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Characterization of Γ-semigroups in terms of anti fuzzy ideals

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Abstract. In this paper the concept of anti fuzzy ideals of a Γ-semigroup has been introduced and some important characterizations have been obtained. Among other results a regular Γ-semigroup has been characterized in terms of anti fuzzy ideals. Finally we show that the set of all the anti fuzzy ideals of a Γ-semigroup forms a semilattice.

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1. Introduction

Uncertainty is an attribute of information and uncertain data are presented in various domains. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [26]. In our daily life, we usually want to seek opinions from professional persons with the best qualifications, for examples, the best medical doctors can provide the best diagnostics, the best pilots can provide the best navigation suggestions for airplanes, etc. It is therefore desirable to incorporate the knowledge of these experts into some automatic systems so that it would become helpful for other people to make appropriate decisions which are (almost) as good as the decisions made by the top experts. With this aim in mind, our task is to design a system that would provide the best advice from the best experts in the field. However, one of the main hurdles of this incorporation is that the experts are usually unable to describe their knowledge by using precise and exact terms. For example, in order to describe the size of certain type of a tumor, a medical doctor

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would rarely use the exact numbers. Instead he would say something like the size is between 1.4 and 1.6 cm. Also, an expert would usually use some words from a natural language, e.g., the size of the tumor is approximately 1.5 cm, with an error of about 0.1 cm. Thus, under such circumstances, the way to formalize the statements given by an expert is one of the main objectives of fuzzy logic.

Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation [9]. The formal study of semigroups began in the early 20th century. The topic of investigations about fuzzy semigroups belongs to the theoretical soft computing (fuzzy structures). Indeed, it is well known that semigroups are basic structures in many applicative branches like automata and formal languages, coding theory, finite state machines and others. Due to these possibilities of applications, semigroups and related structures are presently extensively investigated in fuzzy settings. Azirel Rosenfeld [15] used the idea of fuzzy set to introduce the notions of fuzzy subgroups. Nobuaki Kuroki [12, 13, 14] is the pioneer of fuzzy ideal theory of semigroups. The idea of fuzzy subsemigroup was also introduced by Kuroki [12, 14]. In [13], Kuroki characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. Others who worked on fuzzy semigroup theory, such as X.Y. Xie [25], Y.B. Jun [10], are mentioned in the bibliography. X.Y. Xie [25] introduced the idea of extensions of fuzzy ideals in semigroups.

In 1981 M.K. Sen [19] introduced the notion of Γ-semigroup as a generalization of semigroup and ternary semigroup. We call this Γ-semigroup a both sided Γ-semigroup. In 1986 M.K. Sen and N.K. Saha [21] modified the definition of Sen's Γ-semigroup. This newly defined Γ-semigroup is known as one sided Γ-semigroup. Γ-semigroups have been analyzed by a lot of mathematicians, for instance by Dutta and Adhikari [1, 4, 5], Chattopadhyay [2], Hila [7, 8], Chinram [3], Sen et al. [20, 22]. T.K. Dutta and N.C. Adhikari [1, 4] mostly worked on both sided Γ-semigroups. They defined operator semigroups of such type of Γ-semigroups and established many results and obtained many correspondences between a Γ-semigroup and its operator semigroups. In this paper we have considered both sided Γ-semigroups.

In 2007, Uckun Mustafa, Ali Mehmet and Jun Young Bae [24] introduced the notion of intuitionistic fuzzy ideals in Γ-semigroups. Motivated by Kuroki [12, 13, 14], Mustafa et al. [24], S.K. Sardar et al. [6, 16, 17, 18] have initiated the study of Γ-semigroups in terms of fuzzy sets.

In 2009 M. Shabir and Y. Nawaz [23], M. Khan and T. Asif [11] introduced the concept of anti fuzzy ideals in semigroups and characterized different classes of semigroups by the properties of their anti fuzzy ideals. In this paper the concept of anti fuzzy ideals and normal anti fuzzy ideals of a Γ-semigroup have been introduced and some important characterizations have been obtained. Among other results a regular Γ-semigroup has been characterized in terms of anti fuzzy ideals. Finally we show that the set of all the anti fuzzy ideals of a Γ-semigroup forms a semilattice.

2. Preliminaries

Definition 2.1 ([19])**.** Let S and Γ be two non-empty sets. S is called a Γ-semigroup if there exist mappings from $S \times \Gamma \times S$ to S, written as $(a, \alpha, b) \longrightarrow a\alpha b$, and from

$\Gamma \times S \times \Gamma$ to Γ, written as $(\alpha, a, \beta) \longrightarrow \alpha a\beta$ satisfying the following associative laws $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ and $\alpha(a\beta b)\gamma = (\alpha a\beta)b\gamma = \alpha a(\beta b\gamma)$ for all $a, b, c \in S$ and for all $\alpha, \beta, \gamma \in \Gamma$.

Example 2.2 ([19])**.** Let S be the set of all integers of the form $4n + 1$ and Γ be the set of all integers of the form $4n + 3$ where n is an integer. If $a\alpha b$ is $a + \alpha + b$ and $\alpha a\beta$ is $\alpha + a + \beta$(usual sum of integers) for all $a, b \in S$ and for all $\alpha, \beta \in \Gamma$, then S is a Γ-semigroup.

Definition 2.3 ([5])**.** A left ideal(right ideal) of a Γ-semigroup S is a non-empty subset I of S such that $S\Gamma I \subseteq I$ ($I\Gamma S \subseteq I$). If I is both a left ideal and a right ideal of S, then we say that I is an ideal of S.

Definition 2.4 ([26])**.** A function μ from a non-empty set S to the unit interval $[0, 1]$ is called a fuzzy subset of S.

Unless or otherwise stated throughout this paper S stands for a both sided Γ-semigroup.

3. Anti fuzzy ideals

Definition 3.1. A non-empty fuzzy subset μ of a Γ-semigroup S is called an anti fuzzy left ideal(anti fuzzy right ideal) of S if $\mu(x\gamma y) \leq \mu(y)$(resp. $\mu(x\gamma y) \leq \mu(x)$) $\forall x, y \in S$ and $\forall \gamma \in \Gamma$.

Definition 3.2. A non-empty fuzzy subset μ of a Γ-semigroup S is called an anti fuzzy ideal of S if it is both an anti fuzzy left and an anti fuzzy right ideal of S.

Example 3.3. Let S be the set of all non-positive integers without zero and Γ be the set of all non-positive even integers without zero. Then S is a Γ-semigroup where $a\gamma b$ denote the usual multiplication of integers a, γ, b with $a, b \in S$ and $\gamma \in \Gamma$. Let μ be a fuzzy subset of S, defined as follows

$$\mu(x) = \begin{cases} 0.5 & \text{if } x = -1 \\ 0.3 & \text{if } x = -2 \\ 0.1 & \text{if } x < -2 \end{cases}$$

Then μ is an anti fuzzy ideal of S.

Theorem 3.4. *Let I be a non-empty subset of a Γ-semigroup S and χ be a fuzzy subset of S such that*

$$\chi(x) = \begin{cases} 0 & \text{if } x \in I \\ 1 & \text{if } x \notin I \end{cases}$$

Then I is a left ideal(right ideal, ideal) of S if and only if χ is an anti fuzzy left ideal(resp. anti fuzzy right ideal, anti fuzzy ideal) of S.

Proof. Let I be a left ideal of a Γ-semigroup S. Let $x, y \in S$ and $\gamma \in \Gamma$, then $x\gamma y \in I$ if $y \in I$. It follows that $\chi(x\gamma y) = 0 = \chi(y)$. If $y \notin I$, then $\chi(y) = 1$. In this case $\chi(x\gamma y) \leq 1 = \chi(y)$. Therefore χ is an anti fuzzy left ideal of S.

Conversely, let χ be an anti fuzzy left ideal of S. Let $x \in I$ and $s \in S, \gamma \in \Gamma$, then $\chi(x) = 0$. Also $\chi(s\gamma x) \leq \chi(x) = 0$. Thus $s\gamma x \in I$. So I is a left ideal of S. Similarly we can prove that the other parts of the theorem. □

Definition 3.5. Let μ be a fuzzy subset of a Γ-semigroup S and let $t \in [0, 1]$. Then the set $\mu_t := \{x \in S : \mu(x) \leq t\}$ is called the anti level subset of μ.

Theorem 3.6. *Let S be a Γ-semigroup and μ be a non-empty fuzzy subset of S, then μ is an anti fuzzy left ideal(anti fuzzy right ideal, anti fuzzy ideal) of S if and only if μ_t's are left ideals(resp. right ideals, ideals) of S for all $t \in Im(\mu)$, where $\mu_t = \{x \in S : \mu(x) \leq t\}$.*

Proof. Let μ be an anti fuzzy left ideal of S. Let $t \in \operatorname{Im}\mu$, then there exist some $\alpha \in S$ such that $\mu(\alpha) = t$ and so $\alpha \in \mu_t$. Thus $\mu_t \neq \phi$. Let $s \in S$, $x \in \mu_t$ and $\gamma \in \Gamma$, then $\mu(x) \leq t$. Now $\mu(s\gamma x) \leq \mu(x) \leq t$. Therefore $s\gamma x \in \mu_t$. Thus μ_t is a left ideal of S.

Conversely, let μ_t's are left ideals of S for all $t \in \operatorname{Im}\mu$. Let $x, s \in S$ and $\gamma \in \Gamma$. Now let $\mu(x) = t$, then $x \in \mu_t$. Thus $s\gamma x \in \mu_t$(since μ_t is a left ideal of S). Therefore $\mu(s\gamma x) \leq t = \mu(x)$. Hence μ is an anti fuzzy left ideal of S. Similarly we can prove the other cases. □

Definition 3.7. Let S be a Γ-semigroup and μ be an anti fuzzy left ideal(anti fuzzy right ideal, anti fuzzy ideal) of a Γ-semigroup S. Then the ideals μ_t's are called the anti leveled left ideals(resp. anti leveled right ideals, anti leveled ideals)of μ where $t \in Im(\mu)$.

Theorem 3.8. *Let μ be an anti fuzzy left ideal(anti fuzzy right ideal, anti fuzzy ideal) of a Γ-semigroup S and $t_1 < t_2$. Then $\mu_{t_1} \subseteq \mu_{t_2}$ where $t_1, t_2 \in Im(\mu)$, equality occurs if and only if there is no $x \in S$ such that $t_1 < \mu(x) \leq t_2$.*

Proof. The first part of the theorem follows easily. Now let μ be an anti fuzzy left ideal of S such that $\mu_{t_1} = \mu_{t_2}$. If possible let there exists $x \in S$ such that $t_1 < \mu(x) \leq t_2$. Then $x \in \mu_{t_2}$ but $x \notin \mu_{t_1}$, which is a contradiction. So there exist no $x \in S$ such that $t_1 < \mu(x) \leq t_2$.

Conversely suppose μ is a fuzzy left ideal of S such that there does not exist $x \in S$ with $t_1 < \mu(x) \leq t_2$. Since $t_1 < t_2$, then $\mu_{t_1} \subseteq \mu_{t_2}$. If possible let $\mu_{t_1} \neq \mu_{t_2}$. Then there is some $y \in S$ such that $y \in \mu_{t_2}$ but $y \notin \mu_{t_1}$, *i.e*, $\mu(y) \leq t_2$ but $\mu(y) > t_1$, *i.e*, $t_1 < \mu(y) \leq t_2$, which contradicts our assumption. Hence $\mu_{t_1} = \mu_{t_2}$. By applying similar argument we can prove the theorem when μ is an anti fuzzy right ideal and an anti fuzzy ideal. □

Theorem 3.9. *Let S be a Γ-semigroup. Then a fuzzy set μ is an anti fuzzy left ideal(anti fuzzy right ideal, anti fuzzy ideal) of S if and only if μ^C(where $\mu^C(x) = 1 - \mu(x)\ \forall x \in S$) is a fuzzy left ideal(resp. fuzzy right ideal, fuzzy ideal) of S.*

Proof. Let $x, y \in S$, $\gamma \in \Gamma$ and μ be an anti fuzzy left ideal of S. Then

$$\mu^C(x\gamma y) = 1 - \mu(x\gamma y) \geq 1 - \mu(y) = \mu^C(y).$$

Hence μ^C is an fuzzy left ideal of S. Similarly we can prove the other cases. Converse can easily be verified by routine calculation. □

Theorem 3.10. *If $\{\mu_i : i \in \wedge\}$ is a family of anti fuzzy left ideals(anti fuzzy right ideals, anti fuzzy ideals) of a Γ-semigroup S then so is $\bigcup\limits_{i \in \wedge} \mu_i$.*

Proof. Let $\{\mu_i : i \in \wedge\}$ be a family of anti fuzzy left ideals and $x, y \in S$, $\gamma \in \Gamma$. Then

$$\begin{aligned}(\bigcup_{i\in\wedge}\mu_i)(x\gamma y) &= \sup\{\mu_i(x\gamma y) : i \in \wedge\}\\ &\leq \sup\{\mu_i(y) : i \in \wedge\} = (\bigcup_{i\in\wedge}\mu_i)(y).\end{aligned}$$

Hence $\bigcup_{i\in\wedge}\mu_i$ is an anti fuzzy left ideal of S. Similarly we can prove the other cases. □

Theorem 3.11. *Intersection of a non empty collection of anti fuzzy left ideals(anti fuzzy right ideals, anti fuzzy ideals) of a* Γ*semigroup* S *is an anti fuzzy left ideal(resp. anti fuzzy right ideal, anti fuzzy ideal) of* S *provided it is non-empty.*

Proof. Let $\{\mu_i : i \in \wedge\}$ be a non-empty collection of anti fuzzy left ideals of S such that $\bigcap_{i\in\wedge}\mu_i \neq \varnothing$ and $x, y \in S$, $\gamma \in \Gamma$. Then

$$\begin{aligned}(\bigcap_{i\in\wedge}\mu_i)(x\gamma y) &= \inf\{\mu_i(x\gamma y) : i \in \wedge\}\\ &\leq \inf\{\mu_i(y) : i \in \wedge\} = (\bigcap_{i\in\wedge}\mu_i)(y).\end{aligned}$$

Hence $\bigcap_{i\in\wedge}\mu_i$ is an anti fuzzy left ideal of S. Similarly we can prove the other cases □

Definition 3.12. Let μ be a fuzzy subset of a Γ-semigroup S and $\alpha \in [0, 1 - \sup\{\mu(x) : x \in S\}], \beta \in [0, 1]$. A mapping $\mu^C_{\beta\alpha} : S \to [0, 1]$ is called a fuzzy magnified translation of μ if $\mu^C_{\beta\alpha}(x) = \beta \cdot \mu(x) + \alpha$ for all $x \in S$.

Proposition 3.13. *Let* μ *be an anti fuzzy left ideal(anti fuzzy right ideal, anti fuzzy ideal) of a* Γ*-semigroup* S*. Then the fuzzy magnified translation* $\mu^C_{\beta\alpha}$ *of* μ *is an anti fuzzy left ideal(resp. anti fuzzy right ideal, anti fuzzy ideal) of* S*.*

Proof. Let μ be an anti fuzzy left ideal of S and $x, y \in S$, $\gamma \in \Gamma$. Then

$$\begin{aligned}\mu^C_{\beta\alpha}(x\gamma y) &= \beta \cdot \mu(x\gamma y) + \alpha\\ &\leq \beta \cdot \mu(y) + \alpha \,(\text{since } \mu \text{ is an anti fuzzy left ideal of } S)\\ &= \mu^C_{\beta\alpha}(y).\end{aligned}$$

Hence $\mu^C_{\beta\alpha}$ is an anti fuzzy left ideal of S. Similarly we can prove the other cases. □

Definition 3.14 ([24])**.** A Γ-semigroup S is called left zero(right zero) if $x\gamma y = x$(resp. $x\gamma y = y$) $\forall x, y \in S$, $\forall \gamma \in \Gamma$.

Proposition 3.15. *Let* μ *be an anti fuzzy left ideal(anti fuzzy right ideal) of a left zero(right zero)* Γ*-semigroup* S*. Then fuzzy magnified translation* $\mu^C_{\beta\alpha}$ *of* μ *is a constant function.*

Proof. Let S be a left zero Γ-semigroup and $x, y \in S$. Then $x = x\gamma y$ and $y = y\gamma x$ $\forall\gamma \in \Gamma$. Then

$$\begin{aligned}\mu^C_{\beta\alpha}(x) &= \beta\cdot\mu(x)+\alpha = \beta\cdot\mu(x\gamma y)+\alpha \leq \beta\cdot\mu(y)+\alpha\\ &\qquad\text{(since } \mu \text{ is an anti fuzzy left ideal)}\\ &= \mu^C_{\beta\alpha}(y).\end{aligned}$$

Again

$$\begin{aligned}\mu^C_{\beta\alpha}(y) &= \beta\cdot\mu(y)+\alpha = \beta\cdot\mu(y\gamma x)+\alpha \leq \beta\cdot\mu(x)+\alpha\\ &\qquad\text{(since } \mu \text{ is an ant fuzzy left ideal)}\\ &= \mu^C_{\beta\alpha}(x).\end{aligned}$$

Thus $\mu^C_{\beta\alpha}(x) = \mu^C_{\beta\alpha}(y)$ $\forall x, y \in S$. Hence $\mu^C_{\beta\alpha}$ is a constant function. □

Theorem 3.16. *Let $f : R \to S$ be a homomorphism of Γ-semigroups.*

(1) *If λ is an anti fuzzy left ideal(anti fuzzy right ideal, anti fuzzy ideal) of S then $f^{-1}(\lambda)$ is also an anti fuzzy left ideal(resp. anti fuzzy right ideal, anti fuzzy ideal) of R(where $f^{-1}(\lambda)(r\gamma s) := \lambda(f(r\gamma s))$ for all $r, s \in R$ and $\gamma \in \Gamma$), provided $f^{-1}(\lambda)$ is non-empty.*

(2) *If f is a surjective homomorphism and μ is an anti fuzzy left ideal (anti fuzzy right ideal, anti fuzzy ideal) of R then $f(\mu)$ is also an anti fuzzy left ideal (resp. anti fuzzy right ideal, anti fuzzy ideal) of S(where $f(\mu)(x) := \inf\limits_{f(y)=x} \mu(y), x \in S, y \in R$).*

Proof. (1) Let λ be an anti fuzzy left ideal of S. Let $r, s \in R$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}f^{-1}(\lambda)(r\gamma s) &= \lambda(f(r\gamma s)) = \lambda(f(r)\gamma f(s))\\ &\leq \lambda(f(s)) = f^{-1}(\lambda)(s).\end{aligned}$$

Thus $f^{-1}(\lambda)$ is an anti fuzzy left ideal of R.

(2) Let μ be a fuzzy left ideal of R. Since $f(\mu)(x^{'}) = \inf\limits_{f(x)=x^{'}} \mu(x)$ for $x^{'} \in S$, so $f(\mu)$ is non-empty. Let $x^{'}, y^{'} \in S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}f(\mu)(x^{'}\gamma y^{'}) &= \inf_{f(z)=x^{'}\gamma y^{'}} \mu(z) \leq \inf_{\substack{f(x)=x^{'}\\ f(y)=y^{'}}} \mu(x\gamma y)\\ &\leq \inf_{f(y)=y^{'}} \mu(y) = f(\mu)(y^{'}).\end{aligned}$$

Hence $f(\mu)$ is an anti fuzzy left ideal of S. In a similar way we can prove the theorem when λ, μ are anti fuzzy right ideals and anti fuzzy ideals. □

Proposition 3.17. *Let μ be an anti fuzzy left ideal(anti fuzzy right ideal, anti fuzzy ideal) of a Γ-semigroup S. Then so is μ^α for every real number $\alpha \geq 0$, where μ^α is defined by $\mu^\alpha(x) = (\mu(x))^\alpha$ for all $x \in S$.*

Proof. Let $x, y \in S$, $\gamma \in \Gamma$ and μ be an anti fuzzy left ideal of S. Then $\mu(x\gamma y) \leq \mu(y)$. Then

$$\mu^\alpha(x\gamma y) = (\mu(x\gamma y))^\alpha \leq (\mu(y))^\alpha = \mu^\alpha(y).$$

Consequently, μ^α is an anti fuzzy left ideal of S. Similarly we can prove the other case also. $\square$

4. Normal anti fuzzy ideal

Definition 4.1. An anti fuzzy ideal μ of a Γ-semigroup S is said to be normal if $\mu(0) = 1$.

Definition 4.2. An anti fuzzy ideal μ of a Γ-semigroup S is said to be complete if it is normal and there exists $z \in S$ such that $\mu(z) = 0$.

Example 4.3. Let $S = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ with the following cayley tables:

α	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	b
c	0	0	0	c

β	0	a	b	c
0	0	0	0	0
a	0	b	0	a
b	0	b	0	c
c	0	0	0	b

γ	0	a	b	c
0	0	0	0	0
a	0	b	0	a
b	0	b	0	c
c	0	0	0	b

Then S is a Γ-semigroup. We define a fuzzy subset $\mu : S \to [0, 1]$ as $\mu(0) = \mu(a) = \mu(b) = \mu(c) = 1$. Then μ is a normal anti fuzzy ideal of S.

Theorem 4.4. *Let μ be an anti fuzzy left ideal(anti fuzzy right ideal, anti fuzzy ideal) of a Γ-semigroup S and μ^+ given by $\mu^+(x) = \mu(x) + 1 - \mu(0)$ for all $x \in S$. Then μ^+ is normal anti fuzzy left ideal(resp. anti fuzzy right ideal, anti fuzzy ideal) S which contains μ.*

Proof. Let $x, y \in S$ and $\gamma \in \Gamma$ and μ be an anti fuzzy left ideal of S. Then

$$\mu^+(x\gamma y) = \mu(x\gamma y) + 1 - \mu(0) \leq \mu(y) + 1 - \mu(0) = \mu^+(y).$$

So μ is an anti fuzzy left ideal of S. Again $\mu^+(0) = \mu(0) + 1 - \mu(0) = 1$, hence μ^+ is normal anti fuzzy left ideal of S. Similarly we can prove other cases also. $\square$

By routine calculation we can prove the following proposition.

Proposition 4.5. *If μ is an anti fuzzy ideal of Γ-semigroup S then $(\mu^+)^+ = \mu^+$.*

Theorem 4.6. *μ is a normal anti fuzzy ideal of a Γ-semigroup S if and only if $\mu^+ = \mu$.*

Proof. Let μ be a normal anti fuzzy ideal of S and $x \in S$. Then $\mu^+(x) = \mu(x) + 1 - \mu(0) = \mu(x)$(since μ is a normal anti fuzzy ideal). Converse follows from routine calculation. $\square$

Theorem 4.7. *Let μ be an anti fuzzy left ideal(anti fuzzy right ideal, anti fuzzy ideal) of a Γ-semigroup S and t be a fixed element of S such that $\mu(0) \neq \mu(t)$. Let μ^* be a fuzzy subset of S such that $\mu^*(x) = \frac{\mu(x)-\mu(t)}{\mu(0)-\mu(t)}$ $\forall x \in S$. Then μ^* is a complete anti fuzzy left ideal(resp. anti fuzzy right ideal, anti fuzzy ideal) of S.*

Proof. Let $x, y \in S$, $\gamma \in \Gamma$ and μ be an anti fuzzy left ideal of S. Then

$$\mu^*(x\gamma y) = \frac{\mu(x\gamma y) - \mu(t)}{\mu(0) - \mu(t)} \leq \frac{\mu(y) - \mu(t)}{\mu(0) - \mu(t)} = \mu^*(y).$$

Hence μ^* is an anti fuzzy left ideal of S. Clearly μ^* is a normal anti fuzzy left ideal of S. Since $t \in S$ then $\mu^*(t) = \frac{\mu(t)-\mu(t)}{\mu(0)-\mu(t)} = 0$. Hence μ^* is a complete anti fuzzy left ideal of S. Similarly we can prove the other cases also. □

Theorem 4.8. *Let μ be an anti fuzzy left ideal(anti fuzzy right ideal, anti fuzzy ideal) of a Γ-semigroup S and let $f : [0, 1] \to [0, 1]$ be an increasing function. Then the fuzzy set $\mu_f : S \to [0, 1]$ defined by $\mu_f(x) = f(\mu(x))$ $\forall x \in S$ is an anti fuzzy left ideal(anti fuzzy right ideal, anti fuzzy ideal) of S. In particular, $f(\mu(0)) = 1$ then μ_f is a normal anti fuzzy left ideal(anti fuzzy right ideal, anti fuzzy ideal) of S and if $f(t) \geq t$ for all $t \in [0, 1]$ then $\mu \subseteq \mu_f$.*

Proof. Let $x, y \in S$, $\gamma \in \Gamma$ and μ be an anti fuzzy left ideal of S. Then $\mu_f(x\gamma y) = f(\mu(x\gamma y)) \leq f(\mu(y)) = \mu_f(y)$. So μ_f is an anti fuzzy left ideal of S. Now $\mu_f(0) = f(\mu(0)) = 1$. Hence μ_f is a normal anti fuzzy left ideal of S. Let $t = \mu(x)$. Then $f(t) \geq t$ gives $f(\mu(x)) \geq \mu(x)$ which implies that $\mu_f(x) \geq \mu(x)$ and hence $\mu \subseteq \mu_f$. □

Remark 4.9. Let $N(S)$ denotes the set of all normal fuzzy ideals of S. Then $N(S)$ is a poset under set inclusion.

5. Composition of anti fuzzy ideals

In this section we define the anti fuzzy product of two fuzzy subsets of a Γ-semigroup and characterize regular Γ-semigroups in terms of anti fuzzy ideals.

Definition 5.1. Let S be a Γ-semigroup and μ_1, μ_2 are two fuzzy subsets of S. Then the anti fuzzy product $\mu_1 \circ \mu_2$ of μ_1 and μ_2 is defined as

$$(\mu_1 \circ \mu_2)(x) = \begin{cases} \inf\limits_{x=u\gamma v}[\max\{\mu_1(u), \mu_2(v)\} : u, v \in S; \gamma \in \Gamma] \\ 1, \text{ if for any } u, v \in S \text{ and for any } \gamma \in \Gamma, x \neq u\gamma v \end{cases}$$

Theorem 5.2. *In a Γ-semigroup S the following are equivalent:* (1) *μ is an anti fuzzy left(anti fuzzy right) ideal of S,* (2) *$\chi \circ \mu \supseteq \mu(\mu \circ \chi \supseteq \mu)$, where χ is the characteristic function of S.*

Proof. (1) $\Rightarrow$ (2) : Let μ be an anti fuzzy left ideal of S. Let $a \in S$. Suppose there exist $u, v \in S$ and $\delta \in \Gamma$ such that $a = u\delta v$. Then, since μ is an anti fuzzy left ideal of S, we have

$$\begin{aligned}(\chi \circ \mu)(a) &= \inf_{a=x\gamma y}[\max\{\chi(x), \mu(y)\}] \\ &= \inf_{a=x\gamma y}[\max\{0, \mu(y)\}] = \inf_{a=x\gamma y}\mu(y).\end{aligned}$$

Now, since μ is an anti fuzzy left ideal, $\mu(x\gamma y) \leq \mu(y)$ for all $x, y \in S$ and for all $\gamma \in \Gamma$. So in particular, $\mu(y) \geq \mu(a)$ for all $a = x\gamma y$. Hence $\inf\limits_{a=x\gamma y}\mu(y) \geq \mu(a)$. Thus $\mu(a) \leq (\chi \circ \mu)(a)$. If there do not exist $x, y \in S$, $\gamma \in \Gamma$ such that $a = x\gamma y$ then $(\chi \circ \mu)(a) = 1 \geq \mu(a)$. Hence $\chi \circ \mu \supseteq \mu$. By a similar argument we can show that $\mu \circ \chi \supseteq \mu$ when μ is an anti fuzzy right ideal.

(2) $\Rightarrow$ (1) : Let $\chi \circ \mu \supseteq \mu$. Let $x, y \in S$, $\gamma \in \Gamma$ and $a := x\gamma y$. Then $\mu(x\gamma y) = \mu(a) \leq (\chi \circ \mu)(a)$. Now

$$\begin{aligned}(\chi \circ \mu)(a) &= \inf_{a=u\alpha v}[\max\{\chi(u), \mu(v)\}] \leq \max\{\chi(x), \mu(y)\} \\ &= \max\{0, \mu(y)\} = \mu(y).\end{aligned}$$

Hence $\mu(x\gamma y) \leq \mu(y)$. Hence μ is an anti fuzzy left ideal of S. By a similar argument we can show that if $\mu \circ \chi \supseteq \mu$, then μ is an anti fuzzy right ideal of S. □

Using the above theorem we can deduce the following theorem.

Theorem 5.3. *In a Γ-semigroup S the following are equivalent: (1) μ is an anti fuzzy two-sided ideal of S, (2) $\chi \circ \mu \supseteq \mu$ and $\mu \circ \chi \supseteq \mu$, where χ is the characteristic function of S.*

Proposition 5.4. *Let μ_1 be an anti fuzzy right ideal and μ_2 be an anti fuzzy left ideal of a Γ-semigroup S. Then $\mu_1 \circ \mu_2 \supseteq \mu_1 \cup \mu_2$.*

Proof. Let μ_1 be an anti fuzzy right ideal and μ_2 be an anti fuzzy left ideal of a Γ-semigroup S. Let $x \in S$. Suppose there exist $u_1, v_1 \in S$ and $\gamma_1 \in \Gamma$ such that $x = u_1\gamma_1 v_1$. Then

$$\begin{aligned}(\mu_1 \circ \mu_2)(x) &= \inf_{x=u\gamma v} \max\{\mu_1(u), \mu_2(v)\} \\ &\geq \inf_{x=u\gamma v} \max\{\mu_1(u\gamma v), \mu_2(u\gamma v)\} \\ &= \max\{\mu_1(x), \mu_2(x)\} = (\mu_1 \cup \mu_2)(x).\end{aligned}$$

Suppose there do not exist $u, v \in S$ such that $x = u\gamma v$. Then $(\mu_1 \circ \mu_2)(x) = 1 \geq (\mu_1 \cup \mu_2)(x)$. Thus $\mu_1 \circ \mu_2 \supseteq \mu_1 \cup \mu_2$. □

From the above proposition and the definition of $\mu_1 \cup \mu_2$ the following proposition follows easily.

Proposition 5.5. *If μ_1, μ_2 are anti fuzzy ideals of S then $\mu_1 \circ \mu_2 \supseteq \mu_1 \cup \mu_2 \supseteq \mu_1, \mu_2$.*

Proposition 5.6. *Let S be a regular Γ-semigroup and μ_1, μ_2 be two fuzzy subsets of S. Then $\mu_1 \circ \mu_2 \subseteq \mu_1 \cup \mu_2$.*

Proof. Let $c \in S$. Since S is regular, then there exists an element $x \in S$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $c = c\gamma_1 x\gamma_2 c = c\gamma c$ where $\gamma := \gamma_1 x\gamma_2 \in \Gamma$. Then

$$\begin{aligned}(\mu_1 \circ \mu_2)(c) &= \inf_{c=u\alpha v}\{\max\{\mu_1(u), \mu_2(v)\}\} \\ &\leq \max\{\mu_1(c), \mu_2(c)\} = (\mu_1 \cup \mu_2)(c).\end{aligned}$$

Therefore $\mu_1 \circ \mu_2 \subseteq \mu_1 \cup \mu_2$. □

Definition 5.7. In a Γ-semigroup S an anti fuzzy ideal μ is said to be idempotent if $\mu \circ \mu = \mu$.

Using Propositions 5.5 and 5.6 we can prove the following.

Proposition 5.8. *In a regular Γ-semigroup S every anti fuzzy ideal is idempotent.*

Theorem 5.9. *In a* Γ*-semigroup* S *the following are equivalent.*

(1) S *is regular,*

(2) $\mu_1 \circ \mu_2 = \mu_1 \cup \mu_2$ *for every anti fuzzy right ideal* μ_1 *and every anti fuzzy left ideal* μ_2 *of* S.

Proof. (1) $\Rightarrow$ (2) : Let S be a regular Γ-semigroup. Then by Proposition 5.4 and 5.6 $\mu_1 \circ \mu_2 = \mu_1 \cup \mu_2$.

(2) $\Rightarrow$ (1) : Let S be a Γ-semigroup and for every anti fuzzy right ideal μ_1 and every anti fuzzy left ideal μ_2 of S, $\mu_1 \circ \mu_2 = \mu_1 \cup \mu_2$. Let L and R be respectively a left ideal and a right ideal of S and $x \in R \cap L$. Then $x \in R$ and $x \in L$. Hence $\chi_L(x) = \chi_R(x) = 0$(where $\chi_L(x)$ and $\chi_R(x)$ are respectively the characteristic functions of L and R). Thus

$$(\chi_R \cup \chi_L)(x) = \max\{\chi_R(x), \chi_L(x)\} = 0.$$

Now by Theorem 3.4, χ_L and χ_R are respectively an anti fuzzy left ideal and an anti fuzzy right ideal of S. Hence by hypothesis, $\chi_R \circ \chi_L = \chi_R \cup \chi_L$. Hence

$$(\chi_R \circ \chi_L)(x) = 0$$

$$i.e., \inf_{x=y\gamma z}[\max\{\chi_R(y), \chi_L(z)\} : y, z \in S; \gamma \in \Gamma] = 0.$$

This implies that there exist some $r, s \in S$ and $\gamma_1 \in \Gamma$ such that $x = r\gamma_1 s$ and $\chi_R(r) = 0 = \chi_L(s)$. Hence $r \in R$ and $s \in L$. Hence $x \in R\Gamma L$. Thus $R \cap L \subseteq R\Gamma L$. Also $R\Gamma L \subseteq R \cap L$. Hence $R\Gamma L = R \cap L$. Consequently, the Γ-semigroup S is regular. $\square$

Since commutative semigroup of idempotents is a semilattice [9], so in view of Proposition 5.8 and Theorem 5.9 we obtain the following result.

Theorem 5.10. *The set of all anti fuzzy ideals of a regular* Γ*-semigroup* S *forms a semilattice with respect to the composition.*

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