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Intuitionistic fuzzy $\mathcal{V}$ basically disconnected spaces

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Abstract. In this paper we introduce the new concept of an intuitionistic fuzzy $\mathcal{V}$ space and intuitionistic fuzzy $\mathcal{V}$ basically disconnected spaces. Besides giving some interesting propositions of these spaces. We also prove Tietze extension theorem for an intuitionistic fuzzy $\mathcal{V}$ basically disconnected spaces

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [13]. Fuzzy sets have applications in many fields such as information [10] and control [11]. The theory of fuzzy topological spaces was introduced and developed by Chang [4]. The concept of fuzzy basically disconnected spaces was introduced and studied in [12]. Bruce Hutton [7] constructed an interesting L-fuzzy topological space called the L-fuzzy unit interval which plays the same role in fuzzy topology that the unit interval plays in general topology. Using the concept of L-fuzzy unit interval, Tomasz Kubiaz [8, 9] extended the Urysohn lemma and Tietze extension theorem for the L-fuzzy normal spaces. Atanassov [1] introduced and studied intuitionistic fuzzy sets. On the otherhand, Coker [5] introduced the notions of an intuitionistic fuzzy topological space and some other concepts. In this paper, we introduced the concept of an intuitionistic fuzzy $\mathcal{V}$ space, intuitionistic fuzzy $\mathcal{V}$ basically disconnected spaces, intuitionistic fuzzy real line, intuitionistic fuzzy unit interval, lower(resp. upper) intuitionistic fuzzy $\mathcal{V}$ continuous function and strongly intuitionistic fuzzy $\mathcal{V}$ continuous function are introduced and studied. some interesting Propositions and Remarks are also discussed. Tietze extension theorem for intuitionistic fuzzy $\mathcal{V}$ basically disconnected spaces has been established.

2. Preliminaries

Definition 2.1 ([5]). Let X be a nonempty fixed set and I is the closed interval [0,1]. An intuitionistic fuzzy set(IFS) A is an object having the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$,where the mapping $\mu_A : X \longrightarrow I$ and $\gamma_A : X \longrightarrow I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) for each element $x \in X$ to the set A respectively and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$. Obviously,every fuzzy set A on a nonempty set X is an IFS of the following form, $A = \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle : x \in X\}$. For the sake of simplicity, we shall use the symbol $A = \langle x, \mu_A, \gamma_A\rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$. For a given nonempty set X, the family of all IFSs in X is denoted by ζ^X.

Definition 2.2 ([5]). Let X be a nonempty set and the IFSs A and B in the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$. Then

(i) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$;
(ii) $\overline{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle : x \in X\}$.

Definition 2.3 ([5]). The IFSs $0_\sim$ and $1_\sim$ are defined by $0_\sim=\{\langle x, 0, 1\rangle : x \in X\}$ and $1_\sim=\{\langle x, 1, 0\rangle : x \in X\}$.

Definition 2.4 ([5]). An intuitionistic fuzzy topology (IFT) in Coker's sense on a nonempty set X is a family τ of IFSs in X satisfying the following axioms:

(i) $0_\sim, 1_\sim \in \tau$;
(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$;
(iii) $\cup G_i \in \tau$ for arbitrary family $\{G_i \mid i \in I\} \subseteq \tau$.

In this paper by (X, τ) or simply by X we will denote the Coker's intuitionistic fuzzy topological space (IFTS). Each IFS in τ is called intuitionistic fuzzy open set (IFOS) in X. The complement $\overline{A}$ of an IFOS A in X is called an intuitionistic fuzzy closed set (IFCS) in X.

Definition 2.5 ([5]). Let A be an IFS in IFTS X. Then

$int(A) = \bigcup\{G \mid G$ is an IFOS in X and $G \subseteq A\}$ is called an intuitionistic fuzzy interior of A;

$clA = \bigcap\{G \mid G$ is an IFCS in X and $G \supseteq A\}$ is called an intuitionistic fuzzy closure of A.

Definition 2.6 ([6]). Let a and b be two real numbers in $[0, 1]$ satisfying the inequality $a + b \leq 1$. Then the pair $\langle a, b\rangle$ is called an intuitionistic fuzzy pair. Let $\langle a_1, b_1\rangle$, $\langle a_2, b_2\rangle$ be two intuitionistic fuzzy pairs. Then

(i) $\langle a_1, b_1\rangle \leq \langle a_2, b_2\rangle$ if and only if $a_1 \leq a_2$ and $b_1 \geq b_2$,
(ii) $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle$ if and only if $a_1 = a_2$ and $b_1 = b_2$,
(iii) if $\{\langle a_i, b_i\rangle : i \in J\}$ is a family of intuitionistic fuzzy pairs, then $\vee\langle a_i, b_i\rangle = \langle \vee a_i, \wedge b_i\rangle$ and $\wedge\langle a_i, b_i\rangle = \langle \wedge a_i, \vee b_i\rangle$,
(iv) the complement of an intuitionistic fuzzy pair $\langle a, b\rangle$ is the intuitionistic fuzzy pair defined by $\overline{\langle a, b\rangle} = \langle b, a\rangle$,

(v) $1^{\sim} = \langle 1, 0\rangle$ and $0^{\sim} = \langle 0, 1\rangle$.

Definition 2.7 ([4])**.** Let X be a nonempty set and $A \subset X$. The characteristic function of A is denoted and defined by $\chi_A(x) = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases}$

Definition 2.8 ([2])**.** Let (X,T) be a fuzzy topological space and λ be a fuzzy set in X. Then λ is called fuzzy G_δ if $\lambda = \wedge_{i=1}^{\infty}\lambda_i$ where each $\lambda_i \in T$. The complement of fuzzy G_δ is fuzzy F_σ.

Definition 2.9 ([3])**.** A fuzzy bitopological space (X, τ_1, τ_2) is said to be pairwise fuzzy basically disconnected if τ_1-closure of each τ_2-fuzzy open , τ_2-fuzzy F_σ is τ_2-fuzzy open and τ_2-closure of each τ_1-fuzzy open, τ_1-fuzzy F_σ is τ_1-fuzzy open.

3. Properties of an intuitionistic fuzzy $\mathcal{V}$ basically disconnected spaces

Definition 3.1. Let (X, T) be an intuitionistic fuzzy noncompact space. Let $\mathscr{C}$ be a collection of all intuitionistic fuzzy sets which are both intuitionistic fuzzy closed set and intuitionistic fuzzy compact set in X. Let $U^+ = \{A \in \mathscr{C} : A \cap U \neq 0_\sim, U \in T\}$ and $K^- = \{A \in \mathscr{C} : A \cap K = 0_\sim, K\text{is an intutionistic fuzzy compact set in } X\}$. Then the collection $\mathcal{V} = \{A : \overline{A} \in U^+\} \cup \{B : \overline{B} \in K^-\}$ is said to be intuitionistic fuzzy $\mathcal{V}$ structure on X and the pair $(X, \mathcal{V})$ is said to be an intuitionistic fuzzy $\mathcal{V}$ space.

Notation 3.2. Each member of an intuitionistic fuzzy $\mathcal{V}$ structure is an intuitionistic fuzzy $\mathcal{V}$open set. The complement of an intuitionistic fuzzy $\mathcal{V}$open set is an intuitionistic fuzzy $\mathcal{V}$closed set.

Example 3.3. Let $X = \{a, b\}$ be a nonempty set. Let $A = \langle x, (\frac{a}{0.4}, \frac{b}{0.4}), (\frac{a}{0.4}, \frac{b}{0.4})\rangle$ and $B = \langle x, (\frac{a}{0.5}, \frac{b}{0.5}), (\frac{a}{0.4}, \frac{b}{0.4})\rangle$ be intuitionistic fuzzy sets of X. Then the family $T = \{0_\sim, 1_\sim, A, B\}$ is an intuitionistic fuzzy topology on X. Thus (X, T) is an intuitionistic fuzzy noncompact space. Now, $\mathscr{C} = \{0_\sim, \overline{A}, \overline{B}, 1_\sim\}$ is the collection of all intuitionistic fuzzy sets which are both intuitionistic fuzzy closed set and intuitionistic fuzzy compact set in X. Consider $U^+ = \{\overline{A}, \overline{B}, 1_\sim\}$ and $K^- = \{0_\sim, \overline{A}, \overline{B}, 1_\sim\}$. Then $\mathcal{V} = \{0_\sim, 1_\sim, A, B\}$ is an intuitionistic fuzzy $\mathcal{V}$ structure. Thus, $(X, \mathcal{V})$ is an intuitionistic fuzzy $\mathcal{V}$ space.

Definition 3.4. Let $(X, \mathcal{V})$ be an intuitionistic fuzzy $\mathcal{V}$ space. For an intuitionistic fuzzy set A on X, the intuitionistic fuzzy $\mathcal{V}$ closure of A and the intuitionistic fuzzy $\mathcal{V}$ interior of A are defined by

$$IF\mathcal{V}cl(A) = \bigcap\{B : B = \langle x, \mu_B, \gamma_B\rangle \text{ is an intuitionistic fuzzy } \mathcal{V}\text{closed set in } X \text{ and } A \subseteq B\}$$

$$IF\mathcal{V}int(A) = \bigcup\{B : B = \langle x, \mu_B, \gamma_B\rangle \text{ is an intuitionistic fuzzy } \mathcal{V}\text{open set in } X \text{ and } B \subseteq A\}.$$

Remark 3.5. Let $(X, \mathcal{V})$ be an intuitionistic fuzzy $\mathcal{V}$ space. Then for any intuitionistic fuzzy set A in X,

(i) $IF\mathcal{V}int(A) \subseteq A \subseteq IF\mathcal{V}cl(A)$

(ii) $IF\mathcal{V}cl(\overline{A}) = \overline{IF\mathcal{V}int(A)}$

(ii) $IF\mathcal{V}int(\overline{A}) = \overline{IF\mathcal{V}cl(A)}$

Definition 3.6. Let $(X, \mathcal{V})$ be an intuitionistic fuzzy $\mathcal{V}$ space. An intuitionistic fuzzy set A is said to be intuitionistic fuzzy $\mathcal{V}G_\delta$ set(inshort,$IF\mathcal{V}G_\delta S$) if $A = \bigcap_{i=1}^{\infty} A_i$, where each $A_i \in \mathcal{V}$. The complement of intuitionistic fuzzy $\mathcal{V}G_\delta$ set is said to be an intuitionistic fuzzy $\mathcal{V}F_\sigma$(inshort, $IF\mathcal{V}F_\sigma S$)set.

Notation 3.7. Let $(X, \mathcal{V})$ be an intuitionistic fuzzy $\mathcal{V}$ space.

(i) An intuitionistic fuzzy $\mathcal{V}$open F_σ(inshort, $IF\mathcal{V}oF_\sigma$)set is an intuitionistic fuzzy set which is both intuitionistic fuzzy $\mathcal{V}$open and intuitionistic fuzzy $\mathcal{V}F_\sigma$.

(ii) An intuitionistic fuzzy $\mathcal{V}$closed G_δ(inshort, $IF\mathcal{V}cG_\delta$)set is an intuitionistic fuzzy set which is both intuitionistic fuzzy $\mathcal{V}$closed and intuitionistic fuzzy $\mathcal{V}G_\delta$

(iii) An intuitionistic fuzzy $\mathcal{V}$closed open $G_\delta F_\sigma$(inshort, $IF\mathcal{V}\mathcal{C}\mathcal{O}\mathcal{G}\mathcal{F}$)set is an intuitionistic fuzzy set which is both intuitionistic fuzzy $\mathcal{V}$open F_σ and intuitionistic fuzzy $\mathcal{V}$closed G_δ

Definition 3.8. Let $(X, \mathcal{V})$ be an intuitionistic fuzzy $\mathcal{V}$ space. Then $(X, \mathcal{V})$ is said to be an intuitionistic fuzzy $\mathcal{V}$ basically disconnected space if the intuitionistic fuzzy $\mathcal{V}$ closure of every intuitionistic fuzzy $\mathcal{V}$open F_σ set is an intuitionistic fuzzy $\mathcal{V}$open set.

Example 3.9. In the above Example 3.3, $(X, \mathcal{V})$ is an intuitionistic fuzzy $\mathcal{V}$ basically disconnected space.

Proposition 3.10. *Let $(X, \mathcal{V})$ be an intuitionistic fuzzy $\mathcal{V}$ space. Then the following statements are equivalent*

(i) *$(X, \mathcal{V})$ is an intuitionistic fuzzy $\mathcal{V}$ basically disconnected space*

(ii) *For each intuitionistic fuzzy $\mathcal{V}$closed G_δ set A, we have $IF\mathcal{V}int(A)$ is an intuitionistic fuzzy $\mathcal{V}$closed set*

(iii) *For each $\mathcal{V}$open F_σ set A, we have $IF\mathcal{V}cl(IF\mathcal{V}int(\overline{A})) = \overline{IF\mathcal{V}cl(A)}$*

(iv) *For an intuitionistic fuzzy $\mathcal{V}$open F_σ set A and for any intuitionistic fuzzy set B with $IF\mathcal{V}cl(A) = \overline{B}$, we have $IF\mathcal{V}cl(B) = \overline{IF\mathcal{V}cl(A)}$*

Proof. (i) ⇒ (ii) Let A be any intuitionistic fuzzy $\mathcal{V}$closed G_δ set. Then $\overline{A}$ is an intuitionistic fuzzy $\mathcal{V}$open F_σ set. By assumption(i) $IF\mathcal{V}cl(\overline{A})$ is an intuitionistic fuzzy $\mathcal{V}$open set. Now, $IF\mathcal{V}cl(\overline{A}) = \overline{IF\mathcal{V}int(A)}$. Hence $IF\mathcal{V}int(A)$ is an an intuitionistic fuzzy $\mathcal{V}$closed set.

(ii) ⇒ (iii) Let A be any intuitionistic fuzzy $\mathcal{V}$open F_σ set. Then $\overline{A}$ is an intuitionistic fuzzy $\mathcal{V}$closed G_δ set.By assumption(ii) $IF\mathcal{V}int(\overline{A})$ is an intuitionistic fuzzy $\mathcal{V}$closed set. Consider $IF\mathcal{V}cl(IF\mathcal{V}int(\overline{A})) = IF\mathcal{V}int(\overline{A}) = \overline{IF\mathcal{V}cl(A)}$.

(iii) ⇒ (iv) Let A be an intuitionistic fuzzy $\mathcal{V}$open F_σ set and for any intuitionistic fuzzy set B such that $IF\mathcal{V}cl(A) = \overline{B}$. By (iii),

$$IF\mathcal{V}cl(IF\mathcal{V}int(\overline{A})) = \overline{IF\mathcal{V}cl(A)} = IF\mathcal{V}int(\overline{A})$$

That is,

$$IF\mathcal{V}cl(B) = IF\mathcal{V}int(\overline{A})$$

(iv) $\Rightarrow$ (i) Let A be any intuitionistic fuzzy $\mathcal{V}$open F_σ set. Let $\overline{IF\mathcal{V}cl(A)} = B$. By (iv), it follow that $IF\mathcal{V}cl(B) = \overline{IF\mathcal{V}cl(A)}$. That is, $\overline{IF\mathcal{V}cl(A)}$ is an intuitionistic fuzzy $\mathcal{V}$closed set. This implies that $IF\mathcal{V}cl(A)$ is an intuitionistic fuzzy $\mathcal{V}$open set. Hence, $(X, \mathcal{V})$ is an intuitionistic fuzzy $\mathcal{V}$ basically disconnected space. □

Proposition 3.11. *Let $(X, \mathcal{V})$ be an intuitionistic fuzzy $\mathcal{V}$ space. Then $(X, \mathcal{V})$ is an intuitionistic fuzzy $\mathcal{V}$ basically disconnected space if and only if for each intuitionistic fuzzy $\mathcal{V}$open F_σ set A and intuitionistic fuzzy $\mathcal{V}$closed G_δ set B such that $A \subseteq B$, $IF\mathcal{V}cl(A) \subseteq IF\mathcal{V}int(B)$.*

Proof. Let A be intuitionistic fuzzy $\mathcal{V}$open F_σ set and B be intuitionistic fuzzy $\mathcal{V}$closed G_δ set such that $A \subseteq B$. Then by (ii) of Proposition 3.10, $IF\mathcal{V}int(B)$ is an intuitionistic fuzzy $\mathcal{V}$closed set. Also, since A is an intuitionistic fuzzy $\mathcal{V}$open F_σ set, $IF\mathcal{V}cl(A) \subseteq IF\mathcal{V}int(B)$. Conversely, let B be any intuitiionistic fuzzy $\mathcal{V}$closed G_δ set. Then $IF\mathcal{V}int(B)$ is an intuitionistic fuzzy $\mathcal{V}$open set and $IF\mathcal{V}int(B) \subseteq B$. By assumption, $IF\mathcal{V}cl(IF\mathcal{V}int(B)) \subseteq IF\mathcal{V}int(B)$. Also we know that $IF\mathcal{V}int(B) \subseteq IF\mathcal{V}cl(IF\mathcal{V}int(B))$. This implies that $IF\mathcal{V}cl(IF\mathcal{V}int(B)) = IF\mathcal{V}int(B)$. Therefore, $IF\mathcal{V}int(B)$ is an intuitionistic fuzzy $\mathcal{V}$closed set. Hence by (ii) of Proposition 3.10, it follows that $(X, \mathcal{V})$ is an intuitionistic fuzzy $\mathcal{V}$ basically disconnected space. □

Remark 3.12. Let $(X, \mathcal{V})$ be an intuitionistic fuzzy $\mathcal{V}$ basically disconnected space. Let $\{A_i, \overline{B_i}/i \in N\}$ be collection such that A_i's are intuitionistic fuzzy $\mathcal{V}$open F_σ sets and B_i intuitionistic fuzzy $\mathcal{V}$closed G_δ sets and let A and B be intuitionistic fuzzy $\mathcal{V}$closed open $G_\delta F_\sigma$ sets. If $A_i \subseteq A \subseteq B_j$ and $A_i \subseteq B \subseteq B_j$ for all $i, j \in N$, then there exists an intuitionistic fuzzy $\mathcal{V}$closed open $G_\delta F_\sigma$ set C such that $IF\mathcal{V}cl(A_i) \subseteq C \subseteq IF\mathcal{V}int(B_j)$ for all $i, j \in N$.

Proof. By Proposition 3.11, $IF\mathcal{V}cl(A_i) \subseteq IF\mathcal{V}cl(A) \cap IF\mathcal{V}int(B) \subseteq IF\mathcal{V}int(B_j)$ for all $i, j \in N$. Letting $C = IF\mathcal{V}cl(A) \cap IF\mathcal{V}int(B)$ in the above, we have C is an intuitionistic fuzzy $\mathcal{V}$closed open $G_\delta F_\sigma$ set satisfying the required conditions. □

Proposition 3.13. *Let $(X, \mathcal{V})$ be intuitionistic fuzzy $\mathcal{V}$ basically disconnected space. Let $\{A_q\}_{q\in Q}$ and $\{B_q\}_{q\in Q}$ be monotone increasing collections of an intuitionistic fuzzy $\mathcal{V}$open F_σ sets and intuitionistic fuzzy $\mathcal{V}$closed G_δ sets of $(X, \mathcal{V})$. Suppose that $A_{q_1} \subseteq B_{q_2}$ whenever $q_1 < q_2$ (Q is the set of all rational numbers). Then there exists a monotone increasing collection $\{C_q\}_{q\in Q}$ of an intuitionistic fuzzy $\mathcal{V}$closed open $G_\delta F_\sigma$ sets of $(X, \mathcal{V})$ such that $IF\mathcal{V}cl(A_{q_1}) \subseteq C_{q_2}$ and $C_{q_1} \subseteq IF\mathcal{V}int(B_{q_2})$ whenever $q_1 < q_2$.*

Proof. Let us arrange all rational numbers into a sequence $\{q_n\}$ (without repetitions). For every $n \geq 2$, we shall define inductively a collection $\{C_{q_i}/1 \leq i < n\} \subset \zeta^X$ such that

$$IF\mathcal{V}cl(A_q) \subseteq C_{q_i} \text{if } q < q_i, C_{q_i} \subseteq IF\mathcal{V}int(B_q) \text{if } q_i < q, \text{for all } i < n \qquad (S_n)$$

By Proposition 3.11, the countable collections $\{IF\mathcal{V}cl(A_q)\}$ and $\{IF\mathcal{V}int(B_q)\}$ satisfy $IF\mathcal{V}cl(A_{q_1}) \subseteq IF\mathcal{V}int(B_{q_2})$ if $q_1 < q_2$. By Remark 3.12, there exists an intuitionistic fuzzy $\mathcal{V}$closed open $G_\delta F_\sigma$ set D_1 such that

$$IF\mathcal{V}cl(A_{q_1}) \subseteq D_1 \subseteq IF\mathcal{V}int(B_{q_2})$$

Letting $C_{q_1} = D_1$, we get (S_2). Assume that intuitionistic fuzzy sets C_{q_i} are already defined for $i < n$ and satisfy (S_n). Define $E = \cup\{C_{q_i}/i < n, q_i < q_n\} \cup A_{q_n}$ and $F = \cap\{C_{q_j}/j < n, q_j > q_n\} \cap B_{q_n}$. Then $IF\mathcal{V}cl(C_{q_i}) \subseteq IF\mathcal{V}cl(E) \subseteq IF\mathcal{V}int(C_{q_j})$ and $IF\mathcal{V}cl(C_{q_i}) \subseteq IF\mathcal{V}int(F) \subseteq IF\mathcal{V}int(C_{q_j})$ whenever $q_i < q_n < q_j (i, j < n)$, as well as $A_q \subseteq IF\mathcal{V}cl(E) \subseteq B_{q'}$ and $A_q \subseteq IF\mathcal{V}int(F) \subseteq B_{q'}$ whenever $q < q_n < q'$. This shows that the countable collections $\{C_{q_i}/i < n, q_i < q_n\} \cup \{A_q/q < q_n\}$ and $\{C_{q_j}/j < n, q_j > q_n\} \cup \{B_q/q > q_n\}$ together with E and F fulfil the conditions of Remark 3.12. Hence, there exists an intuitionistic fuzzy $\mathcal{V}$closed open $G_\delta F_\sigma$ set D_n such that $IF\mathcal{V}cl(D_n) \subseteq B_q$ if $q_n < q$, $A_q \subseteq IF\mathcal{V}int(D_n)$ if $q < q_n$, $IF\mathcal{V}cl(C_{q_i}) \subseteq IF\mathcal{V}int(D_n)$ if $q_i < q_n$ $IF\mathcal{V}cl(D_n) \subseteq IF\mathcal{V}int(C_{q_i})$ if $q_n < q_j$ where $1 \leq i, j \leq n-1$. Letting $C_{q_n} = D_n$ we obtain an intuitionistic fuzzy sets $C_{q_1}, C_{q_2}, C_{q_3}, ..., C_{q_n}$ that satisfy $(\mathbf{S_{n+1}})$. Therefore, the collection $\{C_{q_i}/i = 1, 2, ...\}$ has the required property. □

4. Tietze extension theorem for an intuitionistic fuzzy $\mathcal{V}$ basically disconnected spaces

Notation 4.1. The family of all intuitionistic fuzzy sets in $\mathbb{R}$ is denoted by $\zeta^{\mathbb{R}}$

Definition 4.2. An intuitionistic fuzzy real line $\mathbb{R}_{\mathbb{I}}(I)$ is the set of all monotone decreasing intuitionistic fuzzy set $A \in \zeta^{\mathbb{R}}$ satisfying $\cup\{A(t) : t \in \mathbb{R}\} = 1^{\sim}$ and $\cap\{A(t) : t \in \mathbb{R}\} = 0^{\sim}$ after the identification of an intuitionistic fuzzy sets $A, B \in \mathbb{R}_{\mathbb{I}}(I)$ if and only if $A(t-) = B(t-)$ and $A(t+) = B(t+)$ for all $t \in \mathbb{R}$ where $A(t-) = \cap\{A(s) : s < t\}$ and $A(t+) = \cup\{A(s) : s > t\}$.

The intuitionistic fuzzy unit interval $\mathbb{I}_{\mathbb{I}}(I)$ is a subset of $\mathbb{R}_{\mathbb{I}}(I)$ such that $[A] \in \mathbb{I}_{\mathbb{I}}(I)$ if the membership and nonmembership of an intuitionistic fuzzy set $A \in \zeta^{\mathbb{R}}$ are defined by $\mu_A(t) = \begin{cases} 1, & \text{t<0;} \\ 0, & \text{t>1.} \end{cases}$ and $\gamma_A(t) = \begin{cases} 0, & \text{t<0;} \\ 1, & \text{t>1.} \end{cases}$ respectively.

The natural intuitionistic fuzzy topology on $\mathbb{R}_{\mathbb{I}}(I)$ is generated from the subbasis $\{L^{\mathbb{I}}{}_t, R^{\mathbb{I}}{}_t : s < t\}$ where $L^{\mathbb{I}}_t, R^{\mathbb{I}}_t : \mathbb{R}_{\mathbb{I}}(I) \to \mathbb{I}_{\mathbb{I}}(I)$ are given by $L^{\mathbb{I}}_t[A] = \overline{A(t-)}$ and $R^{\mathbb{I}}_t[A] = A(t+)$ respectively.

Definition 4.3. Let $(X, \mathcal{V})$ be an intuitionistic fuzzy $\mathcal{V}$ space. A function $f : X \to \mathbb{R}_{\mathbb{I}}(I)$ is said be lower (resp. upper) intuitionistic fuzzy $\mathcal{V}$ continuous function if $f^{-1}(R^{\mathbb{I}}_t)(f^{-1}(L^{\mathbb{I}}_t))$ is an intuitionistic fuzzy $\mathcal{V}$open F_σ(intuitionistic fuzzy $\mathcal{V}$open F_σ and intuitionistic fuzzy $\mathcal{V}$closed G_δ) set, for each $t \in \mathbb{R}$.

Notation 4.4. Let X be any nonempty set and $A \in \zeta^X$. Then for $x \in X$, $\langle\mu_A(x), \gamma_A(x)\rangle$ is denoted by $A^{\sim}$.

Definition 4.5. Let X be any nonempty set. An intuitionistic fuzzy characteristic function of an intuitionistic fuzzy set $A \in \zeta^X$ is a map $\psi_A : X \to \mathbb{I}_{\mathbb{I}}(I)$ is defined by $\psi_A(x) = A^{\sim}$, for each $x \in X$.

Proposition 4.6. *Let $(X, \mathcal{V})$ be an intuitionistic fuzzy $\mathcal{V}$ space and let A be an intuitionistic fuzzy set in X. Let $f : X \to \mathbb{R}_{\mathbb{I}}(I)$ be such that*

$$f(x)(t) = \begin{cases} 1^{\sim} & if \quad t < 0 \\ A^{\sim} & if \quad 0 \leq t \leq 1 \\ 0^{\sim} & if \quad t > 1 \end{cases}$$

for all $x \in X$ and $t \in \mathbb{R}$. Then f is an lower(resp. upper) intuitionistic fuzzy $\mathcal{V}$ continuous function if and only if A is an intuitionistic fuzzy $\mathcal{V}$open F_σ (intuitionistic fuzzy $\mathcal{V}$open F_σ and intuitionistic fuzzy $\mathcal{V}$closed G_δ)set.

Proof.

$$f^{-1}(R_t^{\mathbb{I}}) = \begin{cases} 1_\sim & \text{if} \quad t < 0 \\ A & \text{if} \quad 0 \leq t \leq 1 \\ 0_\sim & \text{if} \quad t > 1 \end{cases}$$

implies that f is an lower intuitionistic fuzzy $\mathcal{V}$ continuous function if and only if A is an intuitionistic fuzzy $\mathcal{V}$open F_σ set.

$$f^{-1}(\overline{L_t^{\mathbb{I}}}) = \begin{cases} 1_\sim & \text{if} \quad t < 0 \\ A & \text{if} \quad 0 \leq t \leq 1 \\ 0_\sim & \text{if} \quad t > 1 \end{cases}$$

implies that f is an upper intuitionistic fuzzy $\mathcal{V}$ continuous function if and only if A is an intuitionistic fuzzy $\mathcal{V}$closed G_δ set. Hence the proof is complete. □

Remark 4.7. Let $(X, \mathcal{V})$ be an intuitionistic fuzzy $\mathcal{V}$ space. Let ψ_A be an intuitionistic fuzzy characteristic function of an intuitionistic fuzzy set A in X. Then ψ_A is an lower(resp. upper) intuitionistic fuzzy $\mathcal{V}$ continuous function if and only if A is an intuitionistic fuzzy $\mathcal{V}$open F_σ (intuitionistic fuzzy $\mathcal{V}$open F_σ and intuitionistic fuzzy $\mathcal{V}$closed G_δ)set.

Proof. The proof follows from Proposition 4.6. □

Definition 4.8. Let $(X, \mathcal{V})$ be an intuitionistic fuzzy $\mathcal{V}$ space. A function $f : X \to \mathbb{R}_{\mathbb{I}}(I)$ is said to be an strongly intuitionistic fuzzy $\mathcal{V}$ continuous function if $f^{-1}(R_t^{\mathbb{I}})$ is an intuitionistic fuzzy $\mathcal{V}$open F_σ and $f^{-1}(L_t^{\mathbb{I}})$ is both intuitionistic fuzzy $\mathcal{V}$open F_σ and intuitionistic fuzzy $\mathcal{V}$closed G_δ set, for each $t \in \mathbb{R}$.

Notation 4.9. The collection of all strongly intuitionistic fuzzy $\mathcal{V}$ continuous functions in an intuitionistic fuzzy $\mathcal{V}$ space $(X, \mathcal{V})$ with values in $\mathbb{I}_{\mathbb{I}}(I)$ is denoted by $\mathbb{S}_{\mathcal{V}}$.

Proposition 4.10. *Let $(X, \mathcal{V})$ be an intuitionistic fuzzy $\mathcal{V}$ space. Then the following conditions are equivalent*

(i) *$(X, \mathcal{V})$ is an intuitionistic fuzzy $\mathcal{V}$ basically disconnected space*
(ii) *If $g, h : X \to \mathbb{R}_{\mathbb{I}}(I)$, g is lower intuitionistic fuzzy $\mathcal{V}$ continuous function, h is upper intuitionistic fuzzy $\mathcal{V}$ continuous function and $g \subseteq h$, then there exists an $f \in \mathbb{S}_{\mathcal{V}}$ such that $g \subseteq f \subseteq h$*
(iii) *If $\overline{A}$ and B are intuitionistic fuzzy $\mathcal{V}$open F_σ sets such that $B \subseteq A$, then there exists strongly intuitionistic fuzzy $\mathcal{V}$ continuous function $f : X \to \mathbb{R}_{\mathbb{I}}(I)$ such that $B \subseteq f^{-1}(\overline{L_1^{\mathbb{I}}}) \subseteq f^{-1}(R_0^{\mathbb{I}}) \subseteq A$.*

Proof. (i) ⇒ (ii) Define $A_r = h^{-1}(L_r^{\mathbb{I}})$ and $B_r = g^{-1}(\overline{R_r^{\mathbb{I}}})$, for all $r \in \mathbb{Q}$ ($\mathbb{Q}$ is the set of all rationals). Clearly, $\{A_r\}_{r\in\mathbb{Q}}$ and $\{B_r\}_{r\in\mathbb{Q}}$ are monotone increasing families of an intuitionistic fuzzy $\mathcal{V}$open F_σ sets and intuitionistic fuzzy $\mathcal{V}$closed G_δ sets of $(X, \mathcal{V})$. Moreover $A_r \subseteq B_s$ if $r < s$. By Proposition 3.13, there exists a monotone increasing family $\{C_r\}_{r\in\mathbb{Q}}$ of an intuitionistic fuzzy $\mathcal{V}$closed open $G_\delta F_\sigma$ sets of $(X, \mathcal{V})$ such that $IF\mathcal{V}cl(A_r) \subseteq C_s$ and $C_r \subseteq IF\mathcal{V}int(B_s)$ whenever $r < s$

$(r, s \in \mathbb{Q})$. Letting $V_t = \bigcap_{r<t} \overline{C_r}$ for $t \in \mathbb{R}$, we define a monotone decreasing family $\{V_t \mid t \in \mathbb{R}\} \subseteq \zeta^X$. Moreover we have $IF\mathcal{V}cl(V_t) \subseteq IF\mathcal{V}int(V_s)$ whenever $s < t$. We have,

$$\bigcup_{t\in\mathbb{R}} V_t = \bigcup_{t\in\mathbb{R}} \bigcap_{r<t} \overline{C_r} \supseteq \bigcup_{t\in\mathbb{R}} \bigcap_{r<t} \overline{B_r} = \bigcup_{t\in\mathbb{R}} \bigcap_{r<t} g^{-1}(R_r^{\mathbb{I}})$$
$$= \bigcup_{t\in\mathbb{R}} g^{-1}(\overline{L_t^{\mathbb{I}}}) = g^{-1}(\bigcup_{t\in\mathbb{R}} \overline{L_t^{\mathbb{I}}}) = 1_\sim$$

Similarly, $\bigcap_{t\in\mathbb{R}} V_t = 0_\sim$. Now define a function $f : X \to \mathbb{R}_{\mathbb{I}}(I)$ possessing required conditions. Let $f(x)(t) = V_t(x)$, for all $x \in X$ and $t \in \mathbb{R}$. By the above discussion, it follows that f is well defined. To prove f is an strongly intutionistic fuzzy $\mathcal{V}$ continuous function. Observe that $\bigcup_{s>t} V_s = \bigcup_{s>t} IF\mathcal{V}int(V_s)$ and $\bigcap_{s<t} V_s = \bigcap_{s<t} IF\mathcal{V}cl(V_s)$. Then $f^{-1}(R_t^{\mathbb{I}}) = \bigcup_{s>t} V_s = \bigcup_{s>t} IF\mathcal{V}int(V_s)$ is an intuitionistic fuzzy $\mathcal{V}$closed open $G_\delta F_\sigma$ set and $f^{-1}(\overline{L_t^{\mathbb{I}}}) = \bigcap_{s<t} V_s = \bigcap_{s<t} IF\mathcal{V}cl(V_s)$ is an intuitionistic fuzzy $\mathcal{V}$closed open $G_\delta F_\sigma$ set. Therefore, f is strongly intuitionistic fuzzy $\mathcal{V}$ continuous function. To conclude the proof it remains to show that $g \subseteq f \subseteq h$. That is $g^{-1}(\overline{L_t^{\mathbb{I}}}) \subseteq f^{-1}(\overline{L_t^{\mathbb{I}}}) \subseteq h^{-1}(\overline{L_t^{\mathbb{I}}})$ and $g^{-1}(R_t^{\mathbb{I}}) \subseteq f^{-1}(R_t^{\mathbb{I}}) \subseteq h^{-1}(R_t^{\mathbb{I}})$ for each $t \in \mathbb{R}$. We have,

$$g^{-1}(\overline{L_t^{\mathbb{I}}}) = \bigcap_{s<t} g^{-1}(\overline{L_s^{\mathbb{I}}}) = \bigcap_{s<t} \bigcap_{r<s} g^{-1}(R_r^{\mathbb{I}})$$
$$= \bigcap_{s<t} \bigcap_{r<s} \overline{B_r} \subseteq \bigcap_{s<t} \bigcap_{r<s} \overline{C_r} = \bigcap_{s<t} V_s = f^{-1}(\overline{L_t^{\mathbb{I}}})$$

and

$$f^{-1}(\overline{L_t^{\mathbb{I}}}) = \bigcap_{s<t} V_s = \bigcap_{s<t} \bigcap_{r<s} \overline{C_r} \subseteq \bigcap_{s<t} \bigcap_{r<s} \overline{A_r}$$
$$= \bigcap_{s<t} \bigcap_{r<s} h^{-1}(\overline{L_r^{\mathbb{I}}}) = \bigcap_{s<t} h^{-1}(\overline{L_s^{\mathbb{I}}}) = h^{-1}(\overline{L_t^{\mathbb{I}}}).$$

Similarly,

$$g^{-1}(R_t^{\mathbb{I}}) = \bigcup_{s>t} g^{-1}(R_s^{\mathbb{I}}) = \bigcup_{s>t} \bigcup_{r>s} g^{-1}(R_r^{\mathbb{I}})$$
$$= \bigcup_{s>t} \bigcup_{r>s} \overline{B_r} \subseteq \bigcup_{s>t} \bigcap_{r<s} \overline{C_r} = \bigcup_{s>t} V_s = f^{-1}(R_t^{\mathbb{I}})$$

and

$$f^{-1}(R_t^{\mathbb{I}}) = \bigcup_{s>t} V_s = \bigcup_{s>t} \bigcap_{r<s} \overline{C_r} \subseteq \bigcup_{s>t} \bigcup_{r>s} \overline{A_r}$$
$$= \bigcup_{s>t} \bigcup_{r>s} h^{-1}(\overline{L_r^{\mathbb{I}}}) = \bigcup_{s>t} h^{-1}(R_s^{\mathbb{I}}) = h^{-1}(R_t^{\mathbb{I}})$$

Hence, the condition (ii) is proved.

(ii) $\Rightarrow$ (iii) Let $\overline{A}$ be an intuitionistic fuzzy $\mathcal{V}$open F_σ set and B be an intuitionistic fuzzy $\mathcal{V}$closed G_δ set such that $B \subseteq A$. Then $\psi_B \subseteq \psi_A$ where ψ_B, ψ_A are lower and upper intuitionistic fuzzy $\mathcal{V}$ continuous functions respectively. By (ii), there exists a strongly intuitionistic fuzzy $\mathcal{V}$ continuous function $f : X \to \mathbb{I}_{\mathbb{I}}(I)$ such that

$\psi_B \subseteq f \subseteq \psi_A$. Clearly, $f(x) \in \mathbb{I}_{\mathbb{I}}(I)$ for all $x \in X$ and $B = \psi_B^{-1}(\overline{L_1^{\mathbb{I}}}) \subseteq f^{-1}(\overline{L_1^{\mathbb{I}}}) \subseteq f^{-1}(R_0^{\mathbb{I}}) \subseteq \psi A^{-1}(R_0^{\mathbb{I}}) = A$. Therefore, $B \subseteq f^{-1}(\overline{L_1^{\mathbb{I}}}) \subseteq f^{-1}(R_0^{\mathbb{I}}) \subseteq A$.

(iii) $\Rightarrow$ (i) Since $f^{-1}(\overline{L_1^{\mathbb{I}}})$ and $f^{-1}(R_0^{\mathbb{I}})$ are intuitionistic fuzzy $\mathcal{V}$closed open $G_\delta F_\sigma$ sets and by Proposition 3.11, $(X, \mathcal{V})$ is an intuitionistic fuzzy $\mathcal{V}$ basically disconnected space. □

Notation 4.11. Let X be any nonempty set. Let $A \subset X$. Then an intuitionistic fuzzy set χ_A^* is of the form $\langle x, \chi_A(x), 1 - \chi_A(x)\rangle$.

Proposition 4.12. *Let $(X, \mathcal{V})$ be an intuitionistic fuzzy $\mathcal{V}$ basically disconnected space. Let $A \subset X$ such that χ_A^* is an intuitionistic fuzzy $\mathcal{V}$open F_σ set in X. Let $f : (A, \mathcal{V}/A) \to \mathbb{I}_{\mathbb{I}}(I)$ be an strongly intuitionistic fuzzy $\mathcal{V}$ continuous function. Then f has an strongly intuitionistic fuzzy $\mathcal{V}$ continuous extension over $(X, \mathcal{V})$.*

Proof. Let $g, h : X \to \mathbb{I}_{\mathbb{I}}(I)$ be such that $g = f = h$ on A and $g(x) = 0^{\sim}$, $h(x) = 1^{\sim}$ if $x \notin A$. For every $t \in \mathbb{R}$, We have,

$$g^{-1}(R_t^{\mathbb{I}}) = \begin{cases} B_t \cap \chi_A^* & \text{if } t \geq 0\ , \\ 1_{\sim} & \text{if } t < 0, \end{cases}$$

where B_t is an intuitionistic fuzzy $\mathcal{V}$closed open $G_\delta F_\sigma$ set such that $B_t/A = f^{-1}(R_t^{\mathbb{I}})$ and

$$h^{-1}(L_t^{\mathbb{I}}) = \begin{cases} C_t \cap \chi_A^* & \text{if } t \leq 1\ , \\ 1_{\sim} & \text{if } t > 1, \end{cases}$$

where C_t is an intuitionistic fuzzy $\mathcal{V}$closed open $G_\delta F_\sigma$ set such that $C_t/A = f^{-1}(L_t^{\mathbb{I}})$ and. Thus, g is lower intuitionistic fuzzy $\mathcal{V}$ continuous function and h is upper intuitionistic fuzzy $\mathcal{V}$ continuous function with $g \subseteq h$. By Proposition 4.10, there is an strongly intuitionistic fuzzy $\mathcal{V}$ continuous function $F : X \to \mathbb{I}_{\mathbb{I}}(I)$ such that $g \subseteq F \subseteq h$. Hence $F \equiv f$ on A. □

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