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Homology modules of fuzzy soft modules

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Abstract. Molodtsov initiated the concept of soft sets in [17]. Maji et al. defined some operations on soft sets in [15]. In this paper, we introduce the concept of the homology module of fuzzy soft chain complexes. Finally, we investigate about whether or not the exactness of the homology sequence of fuzzy soft modules.

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1. Introduction

Many practical problems in economics, engineering, environment, social science, medical science etc. cannot be dealt with by classical methods, because classical methods have inherent difficulties. The reason for these difficulties may be due to the inadequacy of the theories of parameterization tools. Molodtsov [17] initiated the concept of soft set theory as a new mathematical tool for dealing with uncertainties. Maji et al. [14, 15, 16] research deal with operations over soft set. The investigations over fuzzy soft set were made at the studies [9, 10, 20]. Rosenfeld [19] proposed the concept of fuzzy groups in order to establish the algebraic structures of fuzzy sets. Firstly, fuzzy sets was given by Zadeh[22] and operations fuzzy sets were given in the studies [6, 11, 24]. Aktaş and Çağman [2] defined soft groups and compared soft sets with fuzzy sets and rough sets. After the definition of fuzzy soft groups is given by some authors [4, 10]. F. Feng et al. [7] gave soft semirings and U. Acar et al. [1] introduced initial concepts of soft rings. Definition of fuzzy module is given by some authors [3, 12, 13, 23]. Qiu- Mei Sun et al. [21] defined soft modules and investigated their basic properties. Category of chain complexes of soft modules was given in the study[18].

Ç. Gunduz and S. Bayramov defined fuzzy soft modules and intiutionistic fuzzy soft modules and investigated their basic properties [9].

In this study, the chain complexes category of fuzzy soft modules is defined and the homology module of fuzzy chain complexes is given. This fuzzy soft homology module which is invariant according to fuzzy soft homotopy is proved. Generally, the sequence of homology modules of fuzzy modules is not exact [5]. Thus the homology sequence of fuzzy soft modules is not also exact. At the end of the study, under the some conditions exactness is obtained.

2. Preliminaries

In this section, we recall some basic concepts of soft set theory. Let E be all of convenient paramter set for the universe X.

Definition 2.1 ([15])**.** Let X be an initial universe set and E be a set of parameters. A pair (F, E) is called a soft set over X if only if F is a mapping from E into the set of all subsets of the sets X, i.e., $F : E \to P(X)$, where $P(X)$ is the set of X.

In other words, the soft set is a parameterized family of subsets of the set X. Every set $F(e)$, for every $e \in E$, may be considered as the set of $e-$elements of the soft set (F, E), or as the set of $e-$approximate elements of the soft set.

According to this manner, a soft set (F, E) is given as consisting of collection of approximations:

$$(F, E) = \{F(e) : e \in E\}$$

Definition 2.2 ([16])**.** Let I^X denote the set of all fuzzy sets on X and $A \subset E$. A pair (f, A) is called a fuzzy soft set over X, where f is a mapping from A into I^X. That is, for each $a \in A$, $f(a) = f_a : X \to I$ is a fuzzy set on X.

Definition 2.3 ([16])**.** For two fuzzy soft sets (f, A) and (g, B) over a common universe X, we say that (f, A) is a fuzzy soft subset of (g, B) and write $(f, A) \subseteq (g, B)$ if

(1) $A \subset B$ and
(2) For each $a \in A$, $f_a \leq g_a$, that is, f_a is fuzzy subset of g_a.

Definition 2.4 ([16])**.** Two fuzzy soft sets (f, A) and (g, B) over a common universe X are said to be equal if $(f, A) \subseteq (g, B)$ and $(g, B) \subseteq (f, A)$.

Definition 2.5 ([16])**.** The intersection of two fuzzy soft sets (f, A) and (g, B) over a common universe X is the fuzzy soft set (h, C), where $C = A \cap B$ and $\forall \varepsilon \in C$, $h_c = f_c \wedge g_c$. It is written as $(f, A) \cap (g, B) = (h, C)$.

Definition 2.6 ([16])**.** The union of two fuzzy soft sets (f, A) and (g, B) over a common universe X is the fuzzy soft set (h, C), where $C = A \cup B$ and,

$$h(c) = \begin{cases} f_c, & \text{if } c \in A - B, \\ g_c, & \text{if } c \in B - A, \forall c \in C \\ f_c \vee g_c, & \text{if } c \in A \cap B. \end{cases}$$

This relationship is denoted by $(f, A) \cup (g, B) = (h, C)$.

Definition 2.7 ([16])**.** If (f, A) and (g, B) are two soft sets, then $(f, A)AND(g, B)$ is denoted as $(f, A) \wedge (g, B)$. $(f, A) \wedge (g, B)$ is defined as $(h, A \times B)$ where $h(a, b) = h_{a,b} = f_a \wedge g_b, \ \forall (a, b) \in A \times B$.

Throughout this subsection, let M be a left R-module, A be any nonempty set. $F : A \to P(M)$ refer to a set-valued function and the pair (F, A) is a soft set over M.

Definition 2.8 ([21])**.** Let (F, A) be a soft set over $M.(F, A)$ is said to be a soft module over M if and only if $F(x) < M$ for all $x \in A$.

Definition 2.9 ([21])**.** Let (F, A) and (G, B) be two soft modules over M and N respectively. Then $(F, A) \times (G, B) = (H, A \times B)$ is defined as $H(x, y) = F(x) \times G(y)$ for all $(x, y) \in (A \times B)$.

Definition 2.10 ([21])**.** Let (F, A) and (G, B) be two soft modules over M and N respectively, $f : M \to N$, $g : A \to B$ be two functions. Then we say that (f, g) is a soft homomorphism if the following conditions are satisfied:

(1) $f : M \to N$ is homomorphism of module;
(2) $g : A \to B$ is a mapping;
(3) $f(F(x)) = G(g(x)), \ \forall x \in A$.

Now, R is an ordinary ring. Let M be a left (or right) $R-$module, and let $A \neq \varnothing$ be a set. $PF(M)$ denotes the family of fuzzy sets over M.

Definition 2.11 ([9])**.** Let (F, A) be a soft set over M. Then (F, A) is said to be a fuzzy soft module over M iff $\forall a \in A$, $F(a)$ is a fuzzy submodule of M and denoted as F_a.

Definition 2.12 ([9])**.** Let (F, A) and (H, B) be two fuzzy soft modules over M and N respectively, and let $f : M \to N$ be a homomorphism of modules, and let $g : A \to B$ be a mapping of sets. Then we say that $(f, g) : (F, A) \to (H, B)$ is a fuzzy soft homomorphism of fuzzy soft modules, if the following condition is satisfied:

$$f(F_a) = H(g(a)) = H_{g(a)}.$$

We say that (F, A) is a fuzzy soft homomorphic to (H, B).

Note that for $\forall a \in A, \ f : (M, F_a) \to (N, H_{g(a)})$ is a fuzzy homomorphism of fuzzy modules.

Fuzzy soft modules and their morphisms is consists of a category. This category is denoted FSM.

Let $M^{'} = \ker f$. Define $F^{'} : A \to PF(M^{'})$ by $F^{'}_a = F_a|_{M'}$. Then $(F^{'}, A)$ is a fuzzy soft module over $M^{'}$. It is clear that this module is a fuzzy soft submodule of (F, A).

Definition 2.13 ([9])**.** $(F^{'}, A)$ is said to be kernel of (f, g) and denoted by $\ker(f, g)$.

Now, let $B^{'} = g(A)$. Then for all $b \in B^{'}$, there exists $a \in A$ such that $g(a) = b$. Let $N^{'} = Imf < N$. We define the mapping $H^{'} : B^{'} \to PF(N^{'})$ as $H^{'}(b^{'}) = (H(g(a))|_{N'}$. Since (f, g) is a fuzzy soft homomorphism, $f(F_a) = H_{g(a)}$ is satisfied for all $a \in A$. Then the pair $(H^{'}, B^{'})$ is a fuzzy soft module over $N^{'}$ and $(H^{'}, B^{'})$ is a fuzzy soft submodule of (H, B).

Definition 2.14 ([9]). $(H^{'}, B^{'})$ is said to be image of (f, g) and denoted by $Im(f, g)$.

Proposition 2.15 ([9]). *Let (F, A) be a fuzzy soft module over M and N be an $R-$module and $f : M \to N$ be a homomorphism of $R-$modules. Then $(f(F), A)$ is a fuzzy soft module over N.*

Theorem 2.16 ([9]). *The category of fuzzy soft modules has zero objects, sums, product, kernel and cokernel.*

Definition 2.17 ([8]). Let $IFS(X)$ denote the set of all intiutionistic fuzzy sets on X and $A \subset E$. A pair (F, A) is called an intiutionistic fuzzy soft set over X, where F is a mapping from A into $IFS(X)$. That is, for each $a \in A$, $F(a) = (F_a, F^a) : X \to I$ is an intiutionistic fuzzy set on X, where $F_a, F^a : X \to I$ are fuzzy sets.

By $fgm_{\Lambda}^{\mathbb{Z}}$ we mean the category of fuzzy graded (left) $\Lambda-$modules. An object Q_M in $fgm_{\Lambda}^{\mathbb{Z}}$ is a family $\{Q_{M_i}^{i}\},\ i \in \mathbb{Z}$ of obobjects of $\Lambda-$fzmod, a morphism $\widetilde{\phi} : Q_m \to Q_m$ of degree p is a family

$$\{\widetilde{\phi}_n : Q_{M_i}^{i} \to Q_{M_i+p}^{i}\},\ i \in \mathbb{Z}$$

of fuzzy module homomorphisms[3].

Definition 2.18 ([3]). A fuzzy chain complex $Q_C = \{Q_{C_n}^{n}, \widetilde{\partial}_n\}$ over Λ is an object in $fgm_{\Lambda}^{\mathbb{Z}}$ together with a fuzzy endomorphism $\widetilde{\partial} : Q_C \to Q_C$ of degree -1 with $\widetilde{\partial}\ \widetilde{\partial} = 0$. In fact, we have a family $\{Q_{M_i}^{i}\},\ i \in \mathbb{Z}$ of objects of $\Lambda-$fzmod and a family of morphisms of $\Lambda-$fzmod $\{\widetilde{\partial}_n : Q_{M_n}^{n} \to Q_{Mn-1}^{n-1}\},\ n \in \mathbb{Z}$ such that $\widetilde{\partial}_n\widetilde{\partial}_{n+1} = 0$. If $Q_C = \{Q_{C_n}^{n}, \widetilde{\partial}_n\}$ is a fuzzy complex, then we use the following notation:

$$Q_C :\ \ldots \to Q_{Mn+1}^{n+1} \to Q_{M_n}^{n} \to Q_{Mn-1}^{n-1} \to \ldots$$

The morphism $\widetilde{\partial}$ is called the fuzzy differential (or fuzzy boundary operator). A morphism of fuzzy complexes or a fuzzy chain map $\widetilde{\phi} : Q_C \to v_D$ is a morphism of degree 0 in $fgm_{\Lambda}^{\mathbb{Z}}$ such that $\widetilde{\phi}\widetilde{\partial} = \widetilde{\partial}^{'}\widetilde{\phi}$, where $\widetilde{\partial}^{'}$ denotes the fuzzy differential in v_D.

Remark 2.19 ([3]). Let $Q = \{Q_{C_n}^{n}, \widetilde{\partial}_n\}$ be a fuzzy chain complex. The condition $\widetilde{\partial}\ \widetilde{\partial} = 0$ implies that $Im\widetilde{\partial}_{n+1} \subseteq \ker \widetilde{\partial}_n$, $n \in \mathbb{Z}$. Hence we can associate with Q_C the fuzzy graded module

$$H(Q_C) = \{H_n(Q_C)\},$$

where

$$H_n(Q_C) = \bar{Q_n} \ker \partial_n / Im\partial_{n+1},$$

$\bar{Q_n}$ is the fuzzy quotient of $\ker \widetilde{\partial}_n$ by $Im\partial_{n+1}$, $n \in \mathbb{Z}$. Then $H(Q_C)(H_n(Q_C))$ is called the (nth) fuzzy homology module of Q_C.

Definition 2.20 ([3]). A fuzzy homotopy $\widetilde{\Sigma} : \widetilde{\varphi} \to \widetilde{\psi}$ between two fuzzy chain maps $\widetilde{\varphi}, \widetilde{\psi} : Q_C \to v_D$ is a morphism of degree $+1$ of fuzzy graded modules $\widetilde{\Sigma} : Q_C \to v_D$ such that $\widetilde{\psi} - \widetilde{\varphi} = \widetilde{\partial}\widetilde{\Sigma} + \widetilde{\Sigma}\widetilde{\partial}$, İ.e., such that for $n \in \mathbb{Z}$.

$$\widetilde{\psi} - \widetilde{\varphi} = \widetilde{\partial}_{n+1}\widetilde{\Sigma}_n + \widetilde{\Sigma}_{n-1}\widetilde{\partial}_n.$$

We say that $\widetilde{\varphi}, \widetilde{\psi}$ are fuzzy homotopic, and write $\widetilde{\varphi} \cong \widetilde{\psi}$ if there exists a fuzzy homotopy $\widetilde{\Sigma} : \widetilde{\varphi} \to \widetilde{\psi}$.

Proposition 2.21 ([3]). *If two fuzzy chain maps $\widetilde{\varphi}, \widetilde{\psi} : Q_C \to v_D$ are fuzzy homotopic, than*

$$H(\widetilde{\varphi}) = H(\widetilde{\psi}) : H(Q_C) \to H(v_D).$$

3. Homology modules of fuzzy soft modules

Definition 3.1. Let (F, A) and (G, B) be two fuzzy soft modules over M and N respectively, $f : M \to N$ be homomorphism of modules and $\varphi : A \to B$ be a mapping of sets. If for each $a \in A$

$$\widetilde{f} : (M, F(a)) \to (N, G(\varphi(a)))$$

is a fuzzy isomorphism of fuzzy modules, then the morphism

$$(f, \varphi) : (F, A) \to (G, B)$$

is said to be fuzzy soft isomorphism.

Let $\{(F_i, A)\}_{i\in I}$ be a family of fuzzy soft modules over $\{M_i\}_{i\in I}$ and

$$... \to (F_i, A) \overset{(\widetilde{f_i}, 1_A)}{\longrightarrow} (F_{i-1}, A) \overset{(\widetilde{f_{i-1}}, 1_A)}{\longrightarrow} (F_{i-2}, A) \to ... \tag{3.1}$$

be a sequence of fuzzy soft modules.

Definition 3.2. If for each $a \in A$ the following sequence of fuzzy modules is exact,

$$... \to (M_i, F_i(a)) \overset{\widetilde{f_i}}{\longrightarrow} (M_{i-1}, F_{i-1}(a)) \overset{\widetilde{f_{i-1}}}{\longrightarrow} (M_{i-2}, F_{i-2}(a)) \to ...$$

then the sequence 3.1 is said to be exact sequence of fuzzy soft modules.

Theorem 3.3. *Let*

$$... \to M_i \to M_{i-1} \to M_{i-2} \to ...$$

be a exact sequence of modules. For each $a \in A$, under the condition $F_i(a) = (\chi_{\{0\}})_{M_i}$, the following sequence of fuzzy soft modules is also exact.

$$... \to (M_i, F_i) \longrightarrow (M_{i-1}, F_{i-1}) \longrightarrow (M_{i-2}, F_{i-2}) \to ...$$

Definition 3.4. The sequence

$$\bar{0} \to (F', A) \overset{(\widetilde{f}, 1_A)}{\longrightarrow} (F, A) \overset{(\widetilde{g}, 1_A)}{\longrightarrow} (F'', A) \to \bar{0} \tag{3.2}$$

is said to be a short exact sequence of fuzzy soft modules, where $M = \{0\}$, $F : A \to PF(M)$, $\forall a \in A\ F(a)(m) = 1$.

Theorem 3.5. *Let (F, A) and (G, B) be two fuzzy soft modules over M and N respectively, and $(\widetilde{f}, 1_A) : (f, A) \to (G, B)$ be the homomorphism of fuzzy soft modules. Then the following sequence of fuzzy soft modules is exact.*

$$\bar{0} \to \ker \widetilde{f} \longrightarrow (F, A) \longrightarrow (G, A) \to co\ker \widetilde{f} \to \bar{0} \tag{3.3}$$

Proof. Since for each $a \in A$, $\widetilde{f} : (M, F(a)) \to (N, G(\varphi(a)))$ is a homomorphism of fuzzy modules, the following sequence of fuzzy modules is exact.

$$\bar{0} \to \ker \widetilde{f} \longrightarrow (M, F(a)) \longrightarrow (N, G(\varphi(a))) \to co\ker \widetilde{f} \to \bar{0}$$

Therefore, the sequence 3.3 of fuzzy soft modules is exact. □

Lemma 3.6. *Let the sequence 3.2 be a fuzzy soft short exact sequence. Then $\widetilde{f}$ is a monomorphism and $\widetilde{g}$ is a epimorphism.*

Theorem 3.7. *Let the following diagram of fuzzy soft modules be commutative.*

$$\begin{array}{ccccccccc}
\bar{0} & \to & (F_1, A) & \overset{(\widetilde{f_1}, 1_A)}{\longrightarrow} & (F_2, A) & \overset{(\widetilde{f_2}, 1_A)}{\longrightarrow} & (F_3, A) & \to & \bar{0} \\
 & & {\scriptstyle (\widetilde{\alpha}, \varphi)} \downarrow & & {\scriptstyle (\widetilde{\beta}, \varphi)} \downarrow & & {\scriptstyle (\widetilde{\gamma}, \varphi)} \downarrow & & \\
\bar{0} & \to & (G_1, B) & \overset{(\widetilde{g_1}, 1_B)}{\longrightarrow} & (G_2, B) & \overset{(\widetilde{g_2}, 1_B)}{\longrightarrow} & (G_3, B) & \to & \bar{0}
\end{array}$$

Then

(1) *If $(\widetilde{\alpha}, \varphi)$ and $(\widetilde{\gamma}, \varphi)$ are monomorphisms(epimorphisms), then $(\widetilde{\beta}, \varphi)$ is a fuzzy soft monomorphism(epimorphism).*

(2) *If any two of the three fuzzy soft homomorphisms $(\widetilde{\alpha}, \varphi)$, $(\widetilde{\gamma}, \varphi)$ and $(\widetilde{\beta}, \varphi)$ are fuzzy soft isomorphism, then third is too.*

Proof. For each $a \in A$ and the mapping $\varphi : A \to B$, the following diagram of fuzzy modules is commutative.

$$\begin{array}{ccccccccc}
\bar{0} & \to & (M_1, F_1(a)) & \overset{\widetilde{f_1}}{\longrightarrow} & (M_2, F_2(a)) & \overset{\widetilde{f_2}}{\longrightarrow} & (M_3, F_3(a)) & \to & \bar{0} \\
 & & {\scriptstyle \widetilde{\alpha}} \downarrow & & {\scriptstyle \widetilde{\beta}} \downarrow & & {\scriptstyle \widetilde{\gamma}} \downarrow & & \\
\bar{0} & \to & (N_1, G_1(\varphi(a))) & \overset{\widetilde{g_1}}{\longrightarrow} & (N_2, G_2(\varphi(a))) & \overset{\widetilde{g_2}}{\longrightarrow} & (N_3, G_3(\varphi(a))) & \to & \bar{0}
\end{array}$$

(1) If $\widetilde{\alpha}, \widetilde{\gamma}$ are fuzzy monomorphisms(epimorphisms), then $\widetilde{\beta}$ is a fuzzy monomorphism(epimorphism).

(2) If any two of the three fuzzy homomorphisms $\widetilde{\alpha}, \widetilde{\gamma}$ and $\widetilde{\beta}$ are fuzzy isomorphism, then third is too.

Therefore, for each $a \in A$

(1) Since $\widetilde{\beta}$ is a monomorphism(epimorphism) then $(\widetilde{\beta}, \varphi)$ is also a monomorphism(epimorphism).

(2) We know that when any two of the three fuzzy homomorphisms $\widetilde{\alpha}, \widetilde{\gamma}$ and $\widetilde{\beta}$ fuzzy isomorphism then the third is also fuzzy isomorphism. From here we coincluade that if any two of the three fuzzy soft homomorphisms $(\widetilde{\alpha}, \varphi)$, $(\widetilde{\gamma}, \varphi)$ and $(\widetilde{\beta}, \varphi)$ are fuzzy soft isomorphism, then third is too.

□

Proposition 3.8. *Let the following diagram of fuzzy soft modules be commutative.*

$$\begin{array}{ccccccccccccc} \bar{0} & \to & (F_1, A) & \to & (F_2, A) & \to & (F_3, A) & \to & (F_4, A) & \to & (F_5, A) & \to & \bar{0} \\ & & {\scriptstyle(\widetilde{\alpha_1}, \varphi)} \downarrow & & {\scriptstyle(\widetilde{\alpha_2}, \varphi)} \downarrow & & {\scriptstyle(\widetilde{\alpha_3}, \varphi)} \downarrow & & {\scriptstyle(\widetilde{\alpha_4}, \varphi)} \downarrow & & {\scriptstyle(\widetilde{\alpha_5}, \varphi)} \downarrow & & \\ \bar{0} & \to & (G_1, B) & \to & (G_2, B) & \to & (G_3, B) & \to & (G_4, B) & \to & (G_5, B) & \to & \bar{0} \end{array}$$

Where the rows are fuzzy soft exact sequences. Then

(1) *If $(\widetilde{\alpha_1}, \varphi)$ is fuzzy epimorphism and $(\widetilde{\alpha_2}, \varphi), (\widetilde{\alpha_4}, \varphi)$ are fuzzy soft monomorphisms, then $(\widetilde{\alpha_3}, \varphi)$ is a fuzzy soft monomorphism.*

(2) *If $(\widetilde{\alpha_5}, \varphi)$ is fuzzy soft monomorphism and $(\widetilde{\alpha_2}, \varphi), (\widetilde{\alpha_4}, \varphi)$ are fuzzy soft epimorphisms, then $(\widetilde{\alpha_3}, \varphi)$ is a fuzzy soft epimorphism.*

(3) *If $(\widetilde{\alpha_1}, \varphi)$, $(\widetilde{\alpha_2}, \varphi), (\widetilde{\alpha_4}, \varphi)$ and $(\widetilde{\alpha_5}, \varphi)$ are fuzzy softisomorphisms, then $(\widetilde{\alpha_3}, \varphi)$ is a fuzzy soft isomorphism.*

Proof. For each $a \in A$, the proof is similar to proof of above theorem. □

Definition 3.9. If the fuzzy soft homomorphism

$$(\widetilde{g}, 1_A) : (F, A) \to (F^{''}, A)$$

has a right inverse on the short exact sequence 3.2 of fuzzy soft modules, then sequence is called a splitting sequence.

Theorem 3.10. *The following statements are equivalent.*

(1) *The short exact sequence 3.2 of fuzzy soft modules is a splitting.*

(2) *The fuzzy soft homomorphism $(\widetilde{f}, 1_A) : (F^{'}, A) \to (F, A)$ has a left inverse.*

(3) $(F, A) \xrightarrow{\simeq} (F^{'}, A) \oplus (F^{''}, A)$.

Proof. Since for each $a \in A$ the following short exact sequence of fuzzy modules can be splited,

$$0 \to (M^{'}, F^{'}(a)) \xrightarrow{\widetilde{f}} (M, F(a)) \xrightarrow{\widetilde{g}} (M^{''}, F^{''}(a)) \to 0$$

then the theorem is easily proved. □

Definition 3.11. If for each $a \in A$,

$$\{(M_n, F_n(a)), \partial_n : (M_n, F_n(a)) \to (M_{n-1}, F_{n-1}(a))$$

is a chain complex of fuzzy modules, then

$$\{(\partial_n, 1_A) : (F_n, A) \to (F_{n-1}, A)\}$$

is called a chain complex of fuzzy soft modules.

Let $C = \{(\partial_n, 1_A) : (F_n, A) \to (F_{n-1}, A)\}_n$ and $C^{'} = \{(\partial^{'}_n, 1_B) : (G_n, B) \to (G_{n-1}, B)\}_n$ be two chain complexes of fuzzy soft modules and $g : A \to B$ be a mapping of sets and $\{\varphi_n : M_n \to N_n\}$ be the homomorphisms family of modules.

Definition 3.12. If for each $a \in A$ the following diagram of fuzzy modules is commutative,

$$\begin{array}{ccc} (M_n, F_n(a)) & \overset{\partial_n}{\to} & (M_{n-1}, F_{n-1}(a)) \\ {\scriptstyle (\widetilde{\varphi_n}, g)} \downarrow & & {\scriptstyle (\widetilde{\varphi_{n-1}}, g)} \downarrow \\ (N_n, G_n(g(a))) & \overset{\partial^{'}_n}{\to} & (N_{n-1}, G_{n-1}(g(a))) \end{array}$$

then $(\{\varphi_n\}_n, g) : C \to C^{'}$ is called a morphism of chain complexes of fuzzy soft modules.

Remark 3.13. Chain complexes of fuzzy soft and their morphisms consists a category. This category is denoted by $FSCC$.

Definition 3.14. Let $(\{\varphi_n\}_n, g), (\{\psi_n\}_n, g) : C \to C^{'}$ be two morphisms of chain complexes of fuzzy soft modules and $\widetilde{D} = \{(D_n, g) : (F_n, A) \to (G_{n+1}, B)\}_{n\in\mathbb{Z}}$ be a family of homomorphisms of fuzzy soft modules. If for each $a \in A$, the condition $\widetilde{\partial}^{'}_{n+1} \circ \widetilde{D}_n + \widetilde{D}_{n-1} \circ \widetilde{\partial}_n = \widetilde{\varphi_n} - \widetilde{\psi_n}$ is satisfied that is, if for each $a \in A$, the difference of diagonal lines in the following diagram is equal to sum of edges,

$$\begin{array}{ccccccc} & ... \longleftarrow (M_{n-1}, F_{n-1}(a)) & \overset{\widetilde{\partial}_n}{\longleftarrow} & (M_n, F_n(a)) \longleftarrow ... & & \\ & \widetilde{D}_{n-1} \searrow & & \swarrow \widetilde{\varphi_n}, \widetilde{\psi_n} & \searrow^{\widetilde{D}_n} & \\ \longleftarrow (N_{n-1}, G_{n-1}(g(a))) & \overset{\widetilde{\partial}^{'}_n}{\longleftarrow} & (N_n, G_n(g(a))) & \overset{\widetilde{\partial}^{'}_{n+1}}{\longleftarrow} & (N_{n+1}, G_{n+1}(g(a))) \longleftarrow ... \end{array}$$

then D is called a fuzzy soft chain homotopy and the morphisms $(\{\varphi_n\}_n, g), (\{\psi_n\}_n, g)$ are said to be chain homotopy and the morphisms of chain complexes of fuzzy soft modules. This denoted by $(\{\varphi_n\}_n, g) \simeq (\{\psi_n\}_n, g)$.

Theorem 3.15. *The fuzzy soft chain homotopy relation " $\simeq$ " is an equivalent relation.*

Proof. (1) We show that $(\{\varphi_n\}_n, g) \simeq (\{\varphi_n\}_n, g)$.Let the morphism of modules $D_n : M_n \to N_{n+1}$ be $D_n = 0$.

Now, we check that $\widetilde{D}_n : (M_n, F_n(a)) \to (N_{n+1}, G_{n+1}(g(a)))$ is a fuzzy homotopy of fuzzy modules. That is, let 's check that $G_{n+1}(g(a))(D_n(x)) \geq F_n(a)(x)$. Since $D_n = 0$, then $G_{n+1}(g(a))(0) = 1 \geq F_n(a)(x)$.
Let $(\{\varphi_n\}_n, g) \simeq (\{\psi_n\}_n, g)$. We show that $(\{\psi_n\}_n, g) \simeq (\{\varphi_n\}_n, g)$. Let the homotopy between $(\{\varphi_n\}_n, g)$ and $(\{\psi_n\}_n, g)$ be $\{(D_n, g)\}_n$.

Now, we handle the family $\{(-D_n, g)\}_n$. For each $a \in A$, we show that $-\widetilde{D}_n$ is a homomorphism of fuzzy modules. For $-\widetilde{D}_n : (M_n, F_n(a)) \to (N_{n+1}, G_{n+1}(g(a)))$,

$$G_{n+1}(g(a))(-D_n(x)) = G_{n+1}(g(a))(D_n(x)) \geq F_n(a)(x)$$

is satisfied.

(2) Let $(\{\varphi_n\}_n, g) \simeq (\{\psi_n\}_n, g)$ and $(\{\psi_n\}_n, g) \simeq (\{\chi_n\}_n, g)$. Let the homotopy between $(\{\varphi_n\}_n, g)$ and $(\{\psi_n\}_n, g)$ be $\{(D_n, g)\}_n$ and the homotopy between $(\{\psi_n\}_n, g)$ and $(\{\chi_n\}_n, g)$ be $\{(H_n, g)\}_n$. Then for each $a \in A$,

$$\widetilde{D}_n + \widetilde{H}_n : (M_n, F_n(a)) \to (N_{n+1}, G_{n+1}(g(a)))$$

is a homomorphism of fuzzy modules and

$$\begin{aligned}
& \widetilde{\partial}'_{n+1} \circ (\widetilde{D}_n + \widetilde{H}_n) + (\widetilde{D}_{n-1} + \widetilde{H}_{n-1}) \circ \widetilde{\partial}_n \\
= \; & \widetilde{\partial}'_{n+1} \circ \widetilde{D}_n + \widetilde{D}_{n-1} \circ \widetilde{\partial}_n + \widetilde{\partial}'_{n+1} \circ \widetilde{H}_n + \widetilde{H}_{n-1} \circ \widetilde{\partial}_n \\
= \; & \widetilde{\varphi}_n - \widetilde{\psi}_n + \widetilde{\psi}_n - \widetilde{\chi}_n \\
= \; & \widetilde{\varphi}_n - \widetilde{\chi}_n.
\end{aligned}$$

□

Theorem 3.16. *The relation of fuzzy soft chain homotopy is invariant according to composition.*

Proof. Let $(\{\varphi_n\}_n, g) \simeq (\{\psi_n\}_n, g) : C \to C'$ and $(\{\varphi'_n\}_n, g') \simeq (\{\psi'_n\}_n, g') : C' \to C''$. Let the homotopies between above relations be $\{(D_n, g) : (F_n, A) \to (G_{n+1}, B)\}_n$, $\{(H_n, g) : (G_n, B) \to (K_{n+1}, A')\}_n$. Then $\{(\varphi'_{n+1} \circ D_n, g' \circ g) : (F_n, A) \to (K_{n+1}, A')\}_n$ is a morphisms family of fuzzy soft modules and for each $a \in A$,

$$\begin{aligned}
\text{(3.4)} \qquad \widetilde{\partial}''_{n+1}\widetilde{\varphi}'_{n+1} \circ \widetilde{D}_n + \widetilde{\varphi}'_n \circ \widetilde{D}_{n-1}\,\widetilde{\partial}_n &= \widetilde{\varphi}'_n(\widetilde{\partial}''_{n+1}\widetilde{D}_n + \widetilde{D}_{n-1}\,\widetilde{\partial}_n) \\
&= \widetilde{\varphi}'_n(\widetilde{\varphi}_n - \widetilde{\psi}_n) \\
&= \widetilde{\varphi}'_n\widetilde{\varphi}_n - \widetilde{\varphi}'_n\widetilde{\psi}_n
\end{aligned}$$

is satisfied. If $\{(\psi_n \circ H_n, g' \circ g) : (F_n, A) \to (K_{n+1}, A')\}_n$ is a morphisms family of fuzzy soft modules, then for each $a \in A$,

$$\begin{aligned}
\text{(3.5)} \qquad \widetilde{\partial}''_{n+1}\widetilde{H}_n \circ \widetilde{\psi}_n + \widetilde{H}_{n-1} \circ \widetilde{\psi}_{n-1}\,\widetilde{\partial}_n &= \widetilde{\partial}''_{n+1}\widetilde{H}_n\widetilde{\psi}_n + \widetilde{H}_{n-1}\,\widetilde{\partial}'_n\widetilde{\psi}_n \\
&= (\widetilde{\partial}''_{n+1}\widetilde{H}_n + \widetilde{H}_{n-1}\,\widetilde{\partial}'_n)\widetilde{\psi}_n \\
&= (\widetilde{\varphi}'_n - \widetilde{\psi}'_n)\widetilde{\psi}_n \\
&= \widetilde{\varphi}'_n\widetilde{\psi}_n - \widetilde{\psi}'_n\widetilde{\psi}_n
\end{aligned}$$

Thus, the proof of theorem is obtained from (3.4) and (3.5). □

Let $C = \{(\partial_n, 1_A) : (F_n, A) \to (F_{n-1}, A)\}_n$ be chain complex of fuzzy soft modules. For each $a \in A$, the family

$$\{(M_n, F_n(a)), \widetilde{\partial}_n : (M_n, F_n(a)) \to (M_{n-1}, F_{n-1}(a))\}_n$$

is a chain complex of fuzzy modules. $n-$ dimensionel fuzzy homology module of these complexes defined by $H(C, A) = \{H(C, a)\}$. Here,

$$H(C, a) = \ker \partial_n / Im\partial_{n+1}, \widetilde{F}(a)).$$

$\ker \partial_n$ and $Im\partial_{n+1}$ are the submodules of M_n. $\ker \partial_n / Im\partial_{n+1}$ is not a submodule of M_n but there exists a one to one and surjective mapping between the quotient modules of M_n and submodules M_n. Therefore, we consider $\ker \partial_n / Im\partial_{n+1}$ as a submodule that corresponding to do submodules of M_n. Then for each $a \in A$, $H(C, a)$ to be a fuzzy module. Thus,

$$H(C; -) : A \to PF(M_n)$$

is defined. That is, $(H(C, -); A)$ to be a fuzzy soft module over M_n.

Definition 3.17. $(H(C, -); A)$ is called a fuzzy soft homology module of chain complex C of fuzzy soft modules over M_n.

Now, we show that the corresponding $C \longmapsto (H(C, -); A)$ is a functor. Let $(\{\varphi_n\}_n, g) : C \to C^{'}$ be the morphism of chain complexes of fuzzy soft modules. For each $a \in A$, the family $\{\widetilde{\varphi}_n : (M_n, F_n(a)) \to (N_n, G_n(g(a)))\}$ is a morphism that goes from the chain complex $\{(M_n, F_n(a)),\ \widetilde{\partial}_n : (M_n, F_n(a)) \to (M_{n-1}, F_{n-1}(a))\}_n$ of fuzzy modules to the chain complex $\{(N_n, G_n(g(a))),\ \widetilde{\partial^{'}}_n : (N_n, G_n(g(a))) \to (N_{n-1}, G_{n-1}(g(a)))\}_n$ of fuzzy modules. Then we can define the fuzzy homomorphism $H_n(\{\widetilde{\varphi}_n\}, a) : H_n(C, a) \to H_n(C^{'}, g(a))$ fuzzy modules.

Clearly, $H_n(\{\varphi_n\}_n, g) : (H(C, -); A) \to (H(C^{'}, -); B)$ the homomorphism of fuzzy soft modules.

Theorem 3.18. *The corresponding* $C \longmapsto (H(C, -); A), (\{\varphi_n\}_n, g) \mapsto H_n(\{\varphi_n\}_n, g)$ *is a covariant functor from the category* $FSCC$ *to the category* FSM*.*

Theorem 3.19. *The fuzzy soft homology module is invariant according to chain homotopy relation. That is, If*

$$(\{\varphi_n\}_n, g) \simeq (\{\psi_n\}_n, g) : C \to C^{'}$$

then $H_n(\{\varphi_n\}_n, g) = H_n(\{\psi_n\}_n, g)$*.*

Proof. For each $a \in A$, let us take the family $\{\widetilde{\varphi}_n, \widetilde{\psi}_n : (M_n, F_n(a)) \to (N_n, G_n(g(a)))\}$. Since $\{\widetilde{\varphi}_n\}$ and $\{\widetilde{\psi}_n\}$ are fuzzy chain homotopy mappings of fuzzy chain complexes, then

$$H_n(\{\widetilde{\varphi}_n\}, a) = H_n(\{\widetilde{\psi}_n\}, a) \tag{3.6}$$

is satisfied. Since for each $a \in A$, the equality 3.6 is satisfied then the result of theorem is obtained.

After than, let us take the chain complexes of fuzzy soft modules over the same set A. If C is a chain complex of fuzzy soft modules, then for each $a \in A$ let us denote chain complexes of fuzzy modules by $C(a)$.Let $(\{C_n\}_{n \in \mathbb{Z}}, \{f_n : C_n \to C_{n-1}\}_n)$ be the chain complexes of fuzzy soft modules and their morphisms. □

Definition 3.20. If for each $a \in A$, the sequence

$$... \to C_n(a) \overset{\widetilde{f}_n}{\to} C_{n-1}(a) \overset{\widetilde{f}_{n-1}}{\to} C_{n-2}(a) \to ...$$

of chain complexes of fuzzy modules is exact,then the sequence

$$... \to C_n \overset{f_n}{\to} C_{n-1} \overset{f_{n-1}}{\to} C_{n-2} \to ...$$

of chain complexes of fuzzy soft modules is exact.

Definition 3.21. The exact sequence

$$0 \to C^{'} \to C \to C^{''} \to 0$$

is said to be a short exact sequence of chain complexes of fuzzy soft modules.

Let C and $C^{'}$ be two chain complexes of fuzzy soft modules.

Definition 3.22. For each $a \in A$ and the chain complexes

$$C(a) = \{(M_n, F_n(a)), \ \widetilde{\partial}_n\}, \ C^{'}(a) = \{(M_n', F_n'(a)), \ \widetilde{\partial'}_n\}$$

of fuzzy modules if the pair $(M_n', F_n'(a))$ is fuzzy submodule of the pair $(M_n, F_n(a))$, then $C^{'}$ is said to be a chain subcomplex of C and denoted by $C^{'} < C$.

Let $C^{'} < C$. Since $C^{'}(a) \,\widetilde{<}\, C(a)$ on the category of fuzzy modules for each $a \in A$, we can obtain the quotient complex $C(a)/_{C^{'}(a)}$. Then the chain complexes of fuzzy soft modules which is defined as

$$C/_{C'} : A \to PFCC, \quad (C/_{C'})(a) = C(a)/_{C^{'}(a)}$$

are called the quotient complexes according to $C^{'}$ of C.

Remark 3.23. If $C^{'} < C$, then the sequence

$$0 \to C^{'} \xrightarrow{i} C \xrightarrow{p} C/_{C'} \to 0$$

is the short exact sequence of chain complexes of fuzzy soft modules where i is embedding mapping and p is canonical mapping.

Definition 3.24. The homology module of chain complexes of fuzzy soft modules is called a homology module of the pair $(C^{'}, C)$ and denoted by $H_n(C^{'}, C)$.

Theorem 3.25. *If the short exact sequence*

$$0 \to C^{'} \xrightarrow{(i,1_A)} C \xrightarrow{(p,1_A)} C^{''} \to 0$$

of chain complexes of fuzzy soft modules is splitting, then the homology sequence

$$... \longleftarrow H_{n-1}(C^{'}) \longleftarrow H_n(C^{''}) \longleftarrow H_n(C) \longleftarrow H_n(C^{'}) \longleftarrow ... \tag{3.7}$$

of fuzzy soft modules is exact.

Proof. For each $a \in A$ let us take the following short exact sequence of chain complexes of fuzzy modules

$$0 \to C^{'}(a) \xrightarrow{\widetilde{i}} C(a) \xrightarrow{\widetilde{p}} C^{''}(a) \to 0. \tag{3.8}$$

Generally, The homology sequence of fuzzy modules is not exact. However, from the condition of theorem, the sequence 3.8 is a splitting. Then the following homology sequence of fuzzy modules is exact (see [5]).

$$... \longleftarrow H_{n-1}(C^{'}(a)) \longleftarrow H_n(C^{''}(a)) \longleftarrow H_n(C(a)) \longleftarrow H_n(C^{'}(a)) \longleftarrow ... \tag{3.9}$$

For each $a \in A$, since the sequence 3.9 is exact, then the homology sequence 3.7 of fuzzy soft modules is exact. □

4. Conclusions

In this study, the chain complexes category of fuzzy soft modules is defined and the homology module of fuzzy chain complexes is given.

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