to be a soft topology on X if
(i) $(\widetilde{\phi}, A)$, $(\widetilde{X}, A) \in \tau$, where $\widetilde{\phi}(\alpha) = \phi$ and $\widetilde{X}(\alpha) = X$, $\forall \alpha \in A$;
(ii) the intersection of any two soft sets in τ belongs to τ;
(iii) the union of any number of soft sets in τ belongs to τ.
The triplet (X, A, τ) is called a soft topological space over X. The members of τ are said to be $\tau-$soft open sets or, simply, soft open sets in X.

Proposition 2.9 ([14]). *Let (X, A, τ) be a soft topological space over X. Then the collection $\tau^\alpha = \{F(\alpha) : \ (F, A) \in \tau\}$ defines a topology on X for each $\alpha \in A$.*

Remark 2.10. From Proposition 2.9, it can be seen that if (X, A, τ) is a soft topological space over X in the sense of Shabir and Naz [14], then there is a mapping $\alpha \to \tau^\alpha$ from the parameter set A to the collection of all topologies on X. Thus, the question is whether the converse of the above is true or not. In the next Proposition, we get an affirmative answer to this question, and a relation between the two representations of a soft topology is obtained.

Proposition 2.11. *Let (X, A, τ) be a soft topological space and τ^{α} be the topologies on X as in Proposition 2.9. Let $\tau^* = \{(G, A) \in S(X) :\ G(\alpha) \in \tau^{\alpha},\ \forall\ \alpha \in A\}$. Then τ^* is a soft topology on X with $[\tau^*]^{\alpha} = \tau^{\alpha},\ \forall \alpha \in A$.*

Proof. Since $\widetilde{\phi}(\alpha) = \phi \in \tau^{\alpha},\ \forall \alpha \in A \Rightarrow (\widetilde{\phi}, A) \in \tau^*$, and $\widetilde{X}(\alpha) = X \in \tau^{\alpha},\ \forall \alpha \in A \Rightarrow (\widetilde{X}, A) \in \tau^*$.

Now, let (F_1, A) and $(F_2, A) \in \tau^*$. Then $F_1(\alpha),\ F_2(\alpha) \in \tau^{\alpha},\ \forall \alpha \in A$. Thus $F_1(\alpha) \cap F_2(\alpha) \in \tau^{\alpha},\ \forall \alpha \in A \Rightarrow (F_1 \widetilde{\cap} F_2)(\alpha) \in \tau^{\alpha},\ \forall \alpha \in A$. Therefore, $(F_1, A) \widetilde{\cap} (F_2, A) = (F_1 \widetilde{\cap} F_2, A) \in \tau^*$.

Again, let $(F_i, A) \in \tau^*,\ \forall i \in I$. Then $F_i(\alpha) \in \tau^{\alpha},\ \forall i \in I,\ \forall \alpha \in A \Rightarrow \cup_{i \in I} F_i(\alpha) \in \tau^{\alpha},\ \forall \alpha \in A$. So $(\widetilde{\cup}_{i \in I} F_i)(\alpha) \in \tau^{\alpha},\ \forall \alpha \in A$. Hence $\widetilde{\cup}_{i \in I} (F_i, A) \in \tau^*$. Therefore, τ^* is a soft topology on X.

Next let $U \in \tau^{\alpha}$. Then $\exists\ (F, A) \in \tau$ such that $U = F(\alpha)$. We construct $(G, A) \in S(X)$ such that $G(\alpha) = F(\alpha)$ and $G(\beta) = \phi,\ \forall \beta \neq \alpha$. Then $(G, A) \in \tau^*$ and $U = F(\alpha) = G(\alpha) \in [\tau^*]^{\alpha}$. Therefore, $\tau^{\alpha} \subseteq [\tau^*]^{\alpha}$.........(1)

Also, let $V \in [\tau^*]^{\alpha}$. Then $\exists\ (F, A) \in \tau^*$ such that $V = F(\alpha) \in \tau^{\alpha}$. Therefore, $[\tau^*]^{\alpha} \subseteq \tau^{\alpha}$.........(2)

Thus, from (1) and (2), we get $\tau^{\alpha} = [\tau^*]^{\alpha},\ \forall \alpha \in A$. □

Remark 2.12. It is to be noted that τ and τ^* of Proposition 2.11 may be different. This is exemplified below.

Example 2.13. Let $X = \{a,\ b\}$ and $A = \{\alpha,\ \beta\}$. Then all possible soft sets over X are given below:
$(\widetilde{\phi}, A) = \big\{\{\phi\},\ \{\phi\}\big\}$, $(F_1, A) = \big\{\{\phi\},\ \{a\}\big\}$, $(F_2, A) = \big\{\{\phi\},\ \{b\}\big\}$,
$(F_3, A) = \big\{\{\phi\},\ \{a,\ b\}\big\}$, $(F_4, A) = \big\{\{a\},\ \{\phi\}\big\}$, $(F_5, A) = \big\{\{a\},\ \{a\}\big\}$,
$(F_6, A) = \big\{\{a\},\ \{b\}\big\}$, $(F_7, A) = \big\{\{a\},\ \{a,\ b\}\big\}$, $(F_8, A) = \big\{\{b\},\ \{\phi\}\big\}$,
$(F_9, A) = \big\{\{b\},\ \{a\}\big\}$, $(F_{10}, A) = \big\{\{b\},\ \{b\}\big\}$, $(F_{11}, A) = \big\{\{b\},\ \{a,\ b\}\big\}$,
$(F_{12}, A) = \big\{\{a,\ b\},\ \{\phi\}\big\}$, $(F_{13}, A) = \big\{\{a,\ b\},\ \{a\}\big\}$, $(F_{14}, A) = \big\{\{a,\ b\},\ \{b\}\big\}$,
$(\widetilde{X}, A) = \big\{\{a,\ b\},\ \{a,\ b\}\big\}$.
Let $\tau = \{(\widetilde{\phi}, A),\ (F_2, A),\ (F_6, A),\ (F_{11}, A),\ (\widetilde{X}, A)\}$. Then (X, A, τ) is a soft topological space such that $\tau^{\alpha} = \big\{\phi,\ \{a\},\ \{b\},\ X\big\}$ and $\tau^{\beta} = \big\{\phi,\ \{b\},\ X\big\}$.
Now, let us construct $\tau^* = \{(\widetilde{\phi}, A),\ (F_2, A),\ (F_3, A),\ (F_4, A),\ (F_6, A),\ (F_7, A),\ (F_8, A), (F_{10}, A), (F_{11}, A),\ (F_{12}, A),\ (F_{14}, A),\ (\widetilde{X}, A)\}$. Then $\tau \neq \tau^*$ but $[\tau^*]^{\alpha} = \tau^{\alpha},\ \forall\ \alpha \in A$.

Proposition 2.14 ([14])**.** *Let (X, A, τ_1) and (X, A, τ_2) be two soft topological spaces over X, then $(X, A, \tau_1 \widetilde{\cap} \tau_2)$, where $\tau_1 \widetilde{\cap} \tau_2 = \{(F, A) :\ (F, A) \in \tau_1\ \&\ (F, A) \in \tau_2\}$ is a soft topological space over X. But the union of two soft topological spaces over X may not be a soft topological space over X itself.*

Definition 2.15. Let τ_1 and τ_2 be two soft topologies over X. Then τ_2 is said to be soft finer than τ_1 if $\tau_1 \widetilde{\subseteq} \tau_2$.

Definition 2.16. Let X and Y be two non-empty sets, τ and ν be two soft topologies on X and Y respectively, and $f : X \to Y$ be a mapping. The image of τ and the preimage of ν under f are denoted by $f(\tau)$ and $f^{-1}(\nu)$ respectively, and they are defined by

(i) $f(\tau) = \{(G, A) \in S(Y, A) : \ f^{-1}(G, A) = (f^{-1}(G), A) \in \tau\}$;
(ii) $f^{-1}(\nu) = \{f^{-1}(G, A) = (f^{-1}(G), A) : \ (G, A) \ \in \nu\}$.

Theorem 2.17. *Let X and Y be two non-empty sets, τ and ν be two soft topologies on X and Y respectively, and $f : X \to Y$ be a mapping. Then*
(i) *$f^{-1}(\nu)$ is a soft topology on X;*
(ii) *$f(\tau)$ is a soft topology on Y.*

Proof. Follows from Propositions 2.4, 2.5, and 2.6. □

Definition 2.18 ([15])**.** A soft set (F, A) is called a soft point in (X, A), denoted by e_F, if for the element $e \in A$, $F(e) \neq \phi$ and $F(e') = \phi$, $\forall e' \in A - \{e\}$. The soft point e_F is said to be in the soft set (G, A), denoted by $e_F \widetilde{\in} (G, A)$, if for the element $e \in A$, $F(e) \subseteq G(e)$.

Definition 2.19. ([15]) A soft set (G, A) in a soft topological space (X, A, τ) is called a soft neighbourhood of the soft point $e_F \widetilde{\in} (X, A)$ if there exists a soft open set (H, A) such that $e_F \widetilde{\in} (H, A) \widetilde{\subseteq} (G, A)$.

A soft set (G, A) in a soft topological space (X, A, τ) is called a soft neighbourhood of the soft set (F, A) if there exists a soft open set (H, A) such that $(F, A) \widetilde{\subseteq} (H, A) \widetilde{\subseteq} (G, A)$. The neighbourhood system of a soft point e_F, denoted by $N_\tau(e_F)$, is the family of all its neighbourhoods.

Theorem 2.20 ([15])**.** *The neighbourhood system $N_\tau(e_F)$ at e_F in a soft topological space (X, A, τ) has the following properties:*
(i) *If $(G, A) \in N_\tau(e_F)$, then $e_F \widetilde{\in} (G, A)$;*
(ii) *If $(G, A) \in N_\tau(e_F)$ and $(G, A) \widetilde{\subseteq} (H, A)$, then $(H, A) \in N_\tau(e_F)$;*
(iii) *If (G, A), $(H, A) \in N_\tau(e_F)$, then $(G, A) \widetilde{\cap} (H, A) \in N_\tau(e_F)$;*
(iv) *If $(G, A) \in N_\tau(e_F)$, then there exists a $(M, A) \in N_\tau(e_F)$ such that $(G, A) \in N_\tau(e'_F)$ for each $e'_F \widetilde{\in} (M, A)$.*

Definition 2.21. Let $(X, A.\tau)$ be a soft topological space, and let (G, A) be a soft set over X.

(i) [14] The soft closure of (G, A) is the soft set $\overline{(G, A)} = \widetilde{\cap}\{(S, A) : \ (S, A)$ is soft closed and $(G, A) \widetilde{\subseteq} (S, A)\}$;

(ii) [15] The soft interior of (G, A) is the soft set $(G, A)^0 = \widetilde{\cup}\{(S, A) : \ (S, A)$ is soft open and $(S, A) \widetilde{\subseteq} (G, A)\}$.

Proposition 2.22. *Let $(X, A.\tau)$ be a soft topological space, and let (F, A), (G, A) be a soft sets over X.*
(i) [14] *If $(F, A) \widetilde{\subseteq} (G, A)$, then $\overline{(F, A)} \widetilde{\subseteq} \overline{(G, A)}$;*
(ii) [15] *If $(F, A) \widetilde{\subseteq} (G, A)$, then $(F, A)^0 \widetilde{\subseteq} (G, A)^0$.*

3. Neighbourhood properties of soft topological spaces

In this section (X, A, τ) stands for a soft topological space, where X is an initial universal set, A is a set of parameters, and τ is a soft topology over X.

Definition 3.1. A soft set (E, A) over X is said to be a soft element if $\exists \alpha \in A$ such that $E(\alpha)$ is a singleton, say $\{x\}$, and $E(\beta) = \phi$, $\forall \beta (\neq \alpha) \in A$. Such a soft element is denoted by E^x_α. Let $\mathcal{E}$ be the set of all soft elements of the universal set X.

Definition 3.2. The soft element E_α^x is said to be in the soft set (G, A), denoted by $E_\alpha^x \tilde{\in} (G, A)$, if $x \in G(\alpha)$.

Proposition 3.3. *$E_\alpha^x \tilde{\in} (F, A)$ iff $E_\alpha^x \tilde{\notin}(F^c, A)$.*

Proof. If $E_\alpha^x \tilde{\in} (F, A) \Leftrightarrow x \in F(\alpha) \Leftrightarrow x \notin X - F(\alpha) = F^c(\alpha) \Leftrightarrow E_\alpha^x \tilde{\notin}(F^c, A)$. □

Definition 3.4. A soft set (F, A) is said to be a *soft neighbourhood* (abbreviated as a *soft nbd*) of the soft set (H, A) if there exists a soft set $(G, A) \in \tau$ such that $(H, A) \tilde{\subseteq} (G, A) \tilde{\subseteq} (F, A)$. If $(H, A) = E_\alpha^x$, then (F, A) is said to be a *soft nbd* of the soft element E_α^x.

The *soft neighbourhood system* of a soft element E_α^x, denoted by $N_\tau(E_\alpha^x)$, is the family of all its soft nbds.

Example 3.5. Let $X = \{x, y, z\},\ A = \{\alpha_1, \alpha_2\}$ and
$\tau = \{(\tilde{\phi}, A), (\tilde{X}, A), (F, A) = \{\{x\}, \{y, z\}\},\ (G, A) = \{\{y\}, \{x, z\}\}\}$. Therefore,
$N_\tau(E_{\alpha_1}^x) = \Big\{\{\{x\}, \{y, z\}\},\ \{\{x\}, \{x, y, z\}\},\ \{\{x, y\}, \{y, z\}\},\ \{\{x, y\}, \{x, y, z\}\},$
$\{\{x, z\}, \{y, z\}\},\ \{\{x, z\}, \{x, y, z\}\},\ \{\{x, y, z\}, \{y, z\},\ \{\{x, y, z\}, \{x, y, z\}\}\Big\}$.
$N_\tau(E_{\alpha_2}^x) = \Big\{\{\{y\}, \{x, z\}\},\ \{\{x, y\}, \{x, z\}\},\ \{\{y, z\}, \{x, z\}\},\ \{\{x, y, z\}, \{x, z\}\},$
$\{\{y\}, \{x, y, z\}\},\ \{\{x, y\}, \{x, y, z\}\},\ \{\{y, z\}, \{x, y, z\}\},\ \{\{x, y, z\}, \{x, y, z\}\}\Big\}$.

$N_\tau(E_{\alpha_1}^y) = \Big\{\{\{y\}, \{x, z\}\},\ \{\{y\}, \{x, y, z\}\},\ \{\{x, y\}, \{x, z\}\},\ \{\{x, y\}, \{x, y, z\}\},$
$\{\{y, z\}, \{x, z\}\},\ \{\{y, z\}, \{x, y, z\}\},\ \{\{x, y, z\}, \{x, z\}\},\ \{\{x, y, z\}, \{x, y, z\}\}\Big\}$.
$N_\tau(E_{\alpha_2}^y) = \Big\{\{\{x\}, \{y, z\}\},\ \{\{x, y\}, \{y, z\}\},\ \{\{x, y\}, \{x, y, z\}\},\ \{\{x, z\}, \{y, z\}\},$
$\{\{x, z\}, \{x, y, z\}\},\ \{\{x, y, z\}, \{y, z\}\},\ \{\{x, y, z\}, \{x, y, z\}\},\ \{\{x, y, z\}, \{x, y, z\}\}\Big\}$.

$N_\tau(E_{\alpha_1}^z) = \{\{x, y, z\}, \{x, y, z\}\}$.
$N_\tau(E_{\alpha_2}^z) = \Big\{\{\{x\}, \{y, z\}\},\ \{\{x, y\}, \{y, z\}\},\ \{\{x, z\}, \{y, z\}\},\ \{\{x, y, z\}, \{y, z\}\},$
$\{\{x\}, \{x, y, z\}\},\ \{\{x, y\}, \{x, y, z\}\},\ \{\{x, z\}, \{x, y, z\}\},\ \{\{x, y, z\}, \{x, y, z\}\},$
$\{\{y\}, \{x, z\}\},\ \{\{x, y\}, \{x, z\}\},\ \{\{y, z\}, \{x, z\}\},\ \{\{x, y, z\}, \{x, z\}\},$
$\{\{y\}, \{x, y, z\}\},\ \{\{x, y\}, \{x, y, z\}\},\ \{\{y, z\}, \{x, y, z\}\}\Big\}$.

Proposition 3.6. *A soft set (F, A) over X is soft open iff (F, A) is a nbd of each of its soft elements, i.e. $(F, A) \in \tau$ iff $(F, A) \in N_\tau(E_\alpha^x),\ \forall E_\alpha^x \tilde{\in} (F, A)$.*

Proof. Let $(F, A) \in \tau$ and $E_\alpha^x \tilde{\in} (F, A)$. Then $E_\alpha^x \tilde{\in} (F, A) \tilde{\subseteq} (F, A)$. Therefore, (F, A) is a nbd of $E_\alpha^x,\ \forall E_\alpha^x \tilde{\in} (F, A)$.

Conversely, let (F, A) be a nbd of each of its soft elements. Let $E_\alpha^x \tilde{\in} (F, A)$. Since (F, A) is a nbd of the soft element E_α^x, $\exists\, (G, A) \in \tau$ such that $E_\alpha^x \tilde{\in} (G, A) \tilde{\subseteq} (F, A)$. Since $(F, A) = \cup\{\alpha_F;\ \alpha_F \tilde{\in} (F, A)\}$, it follows that (F, A) is a union of soft open sets and hence (F, A) is soft open. □

Proposition 3.7. *If $\{N_\tau(E_\alpha^x) :\ E_\alpha^x \in \mathcal{E}\}$ be the system of soft nbds. Then,*
(i) $N_\tau(E_\alpha^x) \neq \phi,\ \forall E_\alpha^x \in \mathcal{E}$;

(ii) $E^x_\alpha \widetilde{\in} (F,A),\ \forall\ (F,A) \in N_\tau(E^x_\alpha)$;
(iii) $(F,A) \in N_\tau(E^x_\alpha)$ *and* $(F,A) \widetilde{\subseteq} (G,A) \ \Rightarrow\ (G,A) \in N_\tau(E^x_\alpha)$;
(iv) $(F,A),(G,A) \in N_\tau(E^x_\alpha) \Rightarrow\ (F,A) \widetilde{\cap} (G,A) \in N_\tau(E^x_\alpha)$;
(v) $(F,A) \in N_\tau(E^x_\alpha) \Rightarrow\ \exists\ (G,A) \in N_\tau(E^x_\alpha)$ *such that* $(G,A) \widetilde{\subseteq} (F,A)$ *and* $(G,A) \in N_\tau(E^y_\beta),\ \forall\ E^y_\beta \widetilde{\in} (G,A)$.

Proof. Proofs of (i)-(iv) are straightforward.
(v) Since (F,A) is a nbd of E^x_α, $\exists\ (G,A) \in \tau$ such that $E^x_\alpha \widetilde{\in} (G,A) \widetilde{\subseteq} (F,A)$. Therefore, $E^x_\alpha \widetilde{\in} (G,A) \widetilde{\subseteq} (G,A)$, and hence, $(G,A) \in N_\tau(E^x_\alpha)$ and $(G,A) \widetilde{\subseteq} (F,A)$. Again, since (G,A) is open, by Proposision 3.6., (G,A) is a nbd of each of its soft elements, i.e. $(G,A) \in N_\tau(E^y_\beta),\ \forall\ E^y_\beta \widetilde{\in} (G,A)$. □

Definition 3.8. A mapping $\nu:\ \mathcal{E} \to P(S(X))$ is said to be a *soft nbd operator* on $\mathcal{E}$ if the following conditions hold.
(N(1)) $\nu(E^x_\alpha) \neq \phi,\ \forall E^x_\alpha \in \mathcal{E}$;
(N(2)) $E^x_\alpha \widetilde{\in} (F,A),\ \forall\ (F,A) \in \nu(E^x_\alpha)$;
(N(3)) $(F,A) \in \nu(E^x_\alpha)$ and $(F,A) \widetilde{\subseteq} (G,A) \ \Rightarrow\ (G,A) \in \nu(E^x_\alpha)$;
(N(4)) $(F,A),(G,A) \in \nu(E^x_\alpha) \Rightarrow\ (F,A) \widetilde{\cap} (G,A) \in \nu(E^x_\alpha)$;
(N(5)) $(F,A) \in \nu(E^x_\alpha) \Rightarrow\ \exists\ (G,A) \in \nu(E^x_\alpha)$ such that $(G,A) \widetilde{\subseteq} (F,A)$ and $(G,A) \in \nu(E^y_\beta),\ \forall\ E^y_\beta \widetilde{\in} (G,A)$.

Proposition 3.9. *If (X,A,τ) is a soft topological space, then the mapping $\nu:\ \mathcal{E} \to P(S(X))$ defined by $\nu(E^x_\alpha) = N_\tau(E^x_\alpha),\ \forall E^x_\alpha \in E$, where $N_\tau(E^x_\alpha)$ is the system of all soft nbds of E^x_α, is a soft nbd operator on $\mathcal{E}$.*

Proposition 3.10. *If ν be a soft nbd operator on $\mathcal{E}$, then there exists a soft topology over X such that $\nu(E^x_\alpha)$ is the family of all τ-soft nbds of E^x_α, $\forall E^x_\alpha \in \mathcal{E}$.*

Proof. Let $\tau = \{(F,A) \in S(X):\ (F,A) \in \nu(E^x_\alpha),\ \forall E^x_\alpha \widetilde{\in} (F,A)\}$. As there are no soft elements in $(\widetilde{\phi},A)$, the condition $(\widetilde{\phi},A) \in \nu(E^x_\alpha),\ \forall E^x_\alpha \in (\widetilde{\phi},A)$ is trivially satisfied. Therefore, $(\widetilde{\phi},A) \in \tau$. Let $E^x_\alpha \in (\widetilde{X},A)$. Since ν be a nbd operator, $\nu(E^x_\alpha) \neq \phi$. Let $(F,A) \in \nu(E^x_\alpha)$. Since $(F,A) \widetilde{\subseteq} (\widetilde{X},A)$, by N(3), $(\widetilde{X},A) \in \nu(E^x_\alpha)$. Therefore, $(\widetilde{X},A) \in \tau$. Again, let $(F_i,A) \in \tau,\ \forall i \in I$ and $(F,A) = \cup_{i\in I}(F_i,A)$.
Let $E^x_\alpha \widetilde{\in} (F,A)$. Then $\exists\ i \in I$ such that $E^x_\alpha \widetilde{\in} (F_i,A)$. Since $(F_i,A) \in \tau$, we have, $(F_i,A) \in \nu(E^x_\alpha)$. Also, $(F_i,A) \widetilde{\subseteq} (F,A)$. Thus $(F,A) \in \nu(E^x_\alpha),\ \forall E^x_\alpha \widetilde{\in} (F,A)$ and hence $(F,A) \in \tau$.

Next, let $(F_1,A),(F_2,A) \in \tau$ and $E^x_\alpha \in (F_1,A) \widetilde{\cap} (F_2,A)$. So $(F_1,A),(F_2,A) \in \nu(E^x_\alpha)$. Therefore, $(F_1,A) \widetilde{\cap}(F_2,A) \in \nu(E^x_\alpha)$ and hence $(F_1,A) \widetilde{\cap}(F_2,A) \in \tau$. Hence τ is a soft topology over X. Let (F,A) be a τ-soft nbd of E^x_α. Then $\exists(G,A) \in \tau$ such that $E^x_\alpha \widetilde{\in} (G,A) \widetilde{\subseteq} (F,A)$. Since $(G,A) \in \tau$, from the definition $(G,A) \in \nu(E^x_\alpha)$ and $(G,A) \widetilde{\subseteq} (F,A) \Rightarrow (F,A) \in \nu(E^x_\alpha)$.

Conversely, let $(F,A) \in \nu(E^x_\alpha)$. Since ν is a nbd operator, $\exists(G,A) \in \nu(E^x_\alpha)$ such that $(G,A) \widetilde{\subseteq} (F,A)$ and $(G,A) \in \nu(E^y_\beta),\ \forall E^y_\beta \widetilde{\in} (G,A)$. By the definition of τ, $(G,A) \in \tau$. Again $(G,A) \in \nu(E^x_\alpha) \Rightarrow E^x_\alpha \widetilde{\in} (G,A)$. So $E^x_\alpha \widetilde{\in} (G,A) \widetilde{\subseteq} (F,A)$ and hence (F,A) is a τ-nbd of E^x_α. Thus $\nu(E^x_\alpha)$ is the family of all τ-soft nbds of E^x_α. □

Definition 3.11. Let (X, A, τ) be a soft topological space. A sub-collection β of τ is said to be an open base of τ if every member of τ can be expressed as the union of some members of β.

Example 3.12. Let $X = \{a, b, c, d, e\}$, $A = \{\alpha_1,\ \alpha_2\}$, and $\tau = \Big\{(\ \widetilde{\phi}, A),\ (\ \widetilde{X}, A),$ $(F_1, A) = \big\{\{a\}, \{b\}\big\},\ (F_2, A) = \big\{\{d\}, \{c\}\big\},\ (F_3, A) = \big\{\{b, c\}, \{a, d\}\big\},\ (F_4, A) = \big\{\{a, d\}, \{b, c\}\big\}, (F_5, A) = \big\{\{a, b, c\}, \{a, b, d\}\big\},\ (F_6, A) = \big\{\{b, c, d\}, \{a, c, d\}\big\},\ (F_7, A) = \big\{\{a, b, c, d\}, \{a, b, c, d\}\big\}\Big\}$. Then τ is a soft topology over X.

Let $\beta = \{(\ \widetilde{\phi}, A),\ (\ \widetilde{X}, A),\ (F_1, A),\ (F_2, A),\ (F_3, A)\}$. Then β forms an open base of τ.

Proposition 3.13. *A collection β of soft open sets of a soft topological space (X, A, τ) forms an open base of τ iff $\forall (F, A) \in \tau$ and $\forall E^x_\alpha \ \widetilde{\in}\ (F, A),\ \exists (G, A) \in \beta$ such that $E^x_\alpha \ \widetilde{\in}\ (G, A) \ \widetilde{\subseteq}\ (F, A)$.*

Proof. Let β be an open base of τ. Let $(F, A) \in \tau$ and $E^x_\alpha \ \widetilde{\in}\ (F, A)$. Since β forms an open base of τ, (F, A) can be expressed as the union of some members of β. Since $E^x_\alpha \ \widetilde{\in}\ (F, A),\ \exists\ (G, A) \in \beta$ such that $E^x_\alpha \ \widetilde{\in}\ (G, A) \ \widetilde{\subseteq}\ (F, A)$. Therefore, the given condition is satisfied.

Conversely, let the given condition be satisfied. Let $(F, A) \in \tau$ and $E^x_\alpha \ \widetilde{\in}\ (F, A)$. By the given condition, $\exists\ (G, A) \in \beta$ such that $E^x_\alpha \ \widetilde{\in}\ (G, A) \ \widetilde{\subseteq}\ (F, A)$. So $(F, A) = \cup_{E^x_\alpha \ \widetilde{\in}\ (F,A)} E^x_\alpha \ \widetilde{\subseteq}\ \cup_{E^x_\alpha \ \widetilde{\in}\ (F,A)} (G, A) \ \widetilde{\subseteq}\ (F, A)$. Hence $(F, A) = \cup_{E^x_\alpha \ \widetilde{\in}\ (F,A)} (G, A)$. Thus (F, A) is the union of some members of β.
Therefore, β forms an open base of τ. □

Proposition 3.14. *A collection β of soft subsets over X forms an open base of a soft topology over X iff the following conditions are satisfied.*

(i) $(\ \widetilde{\phi}, A) \in \beta$;

(ii) $(\ \widetilde{X}, A)$ *is union of the members of* β;

(iii) *If $(F_1, A), (F_2, A) \in \beta$ then $(F_1, A) \ \widetilde{\cap}(F_2, A)$ is union of some members of β i.e. $(F_1, A), (F_2, A) \in \beta$ and $E^x_\alpha \ \widetilde{\in}\ (F_1, A) \ \widetilde{\cap}(F_2, A),\ \exists (F_3, A) \in \beta$ such that $E^x_\alpha \ \widetilde{\in}\ (F_3, A) \ \widetilde{\subseteq}\ (F_1, A) \ \widetilde{\cap}(F_2, A)$.*

Proof. Let β form an open base of a soft topology τ over X. Since $(\ \widetilde{\phi}, A), (\ \widetilde{X}, A) \in \tau$, the conditions (i) and (ii) are obviously satisfied.
Next, let $(F_1, A), (F_2, A) \in \beta$ then $(F_1, A), (F_2, A) \in \tau$ and hence $(F_1, A) \ \widetilde{\cap}\ (F_2, A) \in \tau$. Since β forms an open base of τ, $(F_1, A) \ \widetilde{\cap}(F_2, A)$ can be expressed as the union of some members of β. Therefore, the condition (iii) is also satisfied.

Conversely, let β satisfy the conditions (i), (ii) and (iii). Let $\tau = \{(F, A) :\ (F, A)$ can be expressed as the union of some members of $\beta\}$. We shall now show that τ is a soft topology over X. From conditions (i) and (ii), we get $(\ \widetilde{\phi}, A),\ (\ \widetilde{X}, A) \in \tau$.

Let $(F_i, A) \in \tau,\ \forall i \in I$ and $(F, A) = \cup_{i \in I}(F_i, A)$. Since $(F_i, A) \in \tau,\ \exists$ a sub-collection β_i of β such that (F_i, A) is the union of the members of $\beta_i,\ \forall i \in I$. Let $\beta' = \cup_{i \in I}\beta_i$. Then (F, A) is the union of the members of $\beta' \subseteq \beta$. Therefore $(F, A) \in \tau$.

Again, let $(F_1, A), (F_2, A) \in \tau$. Then there exist subcollections $\beta_1,\ \beta_2$ of β such that $(F_1, A) = \ \widetilde{\cup}_{(G,A) \in \beta_1}(G, A),\ (F_2, A) = \ \widetilde{\cup}_{(H,A) \in \beta_2}(H, A)$. Therefore, $(F_1, A) \ \widetilde{\cap}$

$(F_2, A) = (\widetilde{\bigcup}_{(G,A)\in\beta_1}(G, A)) \widetilde{\cap} (\widetilde{\bigcup}_{(H,A)\in\beta_2}(H, A)) = \widetilde{\bigcup}_{(G,A)\in\beta_1} \widetilde{\bigcup}_{(H,A)\in\beta_2}((G, A) \widetilde{\cap} (H, A))$. Since $(G, A) \in \beta_1,\ (H, A) \in \beta_2 \Rightarrow (G, A), (H, A) \in \beta$. Therefore, $(G, A) \widetilde{\cap} (H, A)$ can be expressed as the union of some members of β. Thus $(F_1, A) \widetilde{\cap} (F_2, A)$ can also be expressed as the union of some members of β. Therefore, $(F_1, A) \widetilde{\cap} (F_2, A) \in \tau$, and hence τ is a soft topology over X. Also from the definition of τ it follows that β forms an open base of τ. □

Definition 3.15. Let (X, A, τ) be a soft topological space. A soft element $E^x_\alpha \in \mathcal{E}$ is said to be a limiting soft element of a soft set (F, A) over X if every open soft set containing E^x_α contains at least one soft element of (F, A) other than E^x_α, i.e. if $\forall (G, A) \in \tau$ with $E^x_\alpha \widetilde{\in} (G, A),\ (F, A) \widetilde{\cap} [(G, A) - E^x_\alpha] \neq (\widetilde{\phi}, A)$.

The union of all limiting soft elements of (F, A) is called the derived soft set of (F, A) and is denoted by $(F, A)'$.

Example 3.16. Let $X = \{x, y, z\},\ A = \{\alpha_1, \alpha_2\}$ and $\tau = \Big\{(\widetilde{\phi}, A) = \big\{\{\phi\}, \{\phi\}\big\}$, $(F_1, A) = \big\{\{x\}, \{y\}\big\},\ (F_2, A) = \big\{\{y\}, \{z\}\big\},\ (F_3, A) = \big\{\{x, y\}, \{y, z\}\big\},\ (\widetilde{X}, A) = \big\{\{X\}, \{X\}\big\}\Big\}$. Then τ is a soft topology over X. Let $(F, A) = \big\{\{x, y\}, \{x, z\}\big\}$. Now $E^x_{\alpha_1} \widetilde{\in} (F_1, A) \in \tau$, but $(F, A) \widetilde{\cap} (F_1, A) = E^x_{\alpha_1}$. Therefore, $E^x_{\alpha_1}$ not a limiting soft element of (F, A). The open soft sets containing $E^y_{\alpha_1}$ are $(F_2, A), (F_3, A), (\widetilde{X}, A)$. Also $(F, A) \widetilde{\cap} (F_2, A) \neq E^y_{\alpha_1},\ (F, A) \widetilde{\cap} (F_3, A) \neq E^y_{\alpha_1}$ and $(F, A) \widetilde{\cap} (\widetilde{X}, A) \neq E^y_{\alpha_1}$. Therefore, $E^y_{\alpha_1}$ is a limiting soft element of (F, A). Similarly, we can show that $E^z_{\alpha_1},\ E^x_{\alpha_2},\ E^y_{\alpha_2},\ E^z_{\alpha_2}$ are limiting soft elements of (F, A). Therefore, $(F, A)' = \big\{\{y, z\}, \{x, y, z\}\big\}$.

Proposition 3.17. *A soft set (F, A) of a soft topological space (X, A, τ) is closed iff (F, A) contains all its limiting soft elements.*

Proof. Let (F, A) be closed and $E^x_\alpha \widetilde{\notin} (F, A)$. Since (F, A) is closed, (F^c, A) is open and $E^x_\alpha \widetilde{\in} (F^c, A)$. Since $(F, A) \widetilde{\cap} (F^c, A) = (\widetilde{\phi}, A)$, E^x_α cannot be a limiting soft element of (F, A). Therefore, (F, A) contains all its limiting elements.

Conversely, let (F, A) contains all its limiting elements. Let $E^x_\alpha \widetilde{\in} (F^c, A)$. By our assumption, E^x_α is not a limiting soft elemnt of (F, A). Then $\exists\ (G, A) \in \tau$ such that $E^x_\alpha \widetilde{\in} (G, A)$ and $(F, A) \widetilde{\cap} (G, A) = (\widetilde{\phi}, A)$. Therefore, $(G, A) \widetilde{\subseteq} (F^c, A)$. Therefore, $(F^c, A) = \widetilde{\bigcup}_{E^x_\alpha \widetilde{\in} (F^c,A)} E^x_\alpha \widetilde{\subseteq} \widetilde{\bigcup}_{E^x_\alpha \widetilde{\in} (F^c,A)} (G, A) \widetilde{\subseteq} (F^c, A)$. Thus $(F^c, A) = \widetilde{\bigcup}_{E^x_\alpha \widetilde{\in} (F^c,A)} (G, A)$. Therefore, (F^c, A) is open and hence (F, A) is closed. □

Proposition 3.18. *For any two soft sets $(F, A), (G, A)$ of a soft topological space (X, A, τ), the following conditions hold.*

(i) $(\widetilde{\phi}, A)' = (\widetilde{\phi}, A)$;
(ii) $(F, A) \widetilde{\subseteq} (G, A) \Rightarrow (F, A)' \widetilde{\subseteq} (G, A)'$;
(iii) $[(F, A) \widetilde{\cup} (G, A)]' = (F, A)' \widetilde{\cup} (G, A)'$;
(iv) $[(F, A) \widetilde{\cup} (F, A)']' \widetilde{\subseteq} (F, A) \widetilde{\cup} (F, A)'$.

Proof. (i) and (ii) follow from definition.

(iii) From (ii) it follows that $(F, A)' \widetilde{\cup} (G, A)' \widetilde{\subseteq} [(F, A) \widetilde{\cup} (G, A)]'$...........(1).

Next, let $E^x_\alpha \widetilde{\notin} (F,A)' \widetilde{\cup} (G,A)'$. Then $\exists\ (H,A),\ (K,A) \in \tau$ such that $E^x_\alpha \widetilde{\in} (H,A),\ E^x_\alpha \widetilde{\in} (K,A),\ (F,A) \widetilde{\cap} [(H,A) - E^x_\alpha] = (\widetilde{\phi},A)$ and $(G,A) \widetilde{\cap} [(K,A) - E^x_\alpha] = (\widetilde{\phi},A)$. Let $(V,A) = (H,A) \widetilde{\cap} (K,A)$. Then $E^x_\alpha \widetilde{\in} (V,A) \in \tau$. Also $(V,A) \widetilde{\cap}[((F,A) \widetilde{\cup} (G,A)) - E^x_\alpha] = [(V,A) \widetilde{\cap}((F,A) - E^x_\alpha)] \widetilde{\cup}[(V,A) \widetilde{\cap}((G,A) - E^x_\alpha)] \widetilde{\subseteq} [(H,A) \widetilde{\cap}((F,A) - E^x_\alpha)] \widetilde{\cup}[(G,A) \widetilde{\cap}((G,A) - E^x_\alpha)] = (\widetilde{\phi},A)$. Therefore, E^x_α is not a limiting soft element of $(F,A) \widetilde{\cup}(G,A)$. Therefore, $E^x_\alpha \widetilde{\in} [(F,A)' \widetilde{\cup}(G,A)']$ and hence $[(F,A) \widetilde{\cup}(G,A)]' \widetilde{\subseteq} (F,A)' \widetilde{\cup}(G,A)'$..........(2).

From (1) and (2) we have $[(F,A) \widetilde{\cup}(G,A)]' = (F,A)' \widetilde{\cup}(G,A)'$.

(iv) Let $E^x_\alpha \widetilde{\in} [(F,A) \widetilde{\cup}(F,A)']'$, we have to show that $E^x_\alpha \widetilde{\in} [(F,A) \widetilde{\cup}(F,A)']$.
If $E^x_\alpha \widetilde{\in} (F,A)$, then $E^x_\alpha \widetilde{\in} [(F,A) \widetilde{\cup}(F,A)']$.
Now we consider the case where $E^x_\alpha \widetilde{\notin}(F,A)$. Since $[(F,A) \widetilde{\cup}(F,A)']' = (F,A)' \widetilde{\cup}[(F,A)']'$ and hence $E^x_\alpha \widetilde{\in} [(F,A) \widetilde{\cup}(F,A)']' \Rightarrow E^x_\alpha \widetilde{\in} (F,A)'$ or $E^x_\alpha \widetilde{\in} [(F,A)']'$.
If $E^x_\alpha \widetilde{\in} (F,A)'$, then $E^x_\alpha \widetilde{\in} [(F,A) \widetilde{\cup}(F,A)']$.
If $E^x_\alpha \widetilde{\notin}(F,A)'$ then $E^x_\alpha \widetilde{\in} [(F,A)']'$. Let (G,A) be any soft open set containing E^x_α. Since E^x_α is a limiting soft element of $(F,A)'$, we have that $(G,A) \widetilde{\cap}[(F,A)' - E^x_\alpha] \neq (\widetilde{\phi},A)$. Let $E^y_\beta \widetilde{\in} (G,A) \widetilde{\cap}[(F,A)' - E^x_\alpha]$. Then (G,A) is an open soft set containing E^y_β, which is a limiting soft element of (F,A). Therefore, $(G,A) \widetilde{\cap}[(F,A) - E^y_\beta] \neq (\widetilde{\phi},A)$.
Let $E^z_\gamma \widetilde{\in} (G,A) \widetilde{\cap}[(F,A) - E^y_\beta]$. Now $E^z_\gamma \neq E^x_\alpha$ (as $E^x_\alpha \widetilde{\notin}(F,A)$) and $E^z_\gamma \widetilde{\in} (G,A) \widetilde{\cap}[(F,A) - E^x_\alpha] \neq (\widetilde{\phi},A)$. Therefore, E^x_α is a limiting soft element of (F,A) i.e. $E^x_\alpha \widetilde{\in} (F,A)'$.
Therefore, $[(F,A) \widetilde{\cup}(F,A)']' \widetilde{\subseteq} (F,A) \widetilde{\cup}(F,A)'$. □

Let us now define the closure of a Soft set in terms of soft limit point.

Definition 3.19. The closure of a soft set (F,A) over X denoted by $\overline{(F,A)}$ is defined by $\overline{(F,A)} = (F,A) \widetilde{\cup}(F,A)'$.

Example 3.20. Let $X,\ A,\ \tau$ be the same as in Example 3.16.
If $(F,A) = \{\{x,y\},\{x,z\}$ then $(F,A)' = \{\{y,z\},\{x,y,z\}$ and $\overline{(F,A)} = \{\{x,y,z\},\{x,y,z\}$.

Proposition 3.21. *A soft set (F,A) over X is soft closed iff $(F,A) = \overline{(F,A)}$.*

Proof. Let (F,A) be soft closed. Then (F^c,A) is soft open. Then no soft element belonging to (F^c,A) is a soft limiting element of (F,A).
Therefore, $(F,A)' \widetilde{\subseteq} (F,A) \Rightarrow \overline{(F,A)} = (F,A)$.

Conversely, let $\overline{(F,A)} = (F,A)$. Then $(F,A)' \widetilde{\subseteq} (F,A)$. So, for any $E^x_\alpha \widetilde{\in} (F^c,A)$, $\exists$ soft open set (G,A) such that $E^x_\alpha \widetilde{\in} (G,A) \widetilde{\subseteq} (F^c,A)$. So (F^c,A) can be expressed as a union of soft open sets and hence (F^c,A) is soft open. Therefore, (F,A) is soft closed. □

Remark 3.22. That Definitions 3.19 and 2.21(i) are equivalent is shown by the following proposition.

Proposition 3.23. *The closure of a soft set (F,A) over X is the smallest closed soft set containing (F,A), i.e. it is the meet of all closed soft sets containing (F,A).*

Proof. Since $\overline{[\overline{(F,A)}]} = \overline{(F,A)}$, therefore $\overline{(F,A)}$ is closed. Also $(F,A) \widetilde{\subseteq} \overline{(F,A)}$, therefore $\overline{(F,A)}$ is a closed soft set containing (F,A). Let (G,A) be any closed soft set containing (F,A). Then $(F,A) \widetilde{\subseteq} (G,A) \Rightarrow \overline{(F,A)} \widetilde{\subseteq} \overline{(G,A)} = (G,A)$. Thus $\overline{(F,A)}$ is a smallest closed soft set containing (F,A). □

Proposition 3.24. *For any two soft sets $(F,A),(G,A)$ of a soft topological space (X,A,τ), the following conditions hold.*
(i) $\overline{(\widetilde{\phi},A)} = (\widetilde{\phi},A)$;
(ii) $(F,A) \widetilde{\subseteq} \overline{(F,A)}$;
(iii) $\overline{[(F,A)\,\widetilde{\cup}(G,A)]} = \overline{F,A)}\,\widetilde{\cup}\overline{(G,A)}$;
(iv) $\overline{\overline{(F,A)}} = \overline{(F,A)}$.

Definition 3.25. Let X be a non-empty set. A mapping $C : S(X) \to S(X)$ is said to be a soft closure opetator over X if it satisfies the following conditions:
(i) $C[(\widetilde{\phi},A)] = (\widetilde{\phi},A)$;
(ii) $(F,A) \widetilde{\subseteq} C[(F,A)]$;
(iii) $C[(F,A)\,\widetilde{\cup}(G,A)] = C[(F,A)]\,\widetilde{\cup}C[(G,A)]$;
(iv) $C[C[(F,A)]] = C[(F,A)]$.

Proposition 3.26. *If C is a soft closure operator over X, then there exists a unique soft topology τ over X such that $C[(F,A)]$ is the τ-closure of (F,A).*

Proof. Let $\mathcal{F} = \{(F,A) : C[(F,A)] = (F,A)\}$. Clearly $(\widetilde{\phi},A)$, $(\widetilde{X},A) \in \mathcal{F}$. Let $\{(F_i,A),\ i \in I\}$ be any collection of members of $\mathcal{F}$ and $(F,A) = \widetilde{\cap}_{i\in I}(F_i,A)$. So $\forall i \in I,\ (F,A) \widetilde{\subseteq} (F_i,A) \Rightarrow C[(F,A)] \widetilde{\subseteq} C[(F_i,A)] = (F_i,A)$. Thus $C[(F,A)] \widetilde{\subseteq} (F,A)$ and also $(F,A) \widetilde{\subseteq} C[(F,A)] \Rightarrow C[(F,A)] = (F,A)$. Therefore, $(F,A) \in \mathcal{F}$.

Next, let $(F,A),(G,A) \in \mathcal{F}$ then $C[(F,A)\,\widetilde{\cup}(G,A)] = C[(F,A)]\,\widetilde{\cup}C[(G,A)] = (F,A)\,\widetilde{\cup}(G,A) \Rightarrow (F,A)\,\widetilde{\cup}(G,A) \in \mathcal{F}$. Now, let $\tau = \{(F,A) : (F^c,A) \in \mathcal{F}\}$. Then τ is a soft topology over X. Clearly $\mathcal{F}$ is the family of all τ-closed soft sets over X. □

Definition 3.27. Let (X,A,τ) be a soft topological space. For any soft set (F,A) over X, define $\tau_{(F,A)} = \{(G,A)\,\widetilde{\cap}(F,A) : (G,A) \in \tau\}$. Since $(\widetilde{\phi},A) \in \tau$ and $(\widetilde{\phi},A)\,\widetilde{\cap}(F,A) = (\widetilde{\phi},A) \Rightarrow (\widetilde{\phi},A) \in \tau_{(F,A)}$. Also, since $(\widetilde{X},A) \in \tau$ and $(\widetilde{X},A)\,\widetilde{\cap}(F,A) = (F,A) \Rightarrow (F,A) \in \tau_{(F,A)}$.

Again, let $\{(G_i,A) : i \in I\}$ be any collection of members of $\tau_{(F,A)}$ and $(G,A) = \widetilde{\cup}_{i\in I}(G_i,A)$. Since $\forall i \in I,\ (G_i,A) \in \tau_{(F,A)},\ \exists (H_i,A) \in \tau$ such that $(G_i,A) = (H_i,A)\,\widetilde{\cap}(F,A)$. Therefore, $(G,A) = \widetilde{\cup}_{i\in I}(G_i,A) = \widetilde{\cup}_{i\in I}[(H_i,A)\,\widetilde{\cap}(F,A)] = [\widetilde{\cup}_{i\in I}(H_i,A)]\,\widetilde{\cap}(F,A) \in \tau_{(F,A)}$.

Finally, let $(G_1,A),(G_2,A) \in \tau_{(F,A)}$ then $\exists\ (H_1,A),(H_2,A) \in \tau$ such that $(G_1,A) = (H_1,A)\,\widetilde{\cap}(F,A)$ and $(G_2,A) = (H_2,A)\,\widetilde{\cap}(F,A)$. Therefore, $(G_1,A)\,\widetilde{\cap}(G_2,A) = [(H_1,A)\,\widetilde{\cap}(H_2,A)]\,\widetilde{\cap}(F,A) \in \tau_{(F,A)}$. Therefore, $\tau_{(F,A)}$ is a soft topology on (F,A).

This soft topology is called soft relative topology of τ on (F,A), and $[(F,A),\tau_{(F,A)}]$ is called subspace of (X,A,τ).

Proposition 3.28. *Let (X,A,τ) be a soft topological space and (F,A) be any soft set over X. Then*

(i) $(G,A) \widetilde{\subseteq} (F,A)$ *is closed in* $\tau_{(F,A)}$ *iff* $(G,A) = (F,A) \widetilde{\cap}(K,A)$ *where* (K,A) *is closed in* (X,A,τ)*;*

(ii) *a soft element* E^x_α *of* (F,A) *is a limiting soft element of a soft subset* (G,A) *of* (F,A) *in* $(F,A,\tau_{(F,A)})$ *iff* E^x_α *is a limiting soft element of* (G,A) *in* (X,A,τ)*;*

(iii) $\overline{[(G,A)_{\tau_{(F,A)}}} = \overline{[(G,A)]_\tau} \widetilde{\cap}(F,A)$*;*

(iv) β *be an open base of* τ *then* $\beta_{(F,A)} = \{(G,A) \widetilde{\cap}(F,A) :\ (G,A) \in \beta\}$ *is an open base of* $\tau_{(F,A)}$.

Proof. (i) Let (G,A) be closed in $\tau_{(F,A)}$. Then $(F,A) - (G,A) = (F,A) \widetilde{\cap} (G^c,A) \in \tau_{(F,A)}$. Therefore, $\exists\ (H,A) \in \tau$ such that $(F,A) \widetilde{\cap} (H,A) = (F,A) - (G,A)$. Now $(G,A) = (F,A) - [(F,A) - (G,A)] = (F,A) \widetilde{\cap} [(F,A) \widetilde{\cap} (H,A)]^c = (F,A) \widetilde{\cap} [(F,A)^c \widetilde{\cup} (H,A)^c] = (F,A) \widetilde{\cap} (H^c,A)$, where (H^c,A) is closed in (X,A,τ). Therefore, the given condition is satisfied.

Conversely, let $(G,A) = (F,A) \widetilde{\cap} (K,A)$ where (K,A) is closed in (X,A,τ). Therefore, $(F,A) - (G,A) = (F,A) \widetilde{\cap} (G^c,A) = (F,A) \widetilde{\cap} [(F,A) \widetilde{\cap} (K,A)]^c = (F,A) \widetilde{\cap} [(F,A)^c \widetilde{\cup} (K,A)^c] = (F,A) \widetilde{\cap} (K^c,A)$. Since (K,A) is soft closed in (X,A,τ), (K^c,A) is soft open in (X,A,τ) and hence $(F,A) - (G,A)$ is soft open in $\tau_{(F,A)}$. So (G,A) is soft closed in $\tau_{(F,A)}$.

(ii) Let $E^x_\alpha \widetilde{\in} (F,A)$ be a limiting soft element of soft subset (G,A) of (F,A) in $(F,A,\tau_{(F,A)})$. Let (H,A) be any soft open set in (X,A,τ) containing E^x_α. Then $(H,A) \widetilde{\cap} (F,A)$ is a $\tau_{(F,A)}$ soft open set containing E^x_α. Since E^x_α is a soft limiting element of (G,A) in $(F,A,\tau_{(F,A)})$, $[(H,A) \widetilde{\cap} (F,A)] \widetilde{\cap} [(G,A) - E^x_\alpha] \neq (\widetilde{\phi},A)$. Therefore $(H,A) \widetilde{\cap} [(G,A) - E^x_\alpha] \neq (\widetilde{\phi},A)$. Therefore, E^x_α is a soft limiting element of (G,A) in (X,A,τ).

Conversely, let E^x_α be a soft limiting element of (G,A) in (X,A,τ). Let (H,A) be any soft open set in $\tau_{(F,A)}$ containing E^x_α. Then $\exists\ (K,A) \in \tau$ such that $(H,A) = (F,A) \widetilde{\cap} (K,A)$. Since E^x_α is a soft limiting element of (G,A) in (X,A,τ), $(K,A) \widetilde{\cap} [(G,A) - E^x_\alpha] \neq (\widetilde{\phi},A)$. Therefore, $[(K,A) \widetilde{\cap} (F,A)] \widetilde{\cap} [(G,A) - E^x_\alpha] \neq (\widetilde{\phi},A)$. Therefore, $(H,A) \widetilde{\cap} [(G,A) - E^x_\alpha] \neq (\widetilde{\phi},A)$. Therefore, E^x_α is a soft limiting element of (G,A) in $(F,A,\tau_{(F,A)})$.

(iii) $\overline{(G,A)_{\tau_{(F,A)}}} = \widetilde{\cap}\{(H,A);\ (G,A)\widetilde{\subseteq}\ (H,A)$ and (H,A) is $\tau_{(F,A)}$ closed$\}$ $= \widetilde{\cap}\{(P,A) \widetilde{\cap} (F,A);\ (G,A)\widetilde{\subseteq} (P,A)$ and (P,A) is τ closed$\}$ $= (F,A) \widetilde{\cap} [\widetilde{\cap}\{(P,A);\ (G,A)\widetilde{\subseteq} (P,A)$ and (P,A) is τ closed$\}] = (F,A) \widetilde{\cap} \overline{(G,A)_\tau}$.

(iv) Straightforward. □

Definition 3.29. A soft element E^x_α is said to be a soft interior element of a soft set $(F,A$ if there exists $(G,A) \in \tau$ such that $E^x_\alpha \widetilde{\in} (G,A) \widetilde{\subseteq} (F,A)$. The soft interior of a soft set (F,A) over X, denoted by $(F,A)^0$ is defined to be the union of all soft interior element $E^x_\alpha \widetilde{\in} (F,A)$.

Proposition 3.30. *For any two soft sets* (F,A), (G,A) *over* X,

(i) $(F,A)^0$ *is open;*

(ii) (F,A) *is open iff* $(F,A) = (F,A)^0$*;*

(iii) $[(F,A)^0]^0 = (F,A)^0$*;*

(iv) $(\ \widetilde{\phi},A)^0 = (\ \widetilde{\phi},A)$*;*

(v) $(F,A) \widetilde{\subseteq} (G,A) \Rightarrow (F,A)^0 \widetilde{\subseteq} (G,A)^0$*;*

(vi) $[(F,A)\ \widetilde{\cap}(G,A)]^0 = (F,A)^0\ \widetilde{\cap}(G,A)^0$.

Definition 3.31. Let (X,A,τ) and (Y,A,ν) be two soft topological spaces. A mapping $f:(X,A,\tau)\to(Y,A,\nu)$ is said to be *soft continuous* at $E^x_\alpha \widetilde{\in} S(X)$ if for any soft nbd (G,A) of $f(E^x_\alpha)$ $\exists$ a soft nbd (F,A) of E^x_α such that $f[(F,A)] \widetilde{\subseteq} (G,A)$. f is said soft continuous over (X,A,τ) if it is soft continuous at all E^x_α'$\widetilde{\in} S(X)$.

Remark 3.32. Clearly, a mapping $f:(X,A,\tau)\to(Y,A,\nu)$ is *soft continuous* at a soft element $E^x_\alpha \widetilde{\in} S(X)$ if $\forall\ (G,A)\in\nu$ such that $f(E^x_\alpha)\ \widetilde{\in}\ (G,A)$, $\exists\ (F,A)\in\tau$ such that $E^x_\alpha\ \widetilde{\in}\ (F,A)$ and $f[(F,A)]\ \widetilde{\subseteq}\ (G,A)$.

Proposition 3.33. *Let (X,A,τ) and (Y,A,ν) be two soft topological spaces and $f:(X,A,\tau)\to(Y,A,\nu)$ be a mapping. Then the following conditions are equivalent.*

(i) *f is soft continuous;*

(ii) *For any soft open set (G,A) in (Y,A,ν), $f^{-1}[(G,A)]$ is soft open in (X,A,τ);*

(iii) *For any soft closed set (F,A) in (Y,A,ν), $f^{-1}[(F,A)]$ is soft closed in $(X,A,\tau;$.*

(iv) $\forall\ (F,A)\in S(X)$, $f[\overline{(F,A)}]\ \ \widetilde{\subseteq}\ \overline{f[(F,A)]}$.

Proof. Let f be soft continuous. Also, let $(G,A)\in\nu$ and $E^x_\alpha\ \widetilde{\in}\ f^{-1}[(G,A)] \Rightarrow f(E^x_\alpha)\ \widetilde{\in}\ (G,A)$. Since (G,A) is soft open, (G,A) is a soft nbd of $f(E^x_\alpha)$. Therefore, by the definition, $\exists$ a nbd (F,A) of E^x_α such that $f[(F,A)]\ \widetilde{\subseteq}\ (G,A)$. Therefore, $(F,A)\ \widetilde{\subseteq}\ f^{-1}[(G,A)]$. Since (F,A) is a nbd of E^x_α, $f^{-1}[(G,A)$ is also a nbd of E^x_α. Thus $f^{-1}[(G,A)]$ is a soft nbd of each of its elements. Therefore, $f^{-1}[(G,A)]\in\tau$. Therefore, $(i)\Rightarrow(ii)$.

Next, let (F,A) be soft closed set in (Y,A,ν). Then $[(\widetilde{Y},A)-(F,A)]\in\nu$. Therefore, by (ii) we have $f^{-1}[(\widetilde{Y},A)-(F,A)] = [(\widetilde{X},A)-f^{-1}(F,A)]\in\tau$. Therefore $f^{-1}[(F,A)]$ is closed in (X,A,τ). Therefore, $(ii)\Rightarrow(iii)$.

Again, from (iii), we get that the conceopt of soft continuity is equivalent to inverse images closed soft sets being soft closed. Therefore, $f^{-1}\overline{f[(F,A)]}$ is closed and containing (F,A). So $\overline{(F,A)}\ \widetilde{\subseteq}\ f^{-1}\overline{f[(F,A)]} \Rightarrow f[\overline{(F,A)}]\ \widetilde{\subseteq}\ \overline{f[(F,A)]}$. Therefore $(iii)\Rightarrow(iv)$.

Also let $E^x_\alpha\ \widetilde{\in}\ S(X)$ and (G,A) be any soft nbd of $f(E^x_\alpha)$. Then $\exists\ (H,A)\in\nu$ such that $f(E^x_\alpha)\ \widetilde{\in}\ (H,A)\ \widetilde{\subseteq}\ (G,A)$. Then (H^c,A) is a soft closed in (Y,A,ν). Therefore, from (iv) we have $f[\overline{f^{-1}[(H^c,A)]}]\ \ \widetilde{\subseteq}\ \overline{f[f^{-1}[(H^c,A)]]}\ \ \widetilde{\subseteq}\ \ \overline{(H^c,A)} = (H^c.A)$
$\Rightarrow \overline{f^{-1}[(H^c,A)]}\ \ \widetilde{\subseteq}\ \ f^{-1}[(H^c,A)]$. Therefore, $\overline{f^{-1}[(H^c,A)]} = f^{-1}[(H^c,A)]$. Therefore, $f^{-1}[(H^c,A)]$ is soft closed in $(X,A,\tau) \Rightarrow [f^{-1}[(H,A)]]^c$ is soft closed in (X,A,τ). Therefore, $f^{-1}[(H,A)]\in\tau$. And $E^x_\alpha\ \widetilde{\in}\ f^{-1}[(H,A)] \Rightarrow f^{-1}[(H,A)]$ is a soft nbd of E^x_α such that $f[f^{-1}(H,A)]\ \widetilde{\subseteq}\ (H,A)\ \widetilde{\subseteq}\ (G,A)$. Therefore, f is soft continuous. Therefore, $(iv)\Rightarrow(i)$.
Therefore, the above statements are equivalent. □

Definition 3.34. Let (X,A,τ) and (Y,A,ν) be two soft topological spaces. A mapping $f:(X,A,\tau)\to(Y,A,\nu)$ is said to be

(i) *soft open* if $(F,A)\in\tau\Rightarrow f(F,A)\in\nu$;

(ii) *soft closed* if (F,A) is closed in $(X,A,\tau)\Rightarrow f(F,A)$ is closed in (Y,A,ν).

Proposition 3.35. *Let (X,A,τ) and (Y,A,ν) be two soft topological spaces. A mapping $f:(X,A,\tau)\to(Y,A,\nu)$ is*

(i) *soft open iff* $\forall\ (F,A) \in S(X),\ f[(F,A)^0]\ \widetilde{\subseteq}\ [f(F,A)]^0$;
(ii) *soft closed iff* $\forall\ (F,A) \in S(X),\ \overline{f(F,A)}\ \widetilde{\subseteq}\ f[\overline{(F,A)}]$.

Proof. Proof is similar to that of Proposition 3.33. □

Definition 3.36. Let (X, A, τ) and (Y, A, ν) be two soft topological spaces. A mapping $f : (X, A, \tau) \to (Y, A, \nu)$ is said to be a soft homeomorphism if
(i) $f : X \to Y$ is bijective mapping;
(ii) $f : (X, A, \tau) \to (Y, A, \nu)$, and $f^{-1} : (Y, A, \nu) \to (X, A, \tau)$ are soft continuous.

Proposition 3.37. *Let (X, A, τ) and (Y, A, ν) be two soft topological spaces. For a bijective mapping $f : (X, A, \tau) \to (Y, A, \nu)$, the following statements are equivalent.*
(i) $f : (X, A, \tau) \to (Y, A, \nu)$ *is soft open;*
(ii) $f : (X, A, \tau) \to (Y, A, \nu)$ *is soft closed;*
(iii) $f^{-1} : (Y, A, \nu) \to (X, A, \tau)$, *is soft continuous.*

Proposition 3.38. *Let (X, A, τ) and (Y, A, ν) be two soft topological spaces. For a bijective mapping $f : (X, A, \tau) \to (Y, A, \nu)$, the following statements are equivalent.*
(i) $f : (X, A, \tau) \to (Y, A, \nu)$ *is soft homeomorphism;*
(ii) $f : (X, A, \tau) \to (Y, A, \nu)$ *and* $f^{-1} : (Y, A, \nu) \to (X, A, \tau)$ *are soft continuous;*
(iii) $f : (X, A, \tau) \to (Y, A, \nu)$ *is both soft continuous and soft open;*
(iv) $f : (X, A, \tau) \to (Y, A, \nu)$ *is both soft continuous and soft closed;*
(v) $f[\overline{(F,A)}] = \overline{f(F,A)},\ \forall\ (F,A) \in S(X)$.

4. Conclusion

In this paper, we have introduced the notions of the various facets of a neighbourhood system in soft topological spaces. Using these, ideas, we have studied different soft topological properties in terms of neighbourhoods. As a next step, there are scopes for investigating the neighbourhood properties of soft topological groups at their identity elements. Seperation axioms, connectedness etc. in the setting of soft topological groups as well, we think.

Acknowledgements. The authors express their sincere thanks to the reviewers for their valuable suggestions. The authors are also thankful to the editors-in-chief and managing editors for their important comments which hepled improve the presentation of the paper. The present work is partially supported by the Special Assistance Programme (SAP) of UGC, New Delhi, India [Grant No. F 510/8/DRS/2009 (SAP -II)].

References

[1] H. Aktas and N. Cagman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726–2735.
[2] N. Cagman, S. Karatas and S. Enginoglu, Soft topology, Comput. Math. Appl. 62 (2011) 351–358.
[3] F. Feng, Y. B. Jun and X. Zhao, Soft semirings, Comput. Math. Appl. 56 (2008) 2621–2628.
[4] H. Hazra, P. Majumdar and S. K. Samanta, Soft topology, Fuzzy Inform. Eng. 4(1) (2012) 105–115.
[5] S. Hussain and B. Ahmad, Some properties of soft topological spaces, Comput. Math. Appl. 62 (2011) 4058–4067.
[6] A. Kharal and B. Ahmad, Mappings on soft classes, New Math. Nat. Comput. 7(3) (2011) 471–481.

[7] P. K. Maji, R. Biswas and A. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[8] P. K. Maji, R. Biswas and A. Roy, Fuzzy soft sets, J. Fuzzy Math. 9(3) (2001) 589–602.
[9] P. K. Maji, R. Biswas and A. Roy, Intuitionistic fuzzy soft sets, J. Fuzzy Math. 9(3) (2001) 677–691.
[10] P. Majumdar and S. K. Samanta, On soft mappings, Comput. Math. Appl. 60 (2010) 2666–2672.
[11] D. Molodtsov, Soft set theory-First Results, Comput. Math. Appl. 37 (1999) 19–31.
[12] Sk. Nazmul and S. K. Samanta, Fuzzy soft group, J. Fuzzy Math. 19(1) (2011) 101–114
[13] M. Shabir and M. Ali, Soft ideals and generalized fuzzy ideals in semigroups, New Math. Nat. Comput. 5(3) (2009) 599–615.
[14] M. Shabir, M. Naz, On soft toplogical spaces, Comput. Math. Appl. 61(7) (2011) 1786–1799.
[15] I. Zorlutuna, M. Akdag, W. K. Min and S. Atmaca, Remarks on soft topological spaces, Ann. Fuzzy Math. Inform. 3(2) (2012) 171–185.

Sk. Nazmul (sk.nazmul_math@yahoo.in)
Department of Mathematics, Govt. College of Education, Burdwan, Kazirhat, Lakurdi, Burdwan-713102, West Bengal, India.

S. K. Samanta (syamal_123@yahoo.co.in)
Department of Mathematics, Visva-Bharati, Santiniketan-731235, West Bengal, India.