Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 5, No. 2, (March 2013), pp. 429–439
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

On lacunary statistical convergence in random n-normed space

Ayhan Esi, M. Kemal Özdemir

Received 15 May 2012; Revised 10 June 2012; Accepted 3 August 2012

Abstract. Mursaleen [24] introduced the concepts of statistical convergence in random 2−normed spaces. Recently Mohiuddine and Aiyup [22] defined the notion of lacunary statistical convergence and lacunary statistical Cauchy in random 2-normed spaces. In this paper, we define and study the notion of lacunary statistical convergence and lacunary statistical Cauchy sequences in random $n-$normed spaces and prove some theorems which generalizes Mohiuddine and Aiyup results.

2010 AMS Classification: 40A05, 46A70, 40A99, 46A99

Keywords: Statistical convergence, Lacunary statistical convergence, t-norm, n-norm, Random n-normed space.

Corresponding Author: M. Kemal Özdemir (kozdemir73@gmail.com)

1. Introduction

The concept of statistical convergence play a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modeling, and motion planning in robotics.

The notion of statistical convergence was introduced by Fast [5] and Schoenberg [33] independently. Over the years and under different names statistical convergence has been discussed in the theory of fourier analysis, ergodic theory and number theory. Later on it was further investigated by Fridy [6], Šalát [32], Çakalli [3], Maio and Kocinac [19], Miller [21], Maddox [18], Leindler [17], Mursaleen and Alotaibi [25], Mursaleen and Edely [29], Mursaleen and Edely [31], and many others. In the recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers. Moreover statistical convergence is closely related to the concept of convergence in probability, (see [2]).

The notion of statistical convergence depends on the density of subsets of $\mathbf{N}$. A subset of $\mathbf{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists.}$$

Definition 1.1. A sequence $x = (x_k)$ is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbf{N} : |x_k - \ell| \geq \varepsilon\}) = 0.$$

In this case, we write $S - \lim x = \ell$ or $x_k \to \ell(S)$ and S denotes the set of all statistically convergent sequences.

The probabilistic metric space was introduced by Menger [20] which is an interesting and important generalization of the notion of a metric space. Karakus [14] studied the concept of statistical convergence in probabilistic normed spaces. Subsequently, Esi and Özdemir [4] generalized these results on statistical convergence in probabilistic normed space given by Karakus [14]. The theory of probabilistic normed spaces was initiated and developed in [1], [34], [35], [36], [38] and further it was extended to random/probabilistic 2−normed spaces by Goleţ [9] using the concept of 2−norm which is defined by Gähler [8], and Gürdal and Pehlivan [11] studied statistical convergence in 2−Banach spaces.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_o = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. Let $K \subseteq \mathbb{N}$. The number

$$\delta_\theta(K) = \lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : \ k \in K\}|$$

is said to the $\theta-$density of K, provided the limit exists.

Definition 1.2 ([7])**.** Let θ be a lacunary sequence. Then a sequence $x = (x_k)$ is said to be $S_\theta-$*convergent* to the number ℓ if for every $\varepsilon > 0$, the set $K(\varepsilon)$ has $\theta-$density zero, where

$$K(\varepsilon) = \{k \in \mathbb{N} : \ |x_k - \ell| \geq \varepsilon\}.$$

In this case we write $S_\theta - \lim x = \ell$ or $x_k \to \ell(S_\theta)$.

The existing literature on statistical convergence and its generalizations appears to have been restricted to real or complex sequences, but in recent years these ideas have been also extended to the sequences in fuzzy normed [37] and intuitionistic fuzzy normed spaces [15], [23], [28], [27] and [16]. Further details on generalization of statistical convergence can be found in [26], [29], [30] and [31].

In [12], Gürdal and Pehlivan studied statistical convergence in 2−normed spaces and in 2−Banach spaces in [11]. In fact, Mursaleen [24] studied the concept of statistical convergence of sequences in random 2−normed space. Recently in [13], Hazarika and Esi introduced and studied the concept of generalized Δ^n-statistical convergence of sequences in 2−normed space.

2. Preliminaries

Definition 2.1. A function $f : \mathbf{R} \to \mathbf{R}_0^+$ is called a *distribution function* if it is a non-decreasing and left continuous with $\inf_{t\in\mathbf{R}} f(t) = 0$ and $\sup_{t\in\mathbf{R}} f(t) = 1$. By D^+, we denote the set of all distribution functions such that $f(0) = 0$. If $a \in \mathbf{R}_0^+$, then $H_a \in D^+$, where

$$H_a(t) = \begin{cases} 1 & , \quad \text{if } t > a \\ 0 & , \quad \text{if } t \leq a \end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

A $t-norm$ is a continuous mapping $* : [0,1]\times[0,1] \to [0,1]$ such that $([0,1], *)$ is abelian monoid with unit one and $c*d \geq a*b$ if $c \geq a$ and $d \geq b$ for all $a, b, c \in [0,1]$. A triangle function τ is a binary operation on D^+, which is commutative, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.

Definition 2.2 ([10])**.** Let $n \in \mathbb{N}$ and X be a real vector space of dimension d, where $n \leq d$. A real-valued function $\|., ..., .\|$ on X satisfying the following four conditions:

(i) $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,
(ii) $\|x_1, x_2, ..., x_n\|$ is invariant under permutation,
(iii) $\|\alpha x_1, x_2, ..., x_n\| = |\alpha| \|x_1, x_2, ..., x_n\|, \alpha \in \mathbb{R}$,
(iv) $\|x_1 + x_1^{\prime}, x_2, ..., x_n\| \leq \|x_1, x_2, ..., x_n\| + \|x_1^{\prime}, x_2, ..., x_n\|$
is called an $n-norm$ on X, and the pair $(X, \|., ..., .\|)$ is called an *$n-$normed space.*

A trivial example of $n-$normed space is $X = \mathbb{R}^n$ equipped with the following Euclidean $n-$norm:

$$\|x_1, x_2, ..., x_n\|_E = abs\left(\begin{vmatrix} x_{11}...x_{1n} \\ ... \\ x_{n1}...x_{nn} \end{vmatrix}\right)$$

where $x_i = (x_{i1}, ..., x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$.

By generalizing Definition 2.2, we obtain a satisfactory notion of probabilistic n-normed space as follows:

Definition 2.3. Let X be a real linear space of dimension d greater than n, and let $\mathcal{F}$ be a mapping defined on the Cartesian product of X by itself of n times X^n into D$^+$ such that the following properties are satisfied:

(PnN_1) $\mathcal{F}_{x_1,x_2,...,x_n}(t)$ is invariant under any permutation of $x_1, x_2, ..., x_n \in X$,

(PnN_2) $\mathcal{F}_{x_1,x_2,...,x_n}(t) = \mathcal{F}_{x_1,x_2,...,x_n}(\frac{t}{\varphi(\alpha)})$ for every $x_1, x_2, ..., x_n \in X$ and $\alpha \in \mathbf{R}$,

(PnN_3) $\mathcal{F}_{x_1,x_2,...,x_{n-1},y+z} \geq \tau\left(\mathcal{F}_{x_1,x_2,...,x_{n-1},y}, \mathcal{F}_{x_1,x_2,...,x_{n-1},z}\right)$ for every $x_1, x_2, ..., x_{n-1}, y, z \in X$.

The function $\mathcal{F}$ is called a probabilistic n-norm on X and the triple $(X, \mathcal{F}, \tau)$ is called a probabilistic n-normed space. The triangle inequalities (PnN_3) can be formulated by using a t-norm T.

(PnN_4) $\mathcal{F}_{x_1,x_2,...,x_{n-1},y+z}(t_1 + t_2) \geq T\left(\mathcal{F}_{x_1,x_2,...,x_{n-1},y}(t_1), \mathcal{F}_{x_1,x_2,...,x_{n-1},z}(t_2)\right)$ for every $x_1, x_2, ..., x_{n-1}, y, z \in X$ and $t_1, t_2 \in R^+$. If $(PnN_1), (PnN_2)$ *and* (PnN_4) are satisfied then the triple $(X, \mathcal{F}, \tau)$ is called a generalized probabilistic n-normed saces of Menger type or simply Menger n-normed space.

Definition 2.4. A sequence $x = (x_k)$ in a random $n-$normed space $(X, \mathcal{F}, *)$ is said to be *statistical-convergent* or $S^{RnN}-$*convergent* to some $\ell \in X$ with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $\beta \in (0, 1)$ and for non zero elements $x_1, x_2, ..., x_{n-1} \in X$ such that

$$\delta\left(\{n \in \mathbf{N} : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - \ell; \varepsilon) \leq 1 - \beta\}\right) = 0,$$

In other words we can write the sequence (x_n) *statistical converges* to ℓ in random $n-$normed space $(X, \mathcal{F}, *)$ if

$$\lim_{m\to\infty} \frac{1}{m} |\{n \leq m : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - \ell; \varepsilon) \leq 1 - \beta\}| = 0.$$

or equivalently

$$\delta\left(\{n \in \mathbf{N} : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - \ell; \varepsilon) > 1 - \beta\}\right) = 1,$$

i.e.,

$$S - \lim_{n\to\infty} \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - \ell; \varepsilon) = 1.$$

In this case we write $S^{RnN} - \lim x = \ell$ and ℓ is called the $S^{RnN}-$*limit of* x. Let $S^{RnN}(X)$ denotes the set of all statistical convergent sequences in random $n-$normed space $(X, \mathcal{F}, *)$.

In this paper we define and study lacunary statistical convergence in random $n-$normed space which is quite a new and interesting idea to work with. We show that some properties of lacunary statistical convergence of real numbers also hold for sequences in random $n-$normed spaces. We find some relations related to lacunary statistical convergent sequences in random $n-$normed spaces. Also we find out the relation between lacunary statistical convergent and lacunary statistical Cauchy sequences in these spaces.

3. Lacunary statistical convergence in random $n-$normed space

In this section we define lacunary statistical convergent sequence in random $n-$normed $(X, \mathcal{F}, *)$. Also we obtained some basic properties of this notion in random $n-$normed space.

Definition 3.1. Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_n)$ in a random $n-$normed space $(X, \mathcal{F}, *)$ is said to be *convergent* to $\ell \in X$ with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $\beta \in (0, 1)$ there exists an positive integer $r_0 \in I_r$ such that $\mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - \ell; \varepsilon) > 1 - \beta$, whenever $n \geq r_0$ and for non-zero elements $x_1, x_2, ..., x_{n-1} \in X$. In this case we write $\mathcal{F}_\theta - \lim_n x_n = \ell$, and ℓ is called the $\mathcal{F}_\theta-$limit of $x = (x_n)$.

Definition 3.2. Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_n)$ in a random $n-$normed space $(X, \mathcal{F}, *)$ is said to be *Cauchy* with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $\beta \in (0, 1)$ there exists a subsequence (x_s) of (x_n) such that integer $s \in I_r$ for each r such that $\mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - x_s; \varepsilon) > 1 - \beta$, for non-zero elements $x_1, x_2, ..., x_{n-1} \in X$.

Definition 3.3. A sequence $x = (x_n)$ in a random $n-$normed space $(X, \mathcal{F}, *)$ is said to be *lacunary statistically convergent* or $S_\theta-$*convergent* to $\ell \in X$ with respect to $\mathcal{F}$ if for every $\varepsilon > 0$, $\beta \in (0,1)$ and for non zero elements $x_1, x_2, ..., x_{n-1} \in X$ such that

$$\delta_\theta(\{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - \ell; \varepsilon) \leq 1 - \beta\}) = 0.$$

In other ways we can write the sequence $x = (x_n)$ lacunary statistical converges to ℓ in random $n-$normed space $(X, \mathcal{F}, *)$ if

$$\lim_{r\to\infty} \frac{1}{h_r} |\{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - \ell; \varepsilon) \leq 1 - \beta\}| = 0.$$

or equivalently

$$\delta_\theta\left(\{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - \ell; \varepsilon) > 1 - \beta\}\right) = 1,$$

i.e.,

$$S_\theta - \lim_{n\to\infty} \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - \ell; \varepsilon) = 1.$$

In this case we write $S_\theta^{RnN} - \lim x = \ell$ or $x_n \to \ell(S_\theta^{RnN})$ and

$$S_\theta^{RnN}(X) = \{x = (x_n) : \exists\ \ell \in \mathbf{R}, S_\theta^{RnN} - \lim x = \ell\}.$$

In this case we write $S_\theta^{RnN} - \lim x = \ell$ and ℓ is called the $S_\theta^{RnN}-$*limit of* x. Let $S_\theta^{RnN}(X)$ denotes the set of all statistical convergent sequences in random $n-$normed space $(X, \mathcal{F}, *)$.

Definition 3.4. A sequence $x = (x_k)$ in a random $n-$normed space $(X, \mathcal{F}, *)$ is said to be *lacunary statistical Cauchy* with respect to $\mathcal{F}$ if for every $\varepsilon > 0$, $\beta \in (0,1)$ and for non-zero elements $x_1, x_2, ..., x_{n-1} \in X$ there exists a positive integer $n_o = n_o(\varepsilon)$ such that for all $n, s \geq n_o$

$$\delta_\theta(\{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - x_s; \varepsilon) \leq 1 - \beta\}) = 0.$$

or equivalently

$$\delta_\theta(\{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - x_s; \varepsilon) > 1 - \beta\}) = 1.$$

Definition 3.3, immediately implies the following lemma.

Lemma 3.5. *Let $(X, \mathcal{F}, *)$ be a random $n-$normed space. If $x = (x_n)$ is a sequence in X, then for every $\varepsilon > 0$, $\beta \in (0,1)$ and for non zero elements $x_1, x_2, ..., x_{n-1} \in X$, then the following statements are equivalent:*

(i) $S_\theta - \lim_{n\to\infty} x_n = \ell$.
(ii) $\delta_\theta(\{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - x_s; \varepsilon) \leq 1 - \beta\}) = 0$.
(iii) $\delta_\theta\left(\{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - x_s; \varepsilon) > 1 - \beta\}\right) = 1$.
(iv) $S_\theta - \lim_{n\to\infty} \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - x_s; \varepsilon) = 1$.

Theorem 3.6. *Let $(X, \mathcal{F}, *)$ be a random $n-$normed space. If $x = (x_n)$ is a sequence in X such that $S_\theta^{RnN} - \lim x_n = \ell$ exists, then it is unique.*

Proof. Suppose that there exist elements ℓ_1, ℓ_2 $(\ell_1 \neq \ell_2)$ in X such that

$$S_\theta^{RnN} - \lim_{n\to\infty} x_n = \ell_1; S_\theta^{RnN} - \lim_{n\to\infty} x_n = \ell_2.$$

Let $\varepsilon > 0$ be given. Choose $s > 0$ such that

$$(1 - s) * (1 - s) > 1 - \varepsilon. \tag{3.1}$$

Then, for any $t > 0$ and for non-zero elements $x_1, x_2, ..., x_{n-1} \in X$ we define

$$K_1(s,t) = \left\{ n \in I_r : \mathcal{F}\left(x_1, x_2, ..., x_{n-1}, x_n - l_1; \frac{t}{2}\right) \leq 1 - s \right\};$$

$$K_2(s,t) = \left\{ n \in I_r : \mathcal{F}\left(x_1, x_2, ..., x_{n-1}, x_n - l_2; \frac{t}{2}\right) \leq 1 - s \right\}.$$

Since $S_\theta^{RnN} - \lim_{n\to\infty} x_n = \ell_1$ and $S_\theta^{RnN} - \lim_{n\to\infty} x_n = \ell_2$, we have $\delta_\theta(K_1(s,t)) = 0$ and $\delta_\theta(K_2(s,t)) = 0$ for all $t > 0$. Now let $K(s,t) = K_1(s,t) \cup K_2(s,t)$, then it is easy to observe that $\delta_\theta(K(s,t)) = 0$. But we have $\delta_\theta(K^c(s,t)) = 1$. Now if $n \in K^c(s,t)$ then we have

$$\begin{aligned}\mathcal{F}(x_1, x_2, ..., x_{n-1}, l_1 - l_2; t) \geq \mathcal{F}&\left(x_1, x_2, ..., x_{n-1}, x_n - \ell_1; \frac{t}{2}\right) \\ &* \mathcal{F}\left(x_1, x_2, ..., x_{n-1}, x_n - l_2; \frac{t}{2}\right) \\ &> (1 - s) * (1 - s).\end{aligned}$$

It follows by (3.1) that

$$\mathcal{F}(x_1, x_2, ..., x_{n-1}, \ell_1 - \ell_2; t) > (1 - \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we get $\mathcal{F}(x_1, x_2, ..., x_{n-1}, l_1 - l_2; t) = 1$ for all $t > 0$ and non-zero elements $x_1, x_2, ..., x_{n-1} \in X$. Hence $\ell_1 = \ell_2$. □

Next theorem gives the algebraic characterization of lacunary statistical convergence on random $n-$normed spaces.

Theorem 3.7. *Let $(X, \mathcal{F}, *)$ be a random $n-$normed space, and $x = (x_n)$ and $y = (y_n)$ be two sequences in X.*

(a) *If $S_\theta^{RnN} - \lim x_n = \ell$ and $c(\neq 0) \in R$, then $S_\theta^{RnN} - \lim cx_n = c\ell$.*

(b) *If $S_\theta - \lim x_n = \ell_1$ and $S_\theta^{RnN} - \lim y_n = \ell_2$, then $S_\theta^{RnN} - \lim(x_n + y_n) = \ell_1 + \ell_2$.*

Proof of the theorem is straightforward and thus omitted.

Theorem 3.8. *Let $(X, \mathcal{F}, *)$ be a random $n-$normed space. If $x = (x_n)$ be a sequence in X such that $F_\theta - \lim x_n = \ell$ then $S_\theta^{RnN} - \lim x_n = \ell$.*

Proof. Let $\mathcal{F}_\theta - \lim x_n = \ell$. Then for every $\varepsilon > 0$, $t > 0$ and non zero elements $x_1, x_2, ..., x_{n-1} \in X$, there is a positive integer n_0 such that

$$\mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - \ell; t) > 1 - \varepsilon$$

for all $n \geq n_0$. Since the set

$$K(\varepsilon, t) = \{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - \ell; t) \leq 1 - \varepsilon\}$$

has at most finitely many terms. Also, since every finite subset of $\mathbf{N}$ has δ_θ-density zero, and consequently we have $\delta_\theta(K(\varepsilon, t)) = 0$. This shows that $S_\theta^{RnN} - \lim x_n = \ell$. □

Remark 3.9. The converse of the above theorem is not true in general. It follows from the following example.

Example 3.10. Let $X = \mathbf{R}^2$, with the 2−norm $||x, z|| = |x_1z_2 - x_2z_1|$, $x = (x_1, x_2)$, $z = (z_1, z_2)$ and $a*b = ab$ for all $a, b \in [0, 1]$. Let $\mathcal{F}(x, y; t) = \frac{t}{t+||x,y||}$, for all $x, z \in X$, $z_2 \neq 0$, and $t > 0$. Now we define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} (k,0) & , \quad \text{if } k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r; r \in \mathbb{N} \\ (0,0) & , \quad \text{otherwise} \end{cases}.$$

Now for every $0 < \varepsilon < 1$ and $t > 0$, write

$$\begin{aligned} K(\varepsilon,t) &= \{k \in I_r : \mathcal{F}(x_k - \ell, z; t) \leq 1 - \varepsilon\}, \ \ell = (0,0) \\ &= \left\{k \in I_r : \frac{t}{t + |x_k|} \leq 1 - \varepsilon\right\} = \left\{k \in I_r : |x_k| \geq \frac{t\varepsilon}{1-\varepsilon} > 0\right\} \\ &= \{k \in I_r : x_k = (k,0)\} = \left\{k \in I_r : k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r; r \in \mathbb{N}\right\}, \end{aligned}$$

so we get

$$\frac{1}{h_r}|K(\varepsilon,t)| \leq \frac{1}{h_r}|\left\{k \in I_r : k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r; r \in \mathbb{N}\right\}| \leq \frac{[\sqrt{h_r}]}{h_r}.$$

Taking limit r approaches to ∞, we get

$$\delta_\theta(K(\varepsilon,t)) = \lim_{r\to\infty} \frac{1}{h_r}|K(\varepsilon,t)| \leq \lim_{r\to\infty} \frac{[\sqrt{h_r}]}{h_r} = 0.$$

This shows that $x_k \to 0(S_\theta^{R2N}(X))$.

On the other hand the sequence is not $\mathcal{F}_\theta$−convergent to zero as

$$\mathcal{F}(x_k - \ell, z; t) = \frac{t}{t + |x_k|} = \begin{cases} \frac{t}{t+k} & , \quad \text{if } k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r; r \in \mathbb{N} \\ 1 & , \quad \text{otherwise} \end{cases} \leq 1.$$

Theorem 3.11. *Let $(X, \mathcal{F}, *)$ be a random n−normed space. If $x = (x_n)$ be a sequence in X, then $S^{RnN} - \lim x_n = \ell$ if and only if there exists a subset $K \subseteq N$ such that $\delta_\theta(K) = 1$ and $F_\theta - \lim x_n = \ell$.*

Proof. Suppose first that $S_\theta^{RnN} - \lim x_n = \ell$. Then for any $t > 0$, $s = 1, 2, 3, \ldots$ and non-zero elements $x_1, x_2, ..., x_{n-1} \in X$, let

$$A(s,t) = \left\{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - l; t) > 1 - \frac{1}{s}\right\}$$

and

$$K(s,t) = \left\{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - l; t) \leq 1 - \frac{1}{s}\right\}.$$

Since $S_\theta^{RnN} - \lim x_n = \ell$ it follows that

$$\delta_\theta(K(s,t)) = 0.$$

Now for $t > 0$ and $s = 1, 2, 3, \ldots$, we observe that

$$A(s,t) \supset A(s+1,t)$$

and

$$\delta_\theta(A(s,t)) = 1. \tag{3.2}$$

Now we have to show that, for $n \in A(s,t), \mathcal{F}_\theta - \lim x_n = \ell$. Suppose that for $n \in A(s,t)$, (x_n) not convergent to ℓ with respect to $\mathcal{F}_\theta$. Then there exists some $u > 0$ such that

$$\{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - l; t) \leq 1 - u\}$$

for infinitely many terms x_n. Let

$$A(u,t) = \{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - l; t) > 1 - u\}$$

and

$$u > \frac{1}{s},\ s = 1, 2, 3, \ldots.$$

Then we have

$$\delta_\theta(A(u,t)) = 0.$$

Furthermore, $A(s,t) \subset A(u,t)$ implies that $\delta_\theta(A(s,t)) = 0$, which contradicts (3.2) as $\delta_\theta(A(s,t)) = 1$. Hence $\mathcal{F}_\theta - \lim x_n = \ell$.

Conversely, suppose that there exists a subset $K \subseteq \mathbf{N}$ such that $\delta_\theta(K) = 1$ and $\mathcal{F}_\theta - \lim x_n = \ell$.

Then for every $\varepsilon > 0$, $t > 0$ and non-zero elements $x_1, x_2, ..., x_{n-1} \in X$, we can find out a positive integer k such that

$$\mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - l; t) > 1 - \varepsilon$$

for all $n \geq k$. If we take

$$K(\varepsilon,t) = \{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - l; t) \leq 1 - \varepsilon\}$$

then it is easy to see that

$$K(\varepsilon,t) \subset \mathbf{N} - \{k_{n+1}, k_{n+2}, \ldots\}$$

and consequently

$$\delta_\theta(K(\varepsilon,t)) \leq 1 - 1.$$

Hence $S_\theta^{RnN} - \lim x_n = \ell$. □

Finally, we establish the Cauchy convergence criteria in random $n-$normed spaces.

Theorem 3.12. *Let $(X, \mathcal{F}, *)$ be a random $n-$normed space. Then a sequence (x_n) in X is lacunary statistically convergent if and only if it is lacunary statistically Cauchy.*

Proof. Let (x_n) be a lacunary statistically convergent sequence in X. We assume that $S_\theta^{RnN} - \lim x_n = \ell$. Let $\varepsilon > 0$ be given. Choose $s > 0$ such that (3.1) is satisfied. For $t > 0$ and for non zero elements $x_1, x_2, ..., x_{n-1} \in X$ define

$$A(s,t) = \left\{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - l; \frac{t}{2}) \leq 1 - s\right\}$$

and

$$A^c(s,t) = \left\{n \in I_r : \mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - l; \frac{t}{2}) > 1 - s\right\}.$$

Since $S_\theta^{RnN} - \lim x_n = \ell$ it follows that $\delta_\theta(A(s,t)) = 0$ and consequently $\delta_\theta(A^c(s,t)) = 1$. Let $p \in A^c(s,t)$. Then

$$\mathcal{F}(x_1, x_2, ..., x_{n-1}, x_n - l; \frac{t}{2}) \leq 1 - s. \tag{3.3}$$

If we take

$$B(\varepsilon,t)=\{n\in I_r:\mathcal{F}(x_1,x_2,...,x_{n-1},x_n-x_p;t)\leq 1-\varepsilon\}$$

then to prove the result it is sufficient to prove that $B(\varepsilon,t)\subseteq A(s,t)$. Let $n\in B(\varepsilon,t)$, then for non-zero elements $x_1,x_2,...,x_{n-1}\in X$

$$\mathcal{F}(x_1,x_2,...,x_{n-1},x_n-x_p;t)\leq 1-\varepsilon. \tag{3.4}$$

If $\mathcal{F}(x_1,x_2,...,x_{n-1},x_n-x_p;t)\leq 1-\varepsilon$, then we have $\mathcal{F}(x_1,x_2,...,x_{n-1},x_n-l;\frac{t}{2})\leq 1-s$ and therefore $n\in A(s,t)$. As otherwise i.e., if $\mathcal{F}(x_1,x_2,...,x_{n-1},x_n-l;\frac{t}{2})>1-s$ then by (3.1), (3.3) and (3.4) we get

$$\begin{aligned}1-\varepsilon\geq&\mathcal{F}(x_1,x_2,...,x_{n-1},x_n-x_p;t)\\ \geq&\mathcal{F}(x_1,x_2,...,x_{n-1},x_n-l;\frac{t}{2})*\mathcal{F}(x_1,x_2,...,x_{n-1},x_p-l;\frac{t}{2})\\ >&(1-s)*(1-s)>(1-\varepsilon)\end{aligned}$$

which is not possible. Thus $B(\varepsilon,t)\subset A(s,t)$. Since $\delta_\theta(A(s,t))=0$, it follows that $\delta_\theta(B(\varepsilon,t))=0$. This shows that (x_n) is lacunary statistically Cauchy.

Conversely, suppose (x_n) is lacunary statistically Cauchy but not lacunary statistically convergent. Then there exists positive integer p and for non-zero elements $x_1,x_2,...,x_{n-1}\in X$ such that if we take

$$A(\varepsilon,t)=\{n\in I_r:\mathcal{F}(x_1,x_2,...,x_{n-1},x_n-x_p;t)\leq 1-\varepsilon\}$$

and

$$B(\varepsilon,t)=\{n\in I_r:\mathcal{F}(x_1,x_2,...,x_{n-1},x_n-l;\frac{t}{2})>1-\varepsilon\}$$

then

$$\delta_\theta(A(\varepsilon,t))=0=\delta_\theta(B(\varepsilon,t))$$

and consequently

$$\delta_\theta(A^c(\varepsilon,t))=1=\delta_\theta(B^c(\varepsilon,t)). \tag{3.5}$$

Since

$$\mathcal{F}(x_1,x_2,...,x_{n-1},x_n-x_p;t)\geq 2\mathcal{F}(x_1,x_2,...,x_{n-1},x_n-l;\frac{t}{2})>1-\varepsilon,$$

if $\mathcal{F}(x_1,x_2,...,x_{n-1},x_n-l;\frac{t}{2})>\frac{1-\varepsilon}{2}$ then we have

$$\delta_\theta(\{n\in I_r:\mathcal{F}(x_1,x_2,...,x_{n-1},x_n-x_p;t)>1-\varepsilon\})=0$$

i.e., $\delta_\theta(A^c(\varepsilon,t))=0$, which contradicts (3.5) as $\delta_\theta(A^c(\varepsilon,t))=1$. Hence (x_n) is lacunary statistically convergent. □

Combining Theorem 3.11 and Theorem 3.12 we get the following corollary.

Corollary 3.13. *Let $(X,\mathcal{F},*)$ be a random $n-$normed space and $x=(x_n)$ be a sequence in X. Then the following statements are equivalent:*

(a) *x is lacunary statistically convergent.*
(b) *x is lacunary statistically Cauchy.*
(c) *there exists a subset $K\subseteq N$ such that $\delta_\theta(K)=1$ and $\mathcal{F}_\theta-\lim x_n=\ell$.*

References

[1] C. Alsina, B. Schweizer and A. Sklar, Continuity properties of probabilistic norms, J. Math. Anal. Appl. 208 (1997) 446–452.

[2] J. Connor and M. A. Swardson, Measures and ideals of $C^*(X)$, Ann. New York Acad. Sci. 704 (1993) 80–91.

[3] H. Çakalli, A study on statistical convergence, Funct. Anal. Approx. Comput. 1(2) (2009) 19–24.

[4] A. Esi and M. K. Özdemir, Generalized Δ^m-statistical convergence in probabilistic normed space, J. Comput. Anal. Appl. 13(5) (2011) 923–932.

[5] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.

[6] J. A. Fridy, On statistical convergence, Analysis 5(4) (1985) 301–313.

[7] J. A. Fridy and C. Orhan, Lacunary statistical convergence, Pacific J. Math. 160(1) (1993) 43–51.

[8] S. Gähler, $2-$metrische Raume and ihre topologische Struktur, Math. Nachr. 26 (1963) 115–148.

[9] I. Goleţ, On probabilistic 2-normed spaces, Novi Sad J. Math. 35(1) (2005) 95–102.

[10] H. Gunawan and M. Mashadi, On n-normed space, Int. J. Math. Math. Sci. 27(10) (2001) 631–639.

[11] M. Gürdal and S. Pehlivan, The statistical convergence in 2-Banach spaces, Thai J. Math. 2(1) (2004) 107–113.

[12] M. Gürdal and S. Pehlivan, Statistical convergence in 2-normed spaces, Southeast Asian Bull. Math. 33(2) (2009) 257–264.

[13] B. Hazarika and A. Esi, On generalized Δ^n-Statistical convergence in random 2-normed space, (Submitted).

[14] S. Karakus, Statistical convergence on probabilistic normed spaces, Math. Commun. 12(1) (2007) 11–23.

[15] S. Karakus, K. Demirci and O. Duman, Statistical convergence on intuitionistic fuzzy normed spaces, Chaos Solitons Fractals 35(4) (2008) 763–769.

[16] V. Kumar and M. Mursaleen, On $(\lambda, \mu)-$statistical convergence of double sequences on intuitionistic fuzzy normed spaces, Filomat 25(2) (2011) 109–120.

[17] L. Leindler, Über die verallgemeinerte de la Vallée-Poussinsche Summierbarkeit allgemeiner Orthogonalreihen, Acta Math. Acad. Sci. Hungar. 16 (1965) 375–387.

[18] I. J. Maddox, Statistical convergence in a locally convex space, Math. Proc. Cambridge Philos. Soc. 104(1) (1988) 141–145.

[19] G. D. Maio and L. D. R. Kocinac, Statistical convergence in topology, Topology Appl. 156(1) (2008) 28–45.

[20] K. Menger, Statistical metrics, Proc. Natl. Acad. Sci. USA. 28 (1942) 535–537.

[21] H. I. Miller, A measure theoretical subsequence characterization of statistical convergence, Trans. Amer. Math. Soc. 347(5) (1995) 1811–1819.

[22] S. A. Mohiuddine and M. Aiyup, Lacunary statistical convergence in 2-normed space, Appl. Math. Inf. Sci. 6(3) (2012) 581–585.

[23] S. A. Mohiuddine and Q. M. Danish Lohani, On generalized statistical convergence in intuitionistic fuzzy normed space, Chaos Solitons Fractals 42(3) (2009) 1731–1737.

[24] M. Mursaleen, Statistical convergence in random 2-normed spaces, Acta Sci. Math. (Szeged) 76(1-2) (2010) 101–109.

[25] M. Mursaleen and A. Alotaibi, Statistical summability and approximation by de la Vallee-Pousin mean, Appl. Math. Lett. 24(3) (2011) 320–324.

[26] M. Mursaleen, C. Çakan, S. A. Mohiuddine and E. Savas, Generalized statistical convergence and statistical core of double sequences, Acta Math. Sin. (Engl. Ser.) 26(11) (2010) 2131–2144.

[27] M. Mursaleen and S. A. Mohiuddine, On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space, J. Comput. Appl. Math. 233(2) (2009) 142–149.

[28] M. Mursaleen and S. A. Mohiuddine, Statistical convergence of double sequences in intuitionistic fuzzy normed spaces, Chaos Solitons Fractals 41(5) (2009) 2414–2421.

[29] M. Mursaleen and Osama H. H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl. 288(1) (2003) 223–231.
[30] M. Mursaleen and Osama H. H. Edely, Generalized statistical convergence, Inform. Sci. 162(3-4) (2004) 287–294.
[31] M. Mursaleen and Osama H. H. Edely, On the invariant mean and statistical convergence, Appl. Math. Lett. 22(11) (2009) 1700–1704.
[32] T. Šalát, On statistical convergence of real numbers, Math. Slovaca 30(2) (1980) 139–150.
[33] I. J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly 66 (1959) 361–375.
[34] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960) 313–334.
[35] B. Schweizer and A. Sklar, Probabilistic Metric Spaces, North Holland, New York-Amsterdam-Oxford 1983.
[36] C. Sempi, A short and partial history of probabilistic normed spaces, Mediterr. J. Math. 3(2) (2006) 283–300.
[37] C. Sencimen and S. Pehlivan, Statistical convergence in fuzzy normed linear spaces, Fuzzy Sets and Systems 159(3) (2008) 361–370.
[38] A. N. Serstnev, On the notion of a random normed space, Dokl. Akad. Nauk SSSR. 149 (1963) 280–283.

Ayhan Esi (aesi23@hotmail.com)
Adiyaman University, Science and Art Faculty, Department of Mathematics, 02040, Adiyaman, Turkey

M. Kemal Özdemir (kozdemir73@gmail.com)
Inonu University, Science and Art Faculty, Department of Mathematics, 44280, Malatya, Turkey