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A new method for functions expansion with using triangular fuzzy numbers

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Abstract. In this paper, A new mathematical function expansion for triangular fuzzy numbers of type (m, α, β) are devised. The proposed expressions are direct and do not need the computation of α-cut of the fuzzy number on which the operation is desired. The expressions have been devised using traditional algebraic mathematics and are easy to comprehend. Various numerical examples are also solved to demonstrate the use of contrived expressions. Direct mathematical expressions to evaluate expansion Trigonometric functions of type (m, α, β) are obtained using the basic analytical principles of algebraic mathematics and Taylor series expansion.

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1. Introduction

Algebraic equations involving fuzzy numbers are an important application of fuzzy set theory. Fuzzy numbers can be used for expert system reasoning, fuzzy process modeling and control, and so forth. Previous studies on fuzzy numbers have shown that there are no opposite and reverse fuzzy numbers in the sense of group structure [1, 4, 5]. Fuzzy systems including fuzzy set theory and fuzzy logic have many successful applications. Sophisticated fuzzy set theoretic methods have been applied to various areas ranging from fuzzy topological spaces to quantum optics, medicine and so on [2, 8]. But Most of the recent research work on special behavior of fuzzy numbers like factorial, exponential, logarithmic and trigonometric is limited and much of the work is concentrated on linear fuzzy system of equations and their applications [3, 7]. A good and a detailed representation of fuzzy numbers in the domain of combinatorics and their related arithmetic behavior is described by Kauffman and Gupta [5] but the method of explanation is based upon the α-cut approach and a clear view of its application on triangular fuzzy numbers is not adequately comprehensible.

In this paper a specific attention is given to fuzzy numbers of type (m, α, β) and the related mathematical expressions for evaluation Taylor series expansion. For example used along with simple mathematical multiplication of two triangular fuzzy numbers Bansal [1]. Moreover, Rezvani [9, 10, 11, 12, 13] have a good work on behavior of fuzzy numbers.

The purpose of this paper is to present a calculation method for expansion functions with triangular fuzzy numbers. Section 2 presents basic concepts and definitions of triangular fuzzy numbers.Section 3 we applied numerical example for $\cos(x)$ and $\cosh(x)$. Section 4 introduces how to calculate $\sin(x)$, and $\sinh(x)$, $\arctan(x)$ and use our method to calculate triangular fuzzy numbers. Finally, concluding remarks, figures and examples are given in Section 4.

2. Preliminaries

Definition 2.1. A fuzzy number $\tilde{M}$ is a convex normalized fuzzy set $\tilde{M}$ of the real line $\mathbb{R}$ such thet

1. It exists exactly one $x_0 \in \mathbb{R}$ with $\mu_{\tilde{M}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{M}$).
2. $\mu_{\tilde{M}}(x)$ is piecewise continuous.

Definition 2.2. Generally, a generalized fuzzy number A is described as any fuzzy subset of the real line R, whose membership function $\mu_A(u)$ satisfies the following conditions,

(i) $\mu_A(u)$ is a continuos mapping from R to the closed interval[0,1],
(ii) $\mu_A(u) = 0, -\infty < u \leq \underline{a}$,
(iii) $\mu_A(u) = L(u)$ is strictly increasing on $[\underline{a}, a]$,
(iv) $\mu_A(u) = w, u = a$,
(v) $\mu_A(u) = R(u)$ is strictly decreasing on $[a, \overline{a}]$,
(vi) $\mu_A(u) = 0, \overline{a} \leq u < \infty$
Where $0 < w \leq 1$ and $\underline{a}, a$, and $\overline{a}$ are real numbers.
We call this type of generalized fuzzy number a triangular fuzzy number, and it is denoted bye $A = (\underline{a}, a, \overline{a}; w)_{LR}$.
When $w = 1$, this type of generalized fuzzy number is called normal fuzzy number and is represented by $A = (\underline{a}, a, \overline{a})_{LR}$.
A fuzzy number $\tilde{M}$ is of LR-type if there exist reference functions L (for left), R (for right), and scalars $\alpha > 0$, $\beta > 0$ with

$$\mu_{\tilde{M}}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & x \leq m, \\ R(\frac{x-m}{\beta}) & x \geq m, \end{cases} \tag{2.1}$$

m, called the mean value of $\tilde{M}$, is real number, and α , β are called the left and right spreads, respectively. Symbolically, $\tilde{M}$ is denoted by $(m, \alpha, \beta)_{LR}$.

Theorem 2.3. *Let $\tilde{M}$, $\tilde{N}$ be two fuzzy numbers of LR-type:*
$\tilde{M} = (m, \alpha, \beta)_{LR}$, $\tilde{N} = (n, \lambda, \delta)_{LR}$
Then

1. $(m,\alpha,\beta)_{LR} \oplus (n,\lambda,\delta)_{LR} = (m+n, \alpha+\lambda, \beta+\delta)_{LR}$.
2. $-(m,\alpha,\beta)_{LR} = (-m,\beta,\alpha)_{LR}$
3. $(m,\alpha,\beta)_{LR} \ominus (n,\lambda,\delta)_{LR} = (m-n, \alpha+\delta, \beta+\lambda)_{LR}$.

Definition 2.4. Let $\tilde{M}$, $\tilde{N}$ be fuzzy numbers as in Definition 2.1, then
$(m,\alpha,\beta)_{LR} \odot (n,\lambda,\delta)_{LR} \approx (mn, m\lambda+n\alpha, m\delta+n\beta)_{LR}$
for $\tilde{M}$, $\tilde{N}$ positive;
$(m,\alpha,\beta)_{LR} \odot (n,\lambda,\delta)_{LR} \approx (mn, n\alpha-m\delta, n\beta-m\lambda)_{LR}$
for $\tilde{M}$ positive, $\tilde{N}$ negative, and
$(m,\alpha,\beta)_{LR} \odot (n,\lambda,\delta)_{LR} \approx (mn, -n\beta-m\delta, n\alpha-m\lambda)_{LR}$
for $\tilde{M}$, $\tilde{N}$ negative.

Definition 2.5. Let $\tilde{M}$, $\tilde{N}$ be fuzzy numbers as in Definition 2.4, then
$(m,\alpha,\beta)_{LR} \odot (n,\lambda,\delta)_{LR} \approx (mn, m\lambda+n\alpha, m\delta+n\beta)_{LR}$ for $\tilde{M}$, $\tilde{N}$ positive.

we can have a general formula for the n^{th} $(n>0)$:
Let $\widetilde{a} = (m,\alpha,n)$ be triangular fuzzy number, where f polynomial functions as follows
$f(m,\alpha,n) = a_{1n}m^{n-1}\alpha + a_{2n}m^{n-2}\alpha^2 + a_{3n}m^{n-3}\alpha^3 + \ldots + a_{nn}\alpha^n$,
and $a_{1n} = n$, $a_{nn} = (-1)^{n-1}$, $a_{in} = (-1)^{i-1}\{|a_{i,n-1}| + |a_{i-1,n-1}|\}$, $1 < i < n$.
In this paper, we intend to formula the expansion of a new fuzzy number using the taylor series expansion method. So we define fuzzy taylor series as follows

$$f(m,\alpha,n) = a_{1n}m^{n-1}\alpha + a_{2n}m^{n-2}\alpha^2 + a_{3n}m^{n-3}\alpha^3 + \ldots + a_{nn}\alpha^n,$$

$$f'(m,\alpha,n) = a_{1n}m^{n-1} + 2a_{2n}m^{n-2}\alpha + 3a_{3n}m^{n-3}\alpha^2 + \ldots + na_{nn}\alpha^{n-1},$$

$$f''(m,\alpha,n) = 2a_{2n}m^{n-2} + 6a_{3n}m^{n-3}\alpha + 12a_{4n}m^{n-4}\alpha^2 + \ldots + n(n-1)a_{nn}\alpha^{n-2},$$

$$f^{(3)}(m,\alpha,n) = 6a_{3n}m^{n-3} + 24a_{4n}m^{n-4}\alpha + 60a_{5n}m^{n-5}\alpha^2 + \cdots + n(n-1)(n-2)a_{nn}\alpha^{n-3},$$

$$f^{(4)}(m,\alpha,n) = 24a_{4n}m^{n-4} + 120a_{5n}m^{n-5}\alpha + 360a_{6n}m^{n-6}\alpha^2 + \cdots + n(n-1)(n-2)(n-3)a_{nn}\alpha^{n-4},$$

$$\ldots$$

$$f^{(i)}(m,\alpha,n) = i!a_{in}m^{n-i} + (i+1)!a_{(i+1)n}m^{n-(i+1)}\alpha + (i+2)!a_{(i+2)n}m^{n-(i+2)}\alpha^2 + \cdots + n(n-1)(n-2)(n-3)\ldots(n-(i-1))a_{nn}\alpha^{n-i}$$

$$f(\widetilde{x}) = f(\widetilde{a}) + \tfrac{(\widetilde{x}-\widetilde{a})f'(\widetilde{a})}{1!} + \tfrac{(\widetilde{x}-\widetilde{a})^2 f''(\widetilde{a})}{2!} + \tfrac{(\widetilde{x}-\widetilde{a})^3 f^{(3)}(\widetilde{a})}{3!} + \ldots + \tfrac{(\widetilde{x}-\widetilde{a})^n f^{(n)}(\widetilde{a})}{n!}$$

$$f(\widetilde{x}) = a_{1n}m^{n-1}\alpha + a_{2n}m^{n-2}\alpha^2 + a_{3n}m^{n-3}\alpha^3 + \cdots + a_{nn}\alpha^n + \tfrac{(x-a)(a_{1n}m^{n-1}+2a_{2n}m^{n-2}\alpha+3a_{3n}m^{n-3}\alpha^2+\cdots+na_{nn}\alpha^{n-1})}{1!} + \tfrac{1}{2!}\Big((x-a)^2(2a_{2n}m^{n-2} + 6a_{3n}m^{n-3}\alpha + 12a_{4n}m^{n-4}\alpha^2 + \cdots + n(n-1)a_{nn}\alpha^{n-2})\Big) + \tfrac{1}{3!}\Big((x-a)^3(6a_{3n}m^{n-3} + 24a_{4n}m^{n-4}\alpha + 60a_{5n}m^{n-5}\alpha^2 + \cdots$$

$$+n(n-1)(n-2)a_{nn}\alpha^{n-3})\Big)$$

$\cdots$

$$+\tfrac{1}{n!}\Big((x-a)^n(i!a_{in}m^{n-i}+(i+1)!a_{(i+1)n}m^{n-(i+1)}\alpha$$
$$+(i+2)!a_{(i+2)n}m^{n-(i+2)}\alpha^2+\cdots$$
$$+n(n-1)(n-2)(n-3)...(n-(i-1))a_{nn}\alpha^{n-i})\Big).$$

$(m,\alpha,\beta)^2=(m^2,2m\alpha-\alpha^2,2m\beta+\beta^2)$,
$(m,\alpha,\beta)^3=(m^3,3m^2\alpha-3m\alpha^2+\alpha^3,3m^2\beta+3m\beta^2+\beta^3)$,
$(m,\alpha,\beta)^4=(m^4,4m^3\alpha-6m^2\alpha^2+4m\alpha^3-\alpha^4,4m^3\beta+6m^2\beta^2+4m\beta^3+\beta^4)$,
$(m,\alpha,\beta)^5=(m^5,5m^4\alpha-10m^3\alpha^2+10m^2\alpha^3-5m\alpha^4+\alpha^5,$
$5m^4\beta+10m^3\beta^2+10m^2\beta^3+5m\beta^4+\beta^5)$,
$(m,\alpha,\beta)^6=(m^6,6m^5\alpha-15m^4\alpha^2+20m^3\alpha^3-15m^2\alpha^4+6m\alpha^5-\alpha^6,$
$6m^5\beta+15m^4\beta^2+20m^3\beta^3+15m^2\beta^4+6m\beta^5+\beta^6)$,
$\cdots$.

3. A new fuzzy Taylor series expansion for cos(x) and cosh(x)

Since $cos(x)$ expansion of a real number is defined as follows, we will extend the some formula for a triangular fuzzy number

$$\cos(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...,$$

$$\cos(x)=1+\sum_{i=1}^{\infty}\frac{(-1)^i(x)^{2i}}{(2i)!},$$

For $\widetilde{x}=(m,\alpha,\beta)$, we have

$$\cos(\widetilde{x})=1-\frac{\widetilde{x}^2}{2!}+\frac{\widetilde{x}^4}{4!}-\frac{\widetilde{x}^6}{6!}+...,$$

Or

$$\cos(m,\alpha,\beta)=(1,0,0)+\sum_{i=1}^{\infty}\frac{(-1)^i(m,\alpha,\beta)^{2i}}{(2i)!}.$$

let

$$\cos(m^{'})=1-\frac{m^{'2}}{2!}+\frac{m^{'4}}{4!}-\frac{m^{'6}}{6!}\pm\cdots,$$

now we suppose $m^{'}=(m-\alpha)$, so we have

$$\cos(m-\alpha)=1-\frac{(m-\alpha)^2}{2!}+\frac{(m-\alpha)^4}{4!}-\frac{(m-\alpha)^6}{6!}\pm\cdots,$$

Similarly using the general recursive formula for (m,α,β) we have

$\cos(\widetilde{x})=1-\frac{\widetilde{x}^2}{2!}+\frac{\widetilde{x}^4}{4!}-\frac{\widetilde{x}^6}{6!}+\cdots=1-\frac{(m,\alpha,\beta)^2}{2!}+\frac{(m,\alpha,\beta)^4}{4!}$

$\cos(\widetilde{x})=1-\frac{\widetilde{x}^2}{2!}+\frac{\widetilde{x}^4}{4!}-\frac{\widetilde{x}^6}{6!}+\cdots=1-\frac{(m,\alpha,\beta)^2}{2!}+\frac{(m,\alpha,\beta)^4}{4!}-\frac{(m,\alpha,\beta)^6}{6!}\pm\cdots$

$$= 1 - \tfrac{1}{2!}(m^2, 2m\alpha - \alpha^2, 2m\beta + \beta^2) + \tfrac{1}{4!}(m^4, 4m^3\alpha - 6m^2\alpha^2$$
$$+4m\alpha^3 - \alpha^4, 4m^3\beta + 6m^2\beta^2 + 4m\beta^3 + \beta^4)$$
$$-\tfrac{1}{6!}(m^6, 6m^5\alpha - 15m^4\alpha^2 + 20m^3\alpha^3 - 15m^2\alpha^4 + 6m\alpha^5 - \alpha^6,$$
$$6m^5\beta + 15m^4\beta^2 + 20m^3\beta^3 + 15m^2\beta^4 + 6m\beta^5 + \beta^6) \pm \cdots$$
$$= (1 - \tfrac{m^2}{2!} + \tfrac{m^4}{4!} - \tfrac{m^6}{6!} \pm ...), (\alpha(-\tfrac{2m}{2!} + \tfrac{4m^3}{4!} - \tfrac{6m^5}{6!} \mp \cdots) + \alpha^2(\tfrac{1}{2!} - \tfrac{6m^2}{4!} + \tfrac{15m^4}{6!} \pm \cdots)$$
$$+\alpha^3(\tfrac{4m}{4!} - \tfrac{20m^3}{6!} \pm \cdots)\cdots), (\beta(-\tfrac{2m}{2!} + \tfrac{4m^3}{4!} - \tfrac{6m^5}{6!} \mp \cdots)$$
$$+\beta^2(-\tfrac{1}{2!} + \tfrac{6m^2}{4!} - \tfrac{15m^4}{6!} \mp \cdots) + \beta^3(\tfrac{4m}{4!} - \tfrac{20m^3}{6!} \pm \cdots)\cdots)$$
$= (\cos m, \cos m - \cos m^{'}, \cos m^{''} - \cos m)$. In result

$$\cos(m, \alpha, \beta) = (\cos m, \cos m - \cos m^{'}, \cos m^{''} - \cos m).$$

where we supposed

$$m^{'} = (m - \alpha)$$
$$m^{''} = (m + \beta)$$

Remark 3.1. In similar method, we can obtain $cosh(x)$ as follows

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$
$$\cosh(x) = 1 + \sum_{i=1}^{\infty} \frac{(x)^{2i}}{(2i)!}$$
$$\cosh(\widetilde{x}) = 1 + \frac{\widetilde{x}^2}{2!} + \frac{\widetilde{x}^4}{4!} + \frac{\widetilde{x}^6}{6!} + \cdots$$
$$\cosh(m, \alpha, \beta) = (1, 0, 0) + \textstyle\sum_{i=1}^{\infty} \frac{(m,\alpha,\beta)^{2i}}{(2i)!} = 1 + \frac{(m,\alpha,\beta)^2}{2!} + \frac{(m,\alpha,\beta)^4}{4!} + \frac{(m,\alpha,\beta)^6}{6!} + \cdots$$
$$= 1 + \tfrac{(m^2, 2m\alpha - \alpha^2, 2m\beta + \beta^2)}{2!} + \tfrac{(m^4, 4m^3\alpha - 6m^2\alpha^2 + 4m\alpha^3 - \alpha^4, 4m^3\beta + 6m^2\beta^2 + 4m\beta^3 + \beta^4)}{4!}$$
$$+\tfrac{1}{6!}(m^6, 6m^5\alpha - 15m^4\alpha^2 + 20m^3\alpha^3 - 15m^2\alpha^4 + 6m\alpha^5 - \alpha^6,$$
$$6m^5\beta + 15m^4\beta^2 + 20m^3\beta^3 + 15m^2\beta^4 + 6m\beta^5 + \beta^6) + \cdots$$
$$= (1 + \tfrac{m^2}{2!} + \tfrac{m^4}{4!} + \tfrac{m^6}{6!} + \cdots), (\alpha(\tfrac{2m}{2!} + \tfrac{4m^3}{4!} + \tfrac{6m^5}{6!} + \cdots) - \alpha^2(\tfrac{1}{2!} + \tfrac{6m^2}{4!} + \tfrac{15m^4}{6!} + \cdots)$$
$$+\alpha^3(\tfrac{4m}{4!} + \tfrac{20m^3}{6!} + \cdots)\cdots), (\beta(\tfrac{2m}{2!} + \tfrac{4m^3}{4!} + \tfrac{6m^5}{6!} + \cdots) + \beta^2(\tfrac{1}{2!} + \tfrac{6m^2}{4!} + \tfrac{15m^4}{6!} + \cdots)$$
$$+\beta^3(\tfrac{4m}{4!} + \tfrac{20m^3}{6!} + \cdots)\cdots)$$
$$= (\cosh m, \cosh m - \cosh m^{'}, \cosh m^{''} - \cosh m).$$

In result

$$\cosh(m, \alpha, \beta) = (\cosh m, \cosh m - \cosh m^{'}, \cosh m^{''} - \cosh m).$$

where we supposed

$$m^{'} = (m - \alpha)$$

$$m^{''} = (m + \beta)$$

Example 3.2. Find fuzzy $cos(x)$ expansion in $(18, 14, 9)$.

$$\widetilde{x} = (m, \alpha, \beta) = (18, 14, 9)$$

$$\cos(\widetilde{x}) = \cos(m, \alpha, \beta) = (\cos m, \cos m - \cos(m - \alpha), \cos(m + \beta) - \cos m),$$

$$\cos(18, 14, 9) = (\cos 18, \cos 18 - \cos m^{'}, \cos m^{''} - \cos 18)$$

$$m^{'} = (18 - 14) = 4$$

$$m^{''} = (18 + 9) = 27$$

then

$$\begin{aligned}
\cos(18, 14, 9) &= (\cos 18, \cos 18 - \cos 4, \cos 27 - \cos 18) \\
&= (0.951056516295154, 0.951056516295154 - 0.997564050259824, \\
&\qquad 0.891006524188368 - 0.951056516295154) \\
&= (0.951056516295154, -0.046507533964671, -0.060049992106786) \\
&\simeq (0.95, -0.05, -0.06).
\end{aligned}$$

Example 3.3. Find fuzzy $cosh(x)$ expansion in $(18, 14, 9)$.

$$\widetilde{x} = (m, \alpha, \beta) = (18, 14, 9)$$

$$\cosh(\widetilde{x}) = \cosh(m, \alpha, \beta) = (\cosh m, \cosh m - \cosh(m - \alpha), \cosh(m + \beta) - \cosh m),$$

$$\cosh(18, 14, 9) = (\cosh 18, \cosh 18 - \cosh m^{'}, \cosh m^{''} - \cosh 18)$$

$$m^{'} = (18 - 14) = 4$$

$$m^{''} = (18 + 9) = 27$$

then

$$\cosh(18, 14, 9) = (\cosh 18, \cosh 18 - \cosh 4, \cosh 27 - \cosh 18)$$

$$= (3.282998456866526e + 007, 3.282998456866526e + 007 - 27.308232836016487,$$

$$2.660241203008993e + 011 - 3.282998456866526e + 007)$$

$$= (3.282998456866526e + 007, 3.282995726043243e + 007, 2.659912903163307e + 011).$$

4. Fuzzy Taylor series expansion for sin(x), sinh(x)

In last section we defined a new fuzzy Taylor series expansion, Since $sin(x)$ expansion of a real number is defined as follows, we will extend the some formula for a triangular fuzzy number

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm ...,$$

$$\sin(x) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}(x)^{2i-1}}{(2i-1)!}.$$

For $\widetilde{x} = (m, \alpha, \beta)$, we have

$$\sin(\widetilde{x}) = \frac{\widetilde{x}}{1!} - \frac{\widetilde{x}^3}{3!} + \frac{\widetilde{x}^5}{5!} - \frac{\widetilde{x}^7}{7!} \pm ...,$$

Or

$$\sin(\widetilde{x}) = \sum_{i=1}^{\infty} \frac{(-1)^{i=1}(m, \alpha, \beta)^{2i-1}}{(2i-1)!}.$$

let

$$\sin(m^{'}) = \frac{m}{1!} - \frac{m^{'3}}{3!} + \frac{m^{'5}}{5!} - \frac{m^{'7}}{7!} \pm ...,$$

now we suppose $m^{'} = (m - \alpha)$, so we have

$$\sin(m-\alpha) = \frac{(m-\alpha)}{1!} - \frac{(m-\alpha)^3}{3!} + \frac{(m-\alpha)^5}{5!} - \frac{(m-\alpha)^7}{7!} \pm ...,$$

Similarly using the general recursive formula for (m, α, β) we have

$$\begin{aligned}
\sin(\widetilde{x}) &= \tfrac{\widetilde{x}}{1!} - \tfrac{\widetilde{x}^3}{3!} + \tfrac{\widetilde{x}^5}{5!} - \tfrac{\widetilde{x}^7}{7!} + \cdots = \tfrac{(m,\alpha,\beta)}{1!} - \tfrac{(m,\alpha,\beta)^3}{3!} + \tfrac{(m,\alpha,\beta)^5}{5!} \mp \cdots \\
&= \tfrac{(m,\alpha,\beta)}{1!} - \tfrac{1}{3!}\left(m^3, 3m^2\alpha - 3m\alpha^2 + \alpha^3, 3m^2\beta + 3m\beta^2 + \beta^3\right) \\
&\quad + \tfrac{1}{5!}\Big(m^5, 5m^4\alpha - 10m^3\alpha^2 + 10m^2\alpha^3 - 5m\alpha^4 + \alpha^5, 5m^4\beta \\
&\quad + 10m^3\beta^2 + 10m^2\beta^3 + 5m\beta^4 + \beta^5\Big) \mp \cdots \\
&= (m - \tfrac{m^3}{3!} + \tfrac{m^5}{5!} \mp \cdots), (\alpha(1 - \tfrac{3m^2}{3!} + \tfrac{5m^4}{5!} \mp \cdots) + \alpha^2(\tfrac{3m}{3!} - \tfrac{10m^3}{5!} \pm \cdots)\cdots), \\
&\quad (\beta(1 - \tfrac{3m^2}{3!} + \tfrac{5m^4}{5!} \mp \cdots) + \beta^2(-\tfrac{3m}{3!} + \tfrac{10m^3}{5!} \mp \cdots)\cdots) \\
&= (\sin m, \sin m - \sin m^{'}, \sin m^{''} - \sin m).
\end{aligned}$$

In result

$$\sin(m, \alpha, \beta) = (\sin m, \sin m - \sin m^{'}, \sin m^{''} - \sin m).$$

where we supposed

$$m^{'} = (m - \alpha)$$

$$m^{''} = (m + \beta)$$

Remark 4.1. In similar method, we can obtain $\sin(x)$ as follows

$$\sinh(x) = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ...$$

$$\sinh(x) = \sum_{i=1}^{\infty} \frac{(x)^{2i-1}}{(2i-1)!}$$

$$\begin{aligned}
\sinh(\widetilde{x}) &= \tfrac{\widetilde{x}}{1!} + \tfrac{\widetilde{x}^3}{3!} + \tfrac{\widetilde{x}^5}{5!} + \tfrac{\widetilde{x}^7}{7!} + \cdots \\
&= \tfrac{(m,\alpha,\beta)}{1!} + \tfrac{(m^3, 3m^2\alpha - 3m\alpha^2 + \alpha^3, 3m^2\beta + 3m\beta^2 + \beta^3)}{3!} \\
&\quad + \tfrac{1}{5!}\Big(m^5, 5m^4\alpha - 10m^3\alpha^2 + 10m^2\alpha^3 - 5m\alpha^4 + \alpha^5, 5m^4\beta \\
&\quad + 10m^3\beta^2 + 10m^2\beta^3 + 5m\beta^4 + \beta^5\Big) + \cdots \\
&= (m + \tfrac{m^3}{3!} + \tfrac{m^5}{5!} + ...), (\alpha(1 + \tfrac{3m^2}{3!} + \tfrac{5m^4}{5!} + \cdots) + \alpha^2(-\tfrac{3m}{3!} - \tfrac{10m^3}{5!} - \cdots)\cdots), \\
&\quad (\beta(1 + \tfrac{3m^2}{3!} + \tfrac{5m^4}{5!} + ...) + \beta^2(\tfrac{3m}{3!} + \tfrac{10m^3}{5!} + \cdots)\cdots) \\
&= (\sinh m, \sinh m - \sinh m^{'}, \sinh m^{''} - \sinh m).
\end{aligned}$$

In result

$$\sinh(m,\alpha,\beta) = (\sinh m, \sinh m - \sinh m^{'}, \sinh m^{''} - \sinh m).$$

where we supposed

$$m^{'} = (m - \alpha)$$

$$m^{''} = (m + \beta)$$

Example 4.2. Find fuzzy $sin(x)$ expansion in $(18, 14, 9)$.

$$\widetilde{x} = (m,\alpha,\beta) = (18, 14, 9)$$

$$\sin(\widetilde{x}) = \sin(m,\alpha,\beta) = (\sin m, \sin m - \sin(m-\alpha), \sin(m+\beta) - \sin m),$$

$$\sin(18, 14, 9) = (\sin 18, \sin 18 - \sin m^{'}, \sin m^{''} - \sin 18)$$

$$m^{'} = (18 - 14) = 4$$

$$m^{''} = (18 + 9) = 27$$

then

$$\begin{aligned}
\sin(18, 14, 9) &= (\sin 18, \sin 18 - \sin 4, \sin 27 - \sin 18) \\
&= (0.309016994374947, 0.309016994374947 - 0.069756473744125, \\
&\qquad 0.453990499739547 - 0.309016994374947) \\
&= (0.309016994374947, 0.239260520630822, 0.144973505364599) \\
&\simeq (0.31, 0.24, 0.14).
\end{aligned}$$

Example 4.3. Find fuzzy $sinh(x)$ expansion in $(18, 14, 9)$.

$$\widetilde{x} = (m, \alpha, \beta) = (18, 14, 9)$$

$$\sinh(\widetilde{x}) = \sinh(m, \alpha, \beta) = (\sinh m, \sinh m - \sinh(m - \alpha), \sinh(m + \beta) - \sinh m),$$

$$\sinh(18, 14, 9) = (\sinh 18, \sinh 18 - \sinh m^{'}, \sinh m^{''} - \sinh 18)$$

$$m^{'} = (18 - 14) = 4$$

$$m^{''} = (18 + 9) = 27$$

then

$$\begin{aligned}\sinh(18, 14, 9) &= (\sinh 18, \sinh 18 - \sinh 4, \sinh 27 - \sinh 18)\\ &= (3.282998456866525e + 007, 3.282998456866525e + 007 - 27.289917197127750,\\ &\quad 2.660241203008993e + 011 - 3.282998456866525e + 007)\\ &= (3.282998456866525e + 007, 3.282995727874805e + 007,\\ &\quad 2.659912903163307e + 011)\end{aligned}$$

In similar way, we introduce a new fuzzy **arctan(x)** Taylor series expansion for triangular fuzzy numbers

$$\arctan(x) = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...,$$

$$\arctan(x) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}(x)^{2i-1}}{(2i-1)}.$$

For $\widetilde{x} = (m, \alpha, \beta)$, we have

$$\arctan(\widetilde{x}) = \frac{\widetilde{x}}{1} - \frac{\widetilde{x}^3}{3} + \frac{\widetilde{x}^5}{5} - \frac{\widetilde{x}^7}{7} + ...,$$

Or

$$\arctan(\widetilde{x}) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}(m, \alpha, \beta)^{2i-1}}{(2i-1)}.$$

Similarly using the general recursive formula for (m, α, β) we have

$$\begin{aligned}\arctan(\widetilde{x}) &= \tfrac{\widetilde{x}}{1} - \tfrac{\widetilde{x}^3}{3} + \tfrac{\widetilde{x}^5}{5} - \tfrac{\widetilde{x}^7}{7} + \cdots\\ &= \tfrac{(m,\alpha,\beta)}{1} - \tfrac{(m,\alpha,\beta)^3}{3} + \tfrac{(m,\alpha,\beta)^5}{5} - \tfrac{(m,\alpha,\beta)^7}{7} \pm \cdots\\ &= \tfrac{(m,\alpha,\beta)}{1} - \tfrac{(m^3, 3m^2\alpha - 3m\alpha^2 + \alpha^3, 3m^2\beta + 3m\beta^2 + \beta^3)}{3}\\ &\quad + \tfrac{1}{5}\Big(m^5, 5m^4\alpha - 10m^3\alpha^2 + 10m^2\alpha^3 - 5m\alpha^4 + \alpha^5, 5m^4\beta +\\ &\quad 10m^3\beta^2 + 10m^2\beta^3 + 5m\beta^4 + \beta^5\Big) \mp \cdots\\ &= (\tfrac{m}{1} - \tfrac{m^3}{3} + \tfrac{m^5}{5} \mp \cdots), (\alpha(1 - \tfrac{3m^2}{3} + \tfrac{5m^4}{5} \mp \cdots) + \alpha^2(\tfrac{3m}{3} - \tfrac{10m^3}{5} \pm \cdots)\cdots),\end{aligned}$$

$$(\beta(1-\frac{3m^2}{3}+\frac{5m^4}{5}\mp...)+\beta^2(-\frac{3m}{3}+\frac{10m^3}{5}\mp...)...)$$
$$=(\arctan m, \arctan m-\arctan m^{'}, \arctan m^{''}-\arctan m).$$

In result

$$\arctan(m,\alpha,\beta)=(\arctan m, \arctan m-\arctan m^{'}, \arctan m^{''}-\arctan m).$$

where we supposed

$$m^{'}=(m-\alpha)$$
$$m^{''}=(m+\beta)$$

Example 4.4. Find fuzzy $\arctan(x)$ expansion in $(18, 14, 9)$.

$$\widetilde{x}=(m,\alpha,\beta)=(18,14,9)$$
$$\arctan(\widetilde{x})=\arctan(m,\alpha,\beta)=(\arctan m, \arctan m-\arctan(m-\alpha),$$
$$\arctan(m+\beta)-\arctan m),$$
$$\arctan(18,14,9)=(\arctan 18, \arctan 18-\arctan m^{'}, \arctan m^{''}-\arctan 18)$$
$$m^{'}=(18-14)=4$$
$$m^{''}=(18+9)=27$$

then

$$\arctan(18,14,9)=(\arctan 18, \arctan 18-\arctan 4, \arctan 27-\arctan 18)$$
$$=(86.8202, 86.8202-75.9638, 87.8789-86.8202)=(86.8202, 10.8564, 1.0587).$$

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