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Interval-valued fuzzy completely regular subsemigroups of semigroups

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Abstract. In this paper, we introduce the notion of $(i\text{-}v)$ fuzzy (weakly) completely regular subsemigroup of a semigroup S and study their different properties. We show that every $(i\text{-}v)$ fuzzy completely regular subsemigroup of a semigroup S is a union of $(i\text{-}v)$ fuzzy subgroups of S. We also characterize $(i\text{-}v)$ fuzzy completely regular subsemigroup by $(i\text{-}v)$ fuzzy semilattice of $(i\text{-}v)$ fuzzy subgroups.

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1. Introduction

In 1965, the concept of fuzzy sets first introduced by L. A. Zadeh. At present, it is an important tool in science and engineering (like computer science, control engineering, information sciences etc). In 1975, L. A. Zadeh [15] first introduced the concept of interval-valued fuzzy subset as a generalization of fuzzy subsets. In this concept, the degree of membership of each element is a closed subinterval in [0,1]. The theory based on this extension is known as theory of Interval-valued Fuzzy Sets and the interval-valued fuzzy set is called an interval-valued membership function. A. Rosenfeld, the father of fuzzy abstract algebra, first studied the notion of fuzzy subgroup [12] in 1971. In 1981, N. Kuroki introduced the concept of fuzzy semigroup in [7] and characterized different classes of semigroups [8] in terms of fuzzy ideals in 1991. In [13, 14], the authors introduced the concept of fuzzy (weakly) regular, fuzzy (weakly) completely regular subsemigroups and investigated some of their properties. In [2], the authors characterized a fuzzy completely regular subsemigroup by a fuzzy partition into a family of fuzzy $\varepsilon-$subgroups.

Many researchers [1, 4, 9, 10] have been working on different algebraic structures using this interval-valued fuzzy concept. In [1], Biswas defined the interval-valued fuzzy subgroups of Rosenfeld's nature and investigated some elementary properties. AL. Narayanan and T. Manikantan [10] introduced the notions of an interval-valued fuzzy subsemigroup and various interval-valued fuzzy ideals in semigroups. They [9] also introduced interval-valued fuzzy left (right, two-sided, interior, bi-) ideal generated by an interval-valued fuzzy subset in semigroups and characterized them. We introduced the concept of interval-valued prime fuzzy ideal in [6] and interval-valued semiprime fuzzy ideals in [5] of semigroups and studied their properties using interval-valued fuzzy ideals and interval-valued fuzzy points. The class of completely regular semigroups is a special subclass of regular semigroups. There are many useful tools to characterize various types of fuzzy semigroups by using different types of fuzzy ideals. As a continuation of this process completely regular semigroups have been characterized by different types of ordinary fuzzy ideals and related structures.

In this paper, we introduce the notion of interval-valued fuzzy (weakly) completely regular subsemigroups of a semigroup and study their properties. We also define $(i\text{-}v)$ fuzzy subgroups of an $(i\text{-}v)$ fuzzy subsemigroup of a semigroup. Finally, we have shown that every $(i\text{-}v)$ fuzzy completely regular subsemigroup of a semigroup is an $(i\text{-}v)$ fuzzy semilattice of $(i\text{-}v)$ fuzzy subgroups.

2. Preliminaries

In this section, we give some preliminary definitions of fuzzy algebra which will be used in this paper.

Throughout this paper, we denote the semigroup $(S, \cdot)$ by S and multiplication '$\cdot$' by juxtaposition. We consider the product AB of any two non-empty subsets A and B of a semigroup S, defined by $AB = \{ab : a \in A, b \in B\}$. Also we consider the subsets R_x and C_x of a semigroup S, defined by $R_x = \{y \in S : xyx = x\}$ and $C_x = \{z \in S : xz = zx\}$, where $x \in S$.

Definition 2.1 ([15]). An interval number on $[0, 1]$, denoted by $\widetilde{a}$, is defined as the closed subinterval of $[0, 1]$, where $\widetilde{a} = [a^-, a^+]$ satisfying $0 \leq a^- \leq a^+ \leq 1$.

We denote $D[0, 1]$ as the set of all interval numbers on $[0, 1]$ and also denote the interval numbers $[0, 0]$ and $[1, 1]$ by $\widetilde{0}$ and $\widetilde{1}$ respectively.

Definition 2.2 ([9]). Let $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$ be two interval numbers in $D[0, 1]$. Then

(i) $\widetilde{a} \leq \widetilde{b}$ if $a^- \leq b^-$ and $a^+ \leq b^+$.
(ii) $\widetilde{a} = \widetilde{b}$ if $a^- = b^-$ and $a^+ = b^+$.
(iii) $\widetilde{a} < \widetilde{b}$ if $\widetilde{a} \neq \widetilde{b}$ and $\widetilde{a} \leq \widetilde{b}$.

We write $\widetilde{a} \geq \widetilde{b}$ whenever $\widetilde{b} \leq \widetilde{a}$ and $\widetilde{a} > \widetilde{b}$ whenever $\widetilde{b} < \widetilde{a}$. In this paper we assume that any two interval numbers in $D[0, 1]$ are comparable i.e. for any two interval numbers $\widetilde{a}$ and $\widetilde{b}$ in $D[0, 1]$, we have either $\widetilde{a} \leq \widetilde{b}$ or $\widetilde{a} > \widetilde{b}$.

Definition 2.3 ([4]). The interval Min-norm is a function $Min^i : D[0, 1] \times D[0, 1] \longrightarrow D[0, 1]$, defined by $Min^i(\widetilde{a}, \widetilde{b}) = [min(a^-, b^-), min(a^+, b^+)]$ for all $\widetilde{a}, \widetilde{b} \in D[0, 1]$, where $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$.

Definition 2.4 ([4])**.** The interval Max-norm is a function $Max^i : D[0,1]\times D[0,1] \longrightarrow D[0,1]$, defined by $Max^i(\widetilde{a},\widetilde{b}) = [max(a^-,b^-), max(a^+,b^+)]$ for all $\widetilde{a},\widetilde{b} \in D[0,1]$, where $\widetilde{a} = [a^-,a^+]$ and $\widetilde{b} = [b^-,b^+]$.

Definition 2.5 ([6])**.** Let $\{\widetilde{a}_i : i \in \Lambda\}$ be a family of interval numbers in $D[0,1]$, where $\widetilde{a}_i = [a_i^-, a_i^+]$; $i \in \Lambda$. Then $\sup\limits_{i\in\Lambda}\{\widetilde{a}_i\} = [\sup\limits_{i\in\Lambda} a_i^-, \sup\limits_{i\in\Lambda} a_i^+]$ and $\inf\limits_{i\in\Lambda}\{\widetilde{a}_i\} = [\inf\limits_{i\in\Lambda} a_i^-, \inf\limits_{i\in\Lambda} a_i^+]$.

Definition 2.6 ([6])**.** Let $S(\neq \varnothing)$ be a set and $A \subseteq S$. Then (*i-v*) characteristic function of A, denoted by $\widetilde{\chi}_A$, is an (*i-v*) fuzzy subset of S, defined as :

$$\begin{aligned} \widetilde{\chi}_A(x) &= \widetilde{1}, \quad \text{when } x \in A; \\ &= \widetilde{0}, \quad \text{when } x \in S \setminus A; \end{aligned}$$

where $x \in S$.

Definition 2.7 ([4])**.** Let $\widetilde{\mu}$ be an (*i-v*) fuzzy subset of a set $S(\neq \varnothing)$ and $[a,b] \in D[0,1]$. Then the level subset of $\widetilde{\mu}$, denoted by $\widetilde{U}(\widetilde{\mu},[a,b])$, is defined by

$$\widetilde{U}(\widetilde{\mu},[a,b]) = \{x \in S : \widetilde{\mu}(x) \geq [a,b]\}.$$

Definition 2.8 ([6])**.** Let $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ be two (*i-v*) fuzzy subsets of a semigroup S. Then the product of $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ is an (*i-v*) fuzzy subset of S, defined by :

$$\begin{aligned} (\widetilde{\mu}_1 \; o \; \widetilde{\mu}_2)(x) &= \sup_{x=pq} \left\{Min^i\big(\widetilde{\mu}_1(p), \widetilde{\mu}_2(q)\big)\right\}, \quad \text{when } x = pq \text{ for some } p,q \in S; \\ &= \widetilde{0}, \quad \text{otherwise.} \end{aligned}$$

Definition 2.9 ([9])**.** A non-empty (*i-v*) fuzzy subset $\widetilde{\mu}$ of a semigroup S is called an (*i-v*) fuzzy subsemigroup of S if $\widetilde{\mu}(xy) \geq Min^i\left(\widetilde{\mu}(x), \widetilde{\mu}(y)\right)$ for all $x,y \in S$.

3. (i-v) fuzzy completely regular subsemigroups

Definition 3.1 ([11])**.** A subsemigroup A of a semigroup S is said to be completely regular if for all $x \in A$, there exists $y \in A$ such that $x = xyx$ and $xy = yx$.

Definition 3.2. An (*i-v*) fuzzy subsemigroup $\widetilde{\mu}$ of a semigroup S is said to be an (*i-v*) fuzzy completely regular subsemigroup of S if for all $x \in S$, there exists $x' \in R_x \cap C_x$ such that $\widetilde{\mu}(x') \geq \widetilde{\mu}(x)$, when $\widetilde{\mu}(x) \neq \widetilde{0}$.

Now we can easily prove the following propositions.

Proposition 3.3. *Let A be a non-empty subset of a semigroup S. Then A is a subsemigroup of S if and only if $\widetilde{\chi}_A$ is an (i-v) fuzzy subsemigroup of S.*

Proposition 3.4. *An (i-v) fuzzy subset $\widetilde{\mu}$ of a semigroup S is an (i-v) fuzzy subsemigroup of S if and only if for all $[a,b] \in D[0,1]$, $\widetilde{U}(\widetilde{\mu},[a,b])$ is a subsemigroup of S.*

Proposition 3.5. *Let A be a non-empty subset of a semigroup S. Then A is a completely regular subsemigroup of S if and only if $\widetilde{\chi}_A$ is an (i-v) fuzzy completely regular subsemigroup of S.*

Proof. Let A be a completely regular subsemigroup of S. Then from Proposition 3.3, it follows that $\widetilde{\chi}_A$ is an (i-v) fuzzy subsemigroup of S. Let $x \in S$. If $\widetilde{\chi}_A(x) \neq \widetilde{0}$, then $\widetilde{\chi}_A(x) = \widetilde{1}$. So $x \in A$. Therefore, by our assumption, there exists $x^* \in A$ such that $x = xx^*x$ and $xx^* = x^*x$, i.e. $x^* \in R_x \cap C_x$. Thus $\widetilde{\chi}_A(x^*) = \widetilde{1} = \widetilde{\chi}_A(x)$. Therefore, $\widetilde{\chi}_A$ is an (i-v) fuzzy completely regular subsemigroup of S.

Conversely, let $\widetilde{\chi}_A$ be an (i-v) fuzzy completely regular subsemigroup of S. Then A is a subsemigroup of S, by Proposition 3.3. Let $x \in A$. Then $\widetilde{\chi}_A(x) = \widetilde{1} \neq \widetilde{0}$. Thus by our assumption, there exists $x^* \in R_x \cap C_x$ such that $\widetilde{\chi}_A(x^*) \geq \widetilde{\chi}_A(x)$. Therefore, $\widetilde{\chi}_A(x^*) = \widetilde{1}$ and so $x^* \in A$. Also, we have $x = xx^*x$ and $xx^* = x^*x$. Therefore, A is a completely regular subsemigroup of S. □

Proposition 3.6. *An (i-v) fuzzy subsemigroup $\widetilde{\mu}$ of a semigroup S is an (i-v) fuzzy completely regular subsemigroup of S if and only if for all $[a,b] \in D[0,1]\backslash\{\widetilde{0}\}$, $\widetilde{U}(\widetilde{\mu},[a,b])$ is a completely regular subsemigroup of S.*

Proof. Let $\widetilde{\mu}$ be an (i-v) fuzzy completely regular subsemigroup of S. Let $x \in \widetilde{U}(\widetilde{\mu},[a,b])$. Then $\widetilde{\mu}(x) \geq [a,b] \neq \widetilde{0}$. Thus by our assumption, there exists an $x^* \in R_x \cap C_x$ such that $\widetilde{\mu}(x^*) \geq \widetilde{\mu}(x)$. Therefore, $\widetilde{\mu}(x^*) \geq [a,b]$ and so $x^* \in \widetilde{U}(\widetilde{\mu},[a,b])$. Again, since $x^* \in R_x \cap C_x$, $x = xx^*x$ and $xx^* = x^*x$. Hence, $\widetilde{U}(\widetilde{\mu},[a,b])$ is a completely regular subsemigroup of S.

Conversely, let $\widetilde{U}(\widetilde{\mu},[a,b])$ be a completely regular subsemigroup of S, where $[a,b] \in D[0,1]\backslash\{\widetilde{0}\}$. Let $x \in S$ and consider $\widetilde{\mu}(x) = [a_1,b_1] \neq \widetilde{0}$. Then $x \in \widetilde{U}(\widetilde{\mu},[a_1,b_1])$ and hence by our assumption, there exists $x^* \in \widetilde{U}(\widetilde{\mu},[a_1,b_1])$ such that $x = xx^*x$ and $xx^* = x^*x$. Therefore, $x^* \in R_x \cap C_x$ and $\widetilde{\mu}(x^*) \geq [a_1,b_1]$, i.e. $\widetilde{\mu}(x^*) \geq \widetilde{\mu}(x)$, where $\widetilde{\mu}(x) \neq \widetilde{0}$. Thus $\widetilde{\mu}$ is an (i-v) fuzzy completely regular subsemigroup of S. □

Definition 3.7 ([5])**.** Let A and B be two non-empty sets and $\alpha : A \longrightarrow B$ be a function. Let $\widetilde{\mu}$ and $\widetilde{\sigma}$ be (i-v) fuzzy subsets of A and B respectively. Then image of $\widetilde{\mu}$ under the function α, denoted by $\alpha(\widetilde{\mu})$, is an (i-v) fuzzy subset of B, defined as

$$\begin{aligned} \alpha(\widetilde{\mu})(y) &= \sup_{z \in \alpha^{-1}(y)} \widetilde{\mu}(z), \quad \text{when } \alpha^{-1}(y) \neq \varnothing\,; \\ &= \widetilde{0}, \quad \text{otherwise}\,; \end{aligned}$$

where $y \in B$ and $\alpha^{-1}(y) = \{x \in A : \alpha(x) = y\}$.

Pre-image of $\widetilde{\sigma}$ under the function α, denoted by $\alpha^{-1}(\widetilde{\sigma})$, is an ($i$-$v$) fuzzy subset of A, defined as $\alpha^{-1}(\widetilde{\sigma})(x) = \widetilde{\sigma}(\alpha(x))$ for all $x \in A$.

Proposition 3.8. *Let α be a semigroup surjective homomorphism from S onto T. If $\widetilde{\mu}$ be an (i-v) fuzzy completely regular subsemigroup of S, then $\alpha(\widetilde{\mu})$, image of $\widetilde{\mu}$ under α, is an (i-v) fuzzy completely regular subsemigroup of T.*

Proof. Let $\widetilde{\mu}$ be an (i-v) fuzzy completely regular subsemigroup of S. We have to show that for all $y \in T$, there exists $y^* \in R_y \cap C_y$ such that $\alpha(\widetilde{\mu})(y^*) \geq \alpha(\widetilde{\mu})(y)$, when $\alpha(\widetilde{\mu})(y) \neq \widetilde{0}$, i.e. $\sup_{z^* \in \alpha^{-1}(y^*)} \widetilde{\mu}(z^*) \geq \sup_{z \in \alpha^{-1}(y)} \widetilde{\mu}(z)$, when $\alpha(\widetilde{\mu})(y) \neq \widetilde{0}$. Let $y \in T$. Since α is onto, $\alpha^{-1}(y) \neq \varnothing$. Let $x \in \alpha^{-1}(y) \subseteq S$. If $\widetilde{\mu}(x) \neq \widetilde{0}$, then by our hypothesis, there exists $x^* \in R_x \cap C_x$ such that $\widetilde{\mu}(x^*) \geq \widetilde{\mu}(x)$. Since $x^* \in R_x \cap C_x$, $x = xx^*x$ and $xx^* = x^*x$. Therefore, $\alpha(x) = \alpha(xx^*x) = \alpha(x)\alpha(x^*)\alpha(x)$ and

$\alpha(x)\alpha(x^*) = \alpha(xx^*) = \alpha(x^*x) = \alpha(x^*)\alpha(x)$ (since α is homomorphism). This shows that $\alpha(x^*) \in R_{\alpha(x)} \cap C_{\alpha(x)} = R_y \cap C_y \neq \varnothing$. Let $y^* = \alpha(x^*) \in R_y \cap C_y$. Then $\alpha(\widetilde{\mu})(y^*) = \sup_{u^* \in \alpha^{-1}(\alpha(x^*))} \widetilde{\mu}(u^*) \geq \widetilde{\mu}(x^*) \geq \widetilde{\mu}(x)$, where $x \in \alpha^{-1}(y)$. Therefore, $\alpha(\widetilde{\mu})(y^*) \geq \sup_{x \in \alpha^{-1}(y)} \widetilde{\mu}(x) = \alpha(\widetilde{\mu})(y)$, where $\sup_{x \in \alpha^{-1}(y)} \widetilde{\mu}(x) \neq \widetilde{0}$. Consequently, $\alpha(\widetilde{\mu})$ is an (*i-v*) fuzzy completely regular subsemigroup of T. □

Proposition 3.9. *Let $\alpha : S \longrightarrow T$ be a semigroup isomorphism. If $\widetilde{\mu}$ be an (i-v) fuzzy completely regular subsemigroup of T, then $\alpha^{-1}(\widetilde{\mu})$, pre-image of $\widetilde{\mu}$ under α, is an (i-v) fuzzy completely regular subsemigroup of S.*

Proof. Let $x \in S$. Then $\alpha(x) \in T$. Therefore, by our hypothesis, there exists $y^* \in R_{\alpha(x)} \cap C_{\alpha(x)}$ such that $\widetilde{\mu}(y^*) \geq \widetilde{\mu}(\alpha(x))$, when $\widetilde{\mu}(\alpha(x)) \neq \widetilde{0}$. Since α is onto, there exists $x^* \in S$ such that $\alpha(x^*) = y^*$. Again, since $y^* \in R_{\alpha(x)} \cap C_{\alpha(x)}$, $\alpha(x) = \alpha(x)y^*\alpha(x) \Longrightarrow \alpha(x) = \alpha(x)\alpha(x^*)\alpha(x) \Longrightarrow \alpha(x) = \alpha(xx^*x)$. Also, $\alpha(xx^*) = \alpha(x)\alpha(x^*) = \alpha(x)\, y^* = y^*\, \alpha(x) = \alpha(x^*)\alpha(x) = \alpha(x^*x)$. Since α is injective, it follows that $x = xx^*x$ and $xx^* = x^*x$, i.e. $x^* \in R_x \cap C_x$. Now $\alpha^{-1}(\widetilde{\mu})(x) = \widetilde{\mu}(\alpha(x)) \leq \widetilde{\mu}(y^*) = \widetilde{\mu}(\alpha(x^*)) = \alpha^{-1}(\widetilde{\mu})(x^*)$, where $x^* \in R_x \cap C_x$ and $\alpha^{-1}(\widetilde{\mu})(x) \neq \widetilde{0}$. Thus $\alpha^{-1}(\widetilde{\mu})$ is an (*i-v*) fuzzy completely regular subsemigroup of S. □

Definition 3.10 ([6])**.** Let S be a non-empty set and $x \in S$. Let $\widetilde{a} \in D[0,1]\backslash\{\widetilde{0}\}$. An (*i-v*) fuzzy point $x_{\widetilde{a}}$ of S is an (*i-v*) fuzzy subset of S, defined by

$$x_{\widetilde{a}}(y) = \begin{cases} \widetilde{a}, & \text{if } y = x\,; \\ \widetilde{0}, & \text{otherwise}\,; \end{cases}$$

where $y \in S$.

Definition 3.11 ([6])**.** Let $\widetilde{\mu}$ be a non-empty (*i-v*) fuzzy subset of a non-empty set S and $x_{\widetilde{a}}$ be an (*i-v*) fuzzy point of S. Then the (*i-v*) fuzzy point $x_{\widetilde{a}}$ is said to be contained in $\widetilde{\mu}$ or to belong to $\widetilde{\mu}$ (denoted by $x_{\widetilde{a}} \in \widetilde{\mu}$) if $\widetilde{\mu}(x) \geq \widetilde{a}$.

Lemma 3.12 ([6])**.** *If $x_{\widetilde{a}}\,, y_{\widetilde{b}}$ be (i-v) fuzzy points of a semigroup S, then $x_{\widetilde{a}} \; o \; y_{\widetilde{b}} = (xy)_{Min^i\,(\widetilde{a},\,\widetilde{b})}$.*

Theorem 3.13. *An (i-v) fuzzy subsemigroup $\widetilde{\mu}$ of a semigroup S is an (i-v) fuzzy completely regular subsemigroup of S if and only if for any (i-v) fuzzy point $x_{\widetilde{a}} \in \widetilde{\mu}$, there exists an (i-v) fuzzy point $y_{\widetilde{b}} \in \widetilde{\mu}$ such that $x_{\widetilde{a}} = x_{\widetilde{a}} \; o \; y_{\widetilde{b}} \; o \; x_{\widetilde{a}}$ and $x_{\widetilde{a}} \; o \; y_{\widetilde{b}} = y_{\widetilde{b}} \; o \; x_{\widetilde{a}}$.*

Proof. Let $\widetilde{\mu}$ be an (*i-v*) fuzzy completely regular subsemigroup of S. Let $x_{\widetilde{a}} \in \widetilde{\mu}$ be an (*i-v*) fuzzy point. Then $\widetilde{\mu}(x) \geq \widetilde{a} \neq \widetilde{0}$. Therefore, by our assumption, there exists $y \in R_x \cap C_x$ such that $\widetilde{\mu}(y) \geq \widetilde{\mu}(x)$. Thus, $\widetilde{\mu}(y) \geq \widetilde{a}$ and hence $y_{\widetilde{a}} \in \widetilde{\mu}$. Again, since $y \in R_x \cap C_x$, $x = xyx$ and $xy = yx$. Therefore, for an interval number $\widetilde{b}\,(\geq \widetilde{a})$ in $D[0,1] \setminus \{\widetilde{0}\}$, $x_{\widetilde{a}} = (xyx)_{\widetilde{a}} = (xyx)_{Min^i(\widetilde{a},\widetilde{b})} = x_{\widetilde{a}} \; o \; y_{\widetilde{b}} \; o \; x_{\widetilde{a}}$ and $x_{\widetilde{a}} \; o \; y_{\widetilde{b}} = (xy)_{Min^i(\widetilde{a},\widetilde{b})} = (yx)_{Min^i(\widetilde{b},\widetilde{a})} = y_{\widetilde{b}} \; o \; x_{\widetilde{a}}$, by using Lemma 3.12.

Conversely, let $\widetilde{\mu}$ be an (*i-v*) fuzzy subsemigroup of S such that $\widetilde{\mu}$ satisfies the conditions. Let $x \in S$. If $\widetilde{\mu}(x) \neq \widetilde{0}$, let $\widetilde{\mu}(x) = \widetilde{a}$. Then $x_{\widetilde{a}} \in \widetilde{\mu}$. Therefore, by our hypothesis, there exists an (*i-v*) fuzzy point $y_{\widetilde{b}} \in \widetilde{\mu}$ such that $x_{\widetilde{a}} = x_{\widetilde{a}} \; o \; y_{\widetilde{b}} \; o \; x_{\widetilde{a}}$ and

$x_{\widetilde{a}}\ o\ y_{\widetilde{b}} = y_{\widetilde{b}}\ o\ x_{\widetilde{a}}$. Now $x_{\widetilde{a}} = x_{\widetilde{a}}\ o\ y_{\widetilde{b}}\ o\ x_{\widetilde{a}} \Longrightarrow x_{\widetilde{a}} = (xyx)_{Min^i(\widetilde{a},\widetilde{b})}$ implies that $x = xyx$ and $\widetilde{a} \le \widetilde{b}$. Also, $x_{\widetilde{a}}\ o\ y_{\widetilde{b}} = y_{\widetilde{b}}\ o\ x_{\widetilde{a}}$ implies that $(xy)_{Min^i(\widetilde{a},\widetilde{b})} = (yx)_{Min^i(\widetilde{b},\widetilde{a})}$. Thus it follows that $xy = yx$. So $y \in R_x \cap C_x$ and $\widetilde{\mu}(y) \ge \widetilde{b} \ge \widetilde{a} = \widetilde{\mu}(x)$, i.e. $\widetilde{\mu}(y) \ge \widetilde{\mu}(x)$. Therefore, $\widetilde{\mu}$ is an (*i-v*) fuzzy completely regular subsemigroup of S. □

4. (i-v) fuzzy weakly completely regular subsemigroups

Definition 4.1. An (*i-v*) fuzzy subsemigroup $\widetilde{\mu}$ of a semigroup S is said to be an (*i-v*) fuzzy weakly completely regular subsemigroup of S if for all $x \in S$, $R_x \cap C_x \neq \varnothing$ and $\sup\limits_{x' \in R_x \cap C_x} \widetilde{\mu}(x') \ge \widetilde{\mu}(x)$, when $\widetilde{\mu}(x) \neq \widetilde{0}$.

Definition 4.2. An (*i-v*) fuzzy subsemigroup $\widetilde{\mu}$ of a semigroup S is said to have supremum property if for any $P \subseteq S$, there exists $x_o \in P$ such that $\sup\limits_{x \in P} \widetilde{\mu}(x) = \widetilde{\mu}(x_o)$.

Theorem 4.3. *If $\widetilde{\mu}$ be an (i-v) fuzzy completely regular subsemigroup of a semigroup S, then $\widetilde{\mu}$ is an (i-v) fuzzy weakly completely regular subsemigroup of S.*

Proof. Since $\widetilde{\mu}$ is an (*i-v*) fuzzy completely regular subsemigroup of S, for any $x \in S$, there exists $x^* \in R_x \cap C_x$ such that $\widetilde{\mu}(x^*) \ge \widetilde{\mu}(x)$, when $\widetilde{\mu}(x) \neq \widetilde{0}$. It implies that $R_x \cap C_x \neq \varnothing$ and $\sup\limits_{z \in R_x \cap C_x} \widetilde{\mu}(z) \ge \widetilde{\mu}(x^*) \ge \widetilde{\mu}(x)$, where $\widetilde{\mu}(x) \neq \widetilde{0}$. Thus $\widetilde{\mu}$ is an (*i-v*) fuzzy weakly completely regular subsemigroup of S. □

The converse of the above result may not be true in general. But in particular, we have the following result.

Theorem 4.4. *If $\widetilde{\mu}$ be an (i-v) fuzzy weakly completely regular subsemigroup of a semigroup S and $\widetilde{\mu}$ has supremum property, then $\widetilde{\mu}$ is an (i-v) fuzzy completely regular subsemigroup of S.*

Proof. Since $\widetilde{\mu}$ is an (*i-v*) fuzzy weakly completely regular subsemigroup of S, for any $x \in S$, $R_x \cap C_x \neq \varnothing$ and $\sup\limits_{z \in R_x \cap C_x} \widetilde{\mu}(z) \ge \widetilde{\mu}(x)$, when $\widetilde{\mu}(x) \neq \widetilde{0}$. Since $\widetilde{\mu}$ has supremum property, there exists $x^* \in R_x \cap C_x$ such that $\sup\limits_{z \in R_x \cap C_x} \widetilde{\mu}(z) = \widetilde{\mu}(x^*)$. Therefore, $\widetilde{\mu}(x^*) \ge \widetilde{\mu}(x)$. Thus $\widetilde{\mu}$ is an (*i-v*) fuzzy completely regular subsemigroup of S. □

Proposition 4.5. *Let A be a non-empty subset of a semigroup S. Then A is a completely regular subsemigroup of S if and only if $\widetilde{\chi}_A$ is an (i-v) fuzzy weakly completely regular subsemigroup of S.*

Proposition 4.6. *An (i-v) fuzzy subsemigroup $\widetilde{\mu}$ of a semigroup S is an (i-v) fuzzy weakly completely regular subsemigroup of S if and only if for all $[a,b] \in D[0,1]\backslash\{\widetilde{1}\}$, $\widetilde{U}(\widetilde{\mu}_>, [a,b])$ is a completely regular subsemigroup of S, where $\widetilde{U}(\widetilde{\mu}_>, [a,b]) = \{x \in S : \widetilde{\mu}(x) > [a,b]\}$.*

Proposition 4.7. *Let $\alpha : S \longrightarrow T$ be a semigroup epimorphism. If $\widetilde{\mu}$ be an (i-v) fuzzy weakly completely regular subsemigroup of S, then $\alpha(\widetilde{\mu})$ (image of $\widetilde{\mu}$ under α) is an (i-v) fuzzy weakly completely regular subsemigroup of T.*

Proof. Let $\widetilde{\mu}$ be an (*i-v*) fuzzy weakly completely regular subsemigroup of S. We have to show that for all $y \in T$, $R_y \cap C_y \neq \varnothing$ and $\sup\limits_{y^* \in R_y \cap C_y} \alpha(\widetilde{\mu})(y^*) \geq \alpha(\widetilde{\mu})(y)$, when $\alpha(\widetilde{\mu})(y) \neq \widetilde{0}$, i.e. $\sup\limits_{y^* \in R_y \cap C_y} \big\{ \sup\limits_{z^* \in \alpha^{-1}(y^*)} \widetilde{\mu}(z^*)\big\} \geq \sup\limits_{z \in \alpha^{-1}(y)} \widetilde{\mu}(z)$, when $\alpha(\widetilde{\mu})(y) \neq \widetilde{0}$. Let $y \in T$. Since α is onto, $\alpha^{-1}(y) \neq \varnothing$. Let $x \in \alpha^{-1}(y) \subseteq S$. If $\widetilde{\mu}(x) \neq \widetilde{0}$, then by our hypothesis, we can say that $R_x \cap C_x \neq \varnothing$ and $\sup\limits_{x^* \in R_x \cap C_x} \widetilde{\mu}(x^*) \geq \widetilde{\mu}(x)$. Since $R_x \cap C_x \neq \varnothing$, let $x^* \in R_x \cap C_x$. Then $x = xx^*x$ and $xx^* = x^*x$. Therefore, $\alpha(x) = \alpha(xx^*x) = \alpha(x)\alpha(x^*)\alpha(x)$ and $\alpha(x)\alpha(x^*) = \alpha(xx^*) = \alpha(x^*x) = \alpha(x^*)\alpha(x)$, (since α is homomorphism). This shows that $\alpha(x^*) \in R_{\alpha(x)} \cap C_{\alpha(x)}$, i.e. $R_y \cap C_y \neq \varnothing$. Now

$$\begin{aligned}
\sup_{y^* \in R_y \cap C_y} \alpha(\widetilde{\mu})(y^*) &= \sup_{y^* \in R_y \cap C_y} \big\{ \sup_{z^* \in \alpha^{-1}(y^*)} \widetilde{\mu}(z^*)\big\} \\
&\geq \sup_{\alpha(x^*) \in R_y \cap C_y} \big\{ \sup_{z^* \in \alpha^{-1}(\alpha(x^*))} \widetilde{\mu}(z^*)\big\} \\
&= \sup_{\alpha(x^*) \in R_{\alpha(x)} \cap C_{\alpha(x)}} \big\{ \sup_{z^* \in \alpha^{-1}(\alpha(x^*))} \widetilde{\mu}(z^*)\big\} \\
&\geq \sup_{\alpha(x^*) \in \alpha(R_x) \cap \alpha(C_x)} \big\{ \sup_{z^* \in \alpha^{-1}(\alpha(x^*))} \widetilde{\mu}(z^*)\big\} \\
&\qquad \text{(since } \alpha(R_x) \subseteq R_{\alpha(x)} \text{ for any } x \in S) \\
&\geq \sup_{\alpha(x^*) \in \alpha(R_x \cap C_x)} \big\{ \sup_{z^* \in \alpha^{-1}(\alpha(x^*))} \widetilde{\mu}(z^*)\big\} \\
&\geq \sup_{\alpha(x^*) \in \alpha(R_x \cap C_x)} \widetilde{\mu}(x^*) \\
&\geq \sup_{x^* \in R_x \cap C_x} \widetilde{\mu}(x^*) \\
&\geq \widetilde{\mu}(x).
\end{aligned}$$

Therefore, $\sup\limits_{y^* \in R_y \cap C_y} \alpha(\widetilde{\mu})(y^*) \geq \sup\limits_{x \in \alpha^{-1}(y)} \widetilde{\mu}(x) = \alpha(\widetilde{\mu})(y)$, when $\alpha(\widetilde{\mu})(y) \neq \widetilde{0}$. Thus, it follows from the above that $\alpha(\widetilde{\mu})$ is an (*i-v*) fuzzy weakly completely regular subsemigroup of S. □

Proposition 4.8. *Let $\alpha : S \longrightarrow T$ be a semigroup isomorphism. If $\widetilde{\mu}$ be an (i-v) fuzzy weakly completely regular subsemigroup of T, then $\alpha^{-1}(\widetilde{\mu})$, pre-image of $\widetilde{\mu}$ under α, is an (i-v) fuzzy weakly completely regular subsemigroup of S.*

Proof. Let $x \in S$. Then $\alpha(x) \in T$. Therefore, by our hypothesis, $R_{\alpha(x)} \cap C_{\alpha(x)} \neq \varnothing$ and $\sup\limits_{z \in R_{\alpha(x)} \cap C_{\alpha(x)}} \widetilde{\mu}(z) \geq \widetilde{\mu}(\alpha(x))$, when $\widetilde{\mu}(\alpha(x)) \neq \widetilde{0}$. Since $R_{\alpha(x)} \cap C_{\alpha(x)} \neq \varnothing$, let $y^\star \in R_{\alpha(x)} \cap C_{\alpha(x)}$. Since α is onto, there exists $x^\diamond \in S$ such that $\alpha(x^\diamond) = y^\star$. Again, since $y^\star \in R_{\alpha(x)} \cap C_{\alpha(x)}$, $\alpha(x) = \alpha(x)\, y^\star\, \alpha(x) \Longrightarrow \alpha(x) = \alpha(x)\alpha(x^\diamond)\alpha(x) \Longrightarrow \alpha(x) = \alpha(xx^\diamond x)$. Now $\alpha(xx^\diamond) = \alpha(x)\alpha(x^\diamond) = \alpha(x)\, y^\star = y^\star\, \alpha(x) = \alpha(x^\diamond)\alpha(x) = \alpha(x^\diamond x)$. Since α is injective, it follows that $x = xx^\diamond x$ and $xx^\diamond = x^\diamond x$, i.e. $x^\diamond \in R_x \cap C_x$. Hence $R_x \cap C_x \neq \varnothing$. Now $\sup\limits_{z \in R_x \cap C_x} \alpha^{-1}(\widetilde{\mu})(z) = \sup\limits_{z \in R_x \cap C_x} \widetilde{\mu}(\alpha(z)) \geq \sup\limits_{\substack{x^\diamond \in R_x \cap C_x \\ x^\diamond \in \alpha^{-1}(y^\star)}} \widetilde{\mu}(\alpha(x^\diamond)) = \sup\limits_{y^\star \in R_{\alpha(x)} \cap C_{\alpha(x)}} \widetilde{\mu}(y^\star)$ (since α is injective) $\geq \widetilde{\mu}(\alpha(x)) = \alpha^{-1}(\widetilde{\mu})(x)$, when $\alpha^{-1}(\widetilde{\mu})(x) \neq \widetilde{0}$. Thus $\alpha^{-1}(\widetilde{\mu})$ is an (*i-v*) fuzzy weakly completely regular subsemigroup of S. □

Definition 4.9. Let $\widetilde{\mu}$ be an (*i-v*) fuzzy subset of a semigroup S and $x, z \in S$. We define

$$(x\widetilde{\mu}z)(u) = \begin{cases} \sup\limits_{u=xyz} \widetilde{\mu}(y), & \text{if } u = xyz \text{ for some } y \in S; \\ \widetilde{0}, & \text{otherwise;} \end{cases}$$

where $u \in S$.

Theorem 4.10. *Let $\widetilde{\mu}$ be an (i-v) fuzzy subsemigroup of a semigroup S. Then $\widetilde{\mu}$ is an (i-v) fuzzy weakly completely regular subsemigroup of S if and only if for all $x \in S$, $(x^2\,\widetilde{\mu}\,x^2)(x) \geq \widetilde{\mu}(x)$, when $\widetilde{\mu}(x) \neq \widetilde{0}$.*

Proof. Let $\widetilde{\mu}$ be an (*i-v*) fuzzy weakly completely regular subsemigroup of S. Let $x \in S$. If $\widetilde{\mu}(x) \neq \widetilde{0}$, by our assumption $R_x \cap C_x \neq \varnothing$ and $\sup\limits_{x^* \in R_x \cap C_x} \widetilde{\mu}(x^*) \geq \widetilde{\mu}(x)$. Since $R_x \cap C_x \neq \varnothing$, let $x' \in R_x \cap C_x$. Then $x = xx'x$ and $xx' = x'x$. Therefore, $x = xx'x = xx'(xx'x)x'x = x^2(x')^3x^2 \in x^2Sx^2$. Thus,

$$\left(x^2\,\widetilde{\mu}\,x^2\right)(x) = \sup_{\substack{x\,=x^2zx^2 \\ z\in S}} \widetilde{\mu}(z) \geq \sup_{\substack{x\,=\,x^2(x'x'x')x^2 \\ x' \in R_x \cap C_x}} \widetilde{\mu}(x'x'x') \geq \sup_{x' \in R_x \cap C_x} \widetilde{\mu}(x') \geq \widetilde{\mu}(x).$$

Conversely, let the condition hold for an (*i-v*) fuzzy subsemigroup $\widetilde{\mu}$ of S. Let $x \in S$. If $\widetilde{\mu}(x) \neq \widetilde{0}$, by our assumption, $\left(x^2\,\widetilde{\mu}\,x^2\right)(x) \geq \widetilde{\mu}(x) \neq \widetilde{0}$. Therefore, from Definition 4.9, we can say that there exists $x' \in S$ such that $x = x^2x'x^2$. Now $x = x^2x'x^2 = x(xx'x)x$ and $x(xx'x) = x^2x'(x^2x'x^2) = (x^2x'x^2)x'x^2 = xx'x^2 = (xx'x)x$. Thus, $xx'x \in R_x \cap C_x$ and hence $R_x \cap C_x \neq \varnothing$.

Now
$$\begin{aligned}
\sup_{z \in R_x \cap C_x} \widetilde{\mu}(z) &\geq \sup_{\substack{xx'x \in R_x \cap C_x \\ x' \in S}} \widetilde{\mu}(xx'x) \\
&\geq \sup_{\substack{xx'x \in R_x \cap C_x \\ x' \in S}} \left\{Min^i\left(\widetilde{\mu}(x), \widetilde{\mu}(x')\right)\right\} \\
&\qquad \text{(since } \widetilde{\mu} \text{ is an } (i\text{-}v) \text{ fuzzy subsemigroup of } S) \\
&= Min^i\left(\widetilde{\mu}(x)\,,\ \sup_{\substack{xx'x \in R_x \cap C_x \\ x' \in S}} \left\{ \widetilde{\mu}(x')\right\}\right) \\
&\geq Min^i\left(\widetilde{\mu}(x)\,,\ \sup_{\substack{x\,=\,x^2x'x^2 \\ x' \in S}} \left\{ \widetilde{\mu}(x')\right\}\right) \\
&= Min^i\left(\widetilde{\mu}(x), (x^2\widetilde{\mu}x^2)(x)\right) \\
&\geq Min^i\left(\widetilde{\mu}(x), \widetilde{\mu}(x)\right) \\
&= \widetilde{\mu}(x).
\end{aligned}$$

Therefore, it follows from the above that $\widetilde{\mu}$ is an (*i-v*) fuzzy weakly completely regular subsemigroup of S. □

Theorem 4.11. *Let $\widetilde{\mu}$ be an (i-v) fuzzy subsemigroup of a semigroup S. Then $\widetilde{\mu}$ is an (i-v) fuzzy weakly completely regular subsemigroup of S if and only if for any (i-v) fuzzy point $x_{\widetilde{a}} \in \widetilde{\mu}$, $x_{\widetilde{a}} \in \left({x^2}_{\widetilde{a}}\ o\ \widetilde{\mu}\ o\ {x^2}_{\widetilde{a}}\right)$.*

Definition 4.12. Let $\widetilde{\mu}$ be an (*i-v*) fuzzy subset of a semigroup S and $x \in S$. We define

$$(x\,\widetilde{\mu})(u) = \begin{cases} \sup\limits_{u=xs} \widetilde{\mu}(s), & \text{if there exists } s \in S \text{ such that } u = xs; \\ \widetilde{0}, & \text{otherwise;} \end{cases}$$

and

$$(\widetilde{\mu}\,x)(u) = \begin{cases} \sup\limits_{u=yx} \widetilde{\mu}(y), & \text{if there exists } y \in S \text{ such that } u = yx; \\ \widetilde{0}, & \text{otherwise;} \end{cases}$$

for any $u \in S$.

Theorem 4.13. *An (i-v) fuzzy subsemigroup $\widetilde{\mu}$ of a semigroup S is an (i-v) fuzzy weakly completely regular subsemigroup of S if and only if for all $x \in S$, $\big(x^2\,\widetilde{\mu}\ \cap\ \widetilde{\mu}\,x^2\big)(x) \geq \widetilde{\mu}(x)$, when $\widetilde{\mu}(x) \neq \widetilde{0}$.*

Proof. Let $\widetilde{\mu}$ be an (*i-v*) fuzzy weakly completely regular subsemigroup of S. Let $x \in S$ such that $\widetilde{\mu}(x) \neq \widetilde{0}$. Then $R_x \cap C_x \neq \varnothing$ and $\sup\limits_{x^* \in R_x \cap C_x} \widetilde{\mu}(x^*) \geq \widetilde{\mu}(x)$. Since $R_x \cap C_x \neq \varnothing$, let $x' \in R_x \cap C_x$. Then $x = xx'x$ and $xx' = x'x$. Therefore, $x = x^2x' \in x^2S$ and $x = x'x^2 \in Sx^2$.
Now

$$\begin{aligned}
\big(x^2\,\widetilde{\mu}\ \cap\ \widetilde{\mu}\,x^2\big)(x) &= Min^i\Big(\big(x^2\,\widetilde{\mu}\big)(x), \big(\widetilde{\mu}\,x^2\big)(x)\Big) \\
&= Min^i\big(\sup_{\substack{x=x^2z \\ z \in S}} \widetilde{\mu}(z), \sup_{\substack{x=z^*x^2 \\ z^* \in S}} \widetilde{\mu}(z^*)\big) \\
&\geq Min^i\big(\sup_{\substack{x=x^2z \\ z \in R_x \cap C_x}} \widetilde{\mu}(z), \sup_{\substack{x=z^*x^2 \\ z^* \in R_x \cap C_x}} \widetilde{\mu}(z^*)\big) \\
&\geq Min^i\big(\sup_{x' \in R_x \cap C_x} \widetilde{\mu}(x'), \sup_{x' \in R_x \cap C_x} \widetilde{\mu}(x')\big) \\
&\geq \widetilde{\mu}(x).
\end{aligned}$$

Conversely, let the condition hold for an (*i-v*) fuzzy subsemigroup $\widetilde{\mu}$ of S. Let $x \in S$. If $\widetilde{\mu}(x) \neq \widetilde{0}$, by our assumption, $\big(x^2\,\widetilde{\mu}\ \cap\ \widetilde{\mu}\,x^2\big)(x) \geq \widetilde{\mu}(x)$
$\Longrightarrow Min^i\Big(\big(x^2\,\widetilde{\mu}\big)(x), \big(\widetilde{\mu}\,x^2\big)(x)\Big) \geq \widetilde{\mu}(x)$
$\Longrightarrow \big(x^2\,\widetilde{\mu}\big)(x) \neq \widetilde{0}$ and $\big(\widetilde{\mu}\,x^2\big)(x) \neq \widetilde{0}$. Thus, there exist $x', x'' \in S$ such that $x = x^2x' \in x^2S$ and $x = x''x^2 \in Sx^2$. Also, $\big(x^2\,\widetilde{\mu}\big)(x) = \sup\limits_{\substack{x\,=\,x^2z \\ z \in S}} \widetilde{\mu}(z) \geq \widetilde{\mu}(x)$ and $\big(\widetilde{\mu}\,x^2\big)(x) = \sup\limits_{\substack{x\,=\,z^*x^2 \\ z^* \in S}} \widetilde{\mu}(z^*) \geq \widetilde{\mu}(x)$. Now $xx'x = (x''x^2)x'x = x''(x^2x')x = x''x^2 = x$ and $xx''x = xx''(x^2x') = x(x''x^2)x' = x^2x' = x$. Therefore, $x(x''xx')x = (xx''x)x'x = xx'x = x$, $x(x''xx') = (xx''x)x' = xx' = x''x^2x' = x''x = x''(xx'x) = (x''xx')x$. Thus, $x''xx' \in R_x \cap C_x$ and hence $R_x \cap C_x \neq \varnothing$.
Now

$$\begin{aligned}
\sup_{z \in R_x \cap C_x} \widetilde{\mu}(z) &\geq \sup_{\substack{x''xx' \in R_x \cap C_x \\ x',\, x'' \in S}} \widetilde{\mu}(x''xx') \\
&\geq \sup_{\substack{x''xx' \in R_x \cap C_x \\ x',\, x'' \in S}} \Big\{Min^i\Big(Min^i\big(\widetilde{\mu}(x''), \widetilde{\mu}(x)\big), \widetilde{\mu}(x')\Big)\Big\}
\end{aligned}$$

(since $\widetilde{\mu}$ is an (*i-v*) fuzzy subsemigroup of S)

$$\geq Min^i\Big(\sup_{\substack{x''xx' \in R_x \cap C_x \\ x',\, x'' \in S}} \Big\{Min^i\big(\widetilde{\mu}(x''), \widetilde{\mu}(x')\big)\Big\}, \widetilde{\mu}(x)\Big)$$

$$\geq Min^i\Big(\sup_{\substack{x=x''x^2 \\ x=x^2x' \\ x',\, x'' \in S}} \Big\{Min^i\big(\widetilde{\mu}(x''), \widetilde{\mu}(x')\big)\Big\}, \widetilde{\mu}(x)\Big)$$

$$= Min^i\Big(\ Min^i\Big(\sup_{\substack{x=x^2x' \\ x' \in S}} \{\ \widetilde{\mu}(x')\ \},\ \sup_{\substack{x=x''x^2 \\ x'' \in S}} \{\ \widetilde{\mu}(x'')\ \}\Big), \widetilde{\mu}(x)\Big)$$

$$= \widetilde{\mu}(x).$$

Therefore, it follows from the above that $\widetilde{\mu}$ is an $(i\text{-}v)$ fuzzy weakly completely regular subsemigroup of S. □

Theorem 4.14. *Let $\widetilde{\mu}$ be an $(i\text{-}v)$ fuzzy subsemigroup of a semigroup S. Then $\widetilde{\mu}$ is an $(i\text{-}v)$ fuzzy weakly completely regular subsemigroup of S if and only if $x_{\widetilde{a}} \in \big({x^2}_{\widetilde{a}}\ o\ \widetilde{\mu}\ \cap\ \widetilde{\mu}\ o\ {x^2}_{\widetilde{a}}\big)$ for any $(i\text{-}v)$ fuzzy point $x_{\widetilde{a}} \in \widetilde{\mu}$.*

Definition 4.15. Let X be a set and $A \subseteq X$. For any $\widetilde{a} \in D[0,1]$, we define an $(i\text{-}v)$ fuzzy subset $\widetilde{a}A$ on X as :

$$\begin{aligned}(\widetilde{a}A)(x) &= \widetilde{a}, \text{ when } x \in A; \\ &= \widetilde{0}, \text{ otherwise.}\end{aligned}$$

Proposition 4.16. *Let $\{A_i : i \in \Omega\}$ be a set of sets and $\widetilde{a} \in D[0,1]$. Then $\bigcup\limits_{i \in \Omega} (\widetilde{a}\, A_i) = \widetilde{a}\,(\bigcup\limits_{i \in \Omega} A_i)$.*

Definition 4.17 ([1])**.** An $(i\text{-}v)$ fuzzy subset $\widetilde{\mu}$ of a group G is called an $(i\text{-}v)$ fuzzy subgroup if for any $x, y \in G$,
(i) $\widetilde{\mu}(xy) \geq Min^i\{\ \widetilde{\mu}(x), \widetilde{\mu}(y)\ \}$
(ii) $\widetilde{\mu}(x^{-1}) \geq \widetilde{\mu}(x)$.

Theorem 4.18 ([3])**.** *A semigroup S is completely regular if and only if S is a union of (disjoint) groups.*

Theorem 4.19. *Every $(i\text{-}v)$ fuzzy completely regular subsemigroup $\widetilde{\mu}$ of a semigroup S is a union of $(i\text{-}v)$ fuzzy subgroups of S.*

Proof. Let $\widetilde{\mu}$ be an $(i\text{-}v)$ fuzzy completely regular subsemigroup. Then we can write

$$\widetilde{\mu} = \bigcup_{\widetilde{a}\ \in D[0,1]} \widetilde{a}\,\widetilde{U}(\widetilde{\mu}, \widetilde{a}) = \bigcup_{\substack{\widetilde{a}\ \in D[0,1]\setminus\{\widetilde{0}\} \\ \widetilde{U}(\widetilde{\mu},\widetilde{a})\ \neq\ \varnothing}} \widetilde{a}\,\widetilde{U}(\widetilde{\mu}, \widetilde{a}).$$

Since $\widetilde{\mu}$ is $(i\text{-}v)$ fuzzy completely regular, $\widetilde{U}(\widetilde{\mu}, \widetilde{a})$ is completely regular, by Proposition 3.6 and hence $\widetilde{U}(\widetilde{\mu}, \widetilde{a}) = \bigcup\limits_{i \in \Omega} A_i(\widetilde{a})$, where $A_i(\widetilde{a})$'s $(i \in \Omega)$ are disjoint groups corresponding to $\widetilde{a}$, by Theorem 4.18 . Therefore,

$$\widetilde{\mu} \ = \bigcup_{\substack{\widetilde{a}\ \in D[0,1]\setminus\{\widetilde{0}\} \\ \widetilde{U}(\widetilde{\mu},\widetilde{a})\ \neq\ \varnothing}} \widetilde{a}\widetilde{U}(\widetilde{\mu}, \widetilde{a}) = \bigcup_{\widetilde{a}\ \in D[0,1]\setminus\{\widetilde{0}\}} \widetilde{a}\Big(\bigcup_{i \in \Omega} A_i(\widetilde{a})\ \Big) = \bigcup_{\substack{\widetilde{a}\ \in D[0,1]\setminus\{\widetilde{0}\} \\ i \in \Omega}} \widetilde{a}A_i(\widetilde{a})\ .$$

Since $A_i(\widetilde{a})$'s are disjoint groups, then it is easy to show that $\widetilde{a}A_i(\widetilde{a})$'s are disjoint $(i\text{-}v)$ fuzzy subgroups and hence the result. □

Definition 4.20. Let $\widetilde{\mu}_1$, $\widetilde{\mu}_2$ and $\widetilde{\mu}_3$ be (*i-v*) fuzzy subsets of a semigroup S such that $\widetilde{\mu}_1 \subseteq \widetilde{\mu}_3$ and $\widetilde{\mu}_2 \subseteq \widetilde{\mu}_3$. Then $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are said to be disjoint with respect to $\widetilde{\mu}_3$ if $Min^i\big(\widetilde{\mu}_1(x), \widetilde{\mu}_2(x)\big) \ \leq \ \inf_{u \in S}\{\, \widetilde{\mu}_3(u) \,\}$.

Definition 4.21. Let $\widetilde{\mu}$ be an (*i-v*) fuzzy subset of a semigroup S and G be a subgroup of S. Then an (*i-v*) fuzzy subset $\widetilde{\theta}$ of S is said to be an (*i-v*) fuzzy subgroup of $\widetilde{\mu}$ corresponding to G if

(i) $\widetilde{\theta}(x) \leq \widetilde{\mu}(x)$ for all $x \in S$

(ii) $\widetilde{\theta}(xy) \geq Min^i\big(\widetilde{\theta}(x), \widetilde{\theta}(y)\big)$ for all $x, y \in S$

(iii) $\widetilde{\theta}(x^{-1}) \geq \widetilde{\theta}(x)$ for all $x \in G$.

In [3], we know that a semigroup S is a semilattice of groups if S is a set union of mutually disjoint subgroups $\{S_i : i \in L$ (a semilattice)$\}$ of S such that, for every $j, k \in L$, the products $S_k S_j$ and $S_j S_k$ are both contained in the same subgroup S_l $(l \in L)$. Similar way, we define (*i-v*) fuzzy semilattice of (*i-v*) fuzzy subgroups in the following definition.

Definition 4.22. An (*i-v*) fuzzy subsemigroup $\widetilde{\mu}$ of a semigroup S is called (*i-v*) fuzzy semilattice of (*i-v*) fuzzy subgroups if

(i) there exists a family $\{\widetilde{\mu}_j : j \in L$ (a semilattice)$\}$ of (*i-v*) fuzzy subgroups of $\widetilde{\mu}$ such that it's corresponding family $\{S_j : j \in L\}$ of subgroups of S make a partition of S and $\widetilde{\mu} \ = \ \bigcup_{j \in L} \widetilde{\mu}_j$

(ii) $\widetilde{\mu}_j$ and $\widetilde{\mu}_k$ are disjoint with respect to $\widetilde{\mu}$ for any $j, k (\neq j) \in L$

(iii) for any $j, k \in L$, there exists $l \in L$ such that $\widetilde{\mu}_j \ o \ \widetilde{\mu}_k \subseteq \widetilde{\mu}_l$ and $\widetilde{\mu}_k \ o \ \widetilde{\mu}_j \subseteq \widetilde{\mu}_l$.

Now we characterize the (*i-v*) fuzzy completely regular subsemigroup by (*i-v*) fuzzy semilattice of (*i-v*) fuzzy subgroups as follows :

Theorem 4.23. *An (i-v) fuzzy subsemigroup $\widetilde{\mu}$ of a semigroup S is an (i-v) fuzzy completely regular if and only if $\widetilde{\mu}$ is an (i-v) fuzzy semilattice of (i-v) fuzzy subgroups of $\widetilde{\mu}$.*

Proof. Let $\widetilde{\mu}$ be an (*i-v*) fuzzy completely regular subsemigroup of S. Then S is completely regular. Therefore, from [3], we can write S is a union of mutually disjoint subgroups $\{S_i : i \in L\}$ of S such that for any $j, k \in L$, there exists $l \in L$ such that $S_j\, S_k \subseteq S_l$ and $S_k\, S_j \subseteq S_l$. Now we define (*i-v*) fuzzy subset $\widetilde{\mu}_j$ of S as

$$\begin{aligned} \widetilde{\mu}_j(x) \ &= \ \widetilde{\mu}(x), \ \text{ if } \ x \in S_j \, ; \\ &= \ \widetilde{0}, \quad \text{ otherwise;} \end{aligned}$$

for every $j \in L$ and $x \in S$. Then clearly, $\widetilde{\mu}_j \subseteq \widetilde{\mu}$ and for any two $j, k(\neq j) \in L$, $\widetilde{\mu}_j$ and $\widetilde{\mu}_k$ are pairwise disjoint w.r.t. $\widetilde{\mu}$. Now let $x \in S$. Then $x \in S_l$ for some $l \in L$. Therefore, $(\bigcup_{j \in L} \widetilde{\mu}_j)(x) = \sup_{j \in L} \widetilde{\mu}_j(x) = Max^i\big(\, \widetilde{\mu}_l(x), \ \sup_{j \in L, j \neq l} \widetilde{\mu}_j(x)\big) = \widetilde{\mu}(x)$. Thus, $(\bigcup_{j \in L} \widetilde{\mu}_j) = \widetilde{\mu}$. Again, let $x, y \in S$. If $x, y \in S_j$ for every $j \in L$, then $xy \in S_j$. Therefore, for every $j \in L$, $\widetilde{\mu}_j(xy) = \widetilde{\mu}(xy) \geq Min^i\,\big(\widetilde{\mu}(x), \widetilde{\mu}(y)\big) = Min^i\,\big(\widetilde{\mu}_j(x), \widetilde{\mu}_j(y)\big)$. Again, if either $x \notin S_j$ or $y \notin S_j$, then $\widetilde{\mu}_j(x) = \widetilde{0}$ or $\widetilde{\mu}_j(y) = \widetilde{0}$. Therefore, $\widetilde{\mu}_j(xy) \geq \widetilde{0} = Min^i(\widetilde{\mu}_j(x), \widetilde{\mu}_j(y))$. This implies that for every $j \in L$, $\widetilde{\mu}_j$ is an (*i-v*) fuzzy subsemigroup of S. Again, let $a \in S_j$. Since $\widetilde{\mu}$ is an (*i-v*) fuzzy completely regular subsemigroup, there exists $b \in R_a \cap C_a$ such that $\widetilde{\mu}(b) \geq \widetilde{\mu}(a)$. Let $b \in S_k$ for some $k \in L$. Therefore, $a^2 \in S_j$ and $a^2b = aba = a \in S_j$. Again, since $\{S_j : j \in L\}$ is a partition of S with $S_j\, S_k \subseteq S_l$, $S_k\, S_j \subseteq S_l$, it follows that $S_jS_k \subseteq S_j$ and hence $ba \in S_j$ and $bab \in S_j$. Since S_j is a group with identity e_j,(say) and $aba = a = ae_j$, it follows that $ba = e_j$. Now $a(bab) = (aba)b = ab = ba = e_j$ and $(bab)a = b(aba) = ba = e_j$. For unique inverse in a group, it shows that $a^{-1} = bab$. Therefore, $\widetilde{\mu}_j(a^{-1}) = \widetilde{\mu}_j(bab) = \widetilde{\mu}(bab) \geq Min^i\big(\widetilde{\mu}(b), \widetilde{\mu}(a)\big) = \widetilde{\mu}(a) = \widetilde{\mu}_j(a)$. Thus, every $\widetilde{\mu}_j$ is an (*i-v*) fuzzy subgroup of $\widetilde{\mu}$ corresponding to S_j.

Again, let $z \in S$. Suppose $z \in S_l$ for some $l \in L$. Then for any $j, k \in L$, $(\widetilde{\mu}_j\ o\ \widetilde{\mu}_k)(x) = \sup\limits_{x=pq}\big\{Min^i(\widetilde{\mu}_j(p), \widetilde{\mu}_k(q))\big\} \leq \sup\limits_{x=pq}\big\{Min^i(\widetilde{\mu}(p), \widetilde{\mu}(q))\big\} \leq \sup\limits_{x=pq}\big\{\widetilde{\mu}(pq)\big\}$ $= \sup\ \widetilde{\mu}(x) = \widetilde{\mu}(x) = \widetilde{\mu}_l(x)$. Again, if $x \notin S_l$, then there exists no such $p \in S_j$ and $q \in S_k$ that $x = pq$. Therefore, $(\widetilde{\mu}_j\ o\ \widetilde{\mu}_k)(x) = \sup\limits_{x=pq}\big\{Min^i(\widetilde{\mu}_j(p), \widetilde{\mu}_k(q))\big\} =$ $\widetilde{0} = \widetilde{\mu}_l(x)$. Thus, it follows that $\widetilde{\mu}_j\ o\ \widetilde{\mu}_k \subseteq \widetilde{\mu}_l$. In a similar way, we can prove that $\widetilde{\mu}_k\ o\ \widetilde{\mu}_j \subseteq \widetilde{\mu}_l$. Therefore, from above, it follows that $\widetilde{\mu}$ is an (*i-v*) fuzzy semilattice of (*i-v*) fuzzy subgroups.

Conversely, let $\widetilde{\mu}$ be an (*i-v*) fuzzy semilattice of (*i-v*) fuzzy subgroups. Then there exists a family $\{\widetilde{\mu}_j : j \in L\}$ of (*i-v*) fuzzy subgroups of $\widetilde{\mu}$ such that its corresponding family $\{S_j : j \in L\}$ of subgroups of S forms a partition of S and

(i) $\widetilde{\mu} = \bigcup\limits_{j \in L} \widetilde{\mu}_j$

(ii) $\widetilde{\mu}_j$ and $\widetilde{\mu}_k$ are disjoint with respect to $\widetilde{\mu}$ for any $j, k(\neq j) \in L$

(iii) for any $j, k \in L$, there exists $l \in L$ such that $\widetilde{\mu}_j\ o\ \widetilde{\mu}_k \subseteq \widetilde{\mu}_l$ and $\widetilde{\mu}_k\ o\ \widetilde{\mu}_j \subseteq \widetilde{\mu}_l$.

Let $x \in S$. Then $x \in S_l$ for some $l \in L$. Since S_l is a subgroup of S, $x^{-1} \in S_l$. Therefore, $\widetilde{\mu}_l(x^{-1}) \geq \widetilde{\mu}_l(x)$ and also, $x = xx^{-1}x$ and $xx^{-1} = x^{-1}x$, i.e. $x^{-1} \in R_x \cap C_x$. Now $\widetilde{\mu}(x) = (\bigcup\limits_{j \in L} \widetilde{\mu}_j)(x) = \sup\limits_{j \in L} \widetilde{\mu}_j(x) = Max^i\big(\widetilde{\mu}_l(x), \sup\limits_{j \neq l,\, j \in L} \widetilde{\mu}_j(x)\big) = \widetilde{\mu}_l(x) \leq$ $\widetilde{\mu}_l(x^{-1}) = Max^i\big(\widetilde{\mu}_l(x^{-1}), \sup\limits_{j \neq l,\, j \in L} \widetilde{\mu}_j(x^{-1})\big) = \sup\limits_{j \in L} \widetilde{\mu}_j(x^{-1}) = (\bigcup\limits_{j \in L} \widetilde{\mu}_j)(x^{-1}) =$ $\widetilde{\mu}(x^{-1})$. Hence $\widetilde{\mu}$ is (*i-v*) fuzzy completely regular. □

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